General
Variational Methods,
SS09
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Exercises and Examples

Exercise 1:

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $X = W_0^{1,2}(\Omega, \mathbb{R})$. For $z \in X$ and $\ell \in C^1([0,T]; X')$ we define the energy via

$$\mathcal{E}(t,z) = \frac{1}{2}\int_\Omega |\nabla z(x)|^2 \,\mathrm{d}x + \int_\Omega \chi_{[-1,1]}(z(x))\,\mathrm{d}x - \langle \ell(t), z\rangle_{X',X}.$$

The dissipation potential is given by

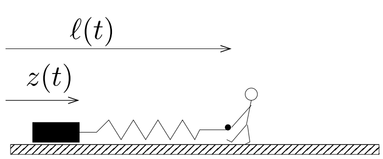
$$\mathcal{R}(\zeta) = \int_\Omega \mu\,|\zeta(x)|\,\mathrm{d}x.$$

Verify the conditions (A0)–(A3) introduced in the lecture.

Exercise 2: Unidirectional processes

Within the global energetic formulation it is possible to model unidirectional processes. This means that a certain state variable e.g. can increase only, but not decrease. This behavior can be modeled by a suitable choice of the dissipation potential.

Example: Again the gliding mass problem (see the figure below):



Energy: $\mathcal{E}(t,z) = \frac{\kappa}{2}(\ell(t) - z)^2$ for $z \in \mathbb{R} = X$.
Dissipation: for some reason, the mass can move to the right, only. Hence $\partial_t z(t) \geq 0$ for almost every t and

$$\mathcal{R}(\zeta) = \begin{cases} \mu\zeta & \text{if } \zeta \geq 0 \\ \infty & \text{otherwise} \end{cases},$$

$\kappa, \mu > 0$ given constants.

Verify the conditions (A0)–(A3) and show that the solutions of the global energetic model formulated with these functionals $\mathcal{E}$ and $\mathcal{R}$ satisfy $\partial_t z(t) \geq 0$ a.e. in Ω.

Derive the formulation of the problem as a differential inclusion.

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Exercise 3: Quadratic energies

Let $X = V \times Z$, where V is a real, reflexive and separable Banach space and Z is a separable Hilbert space. Let V' be the dual of V, while we identify Z with its dual.

Assume that $A \in \mathrm{Lin}(X, X')$ is a linear, bounded operator having the property

$$\langle Aq_1, q_2\rangle_{X',X} = \langle Aq_2, q_1\rangle_{X',X}$$

for all $q_i \in X$ and being X-coercive, i.e. there exists a constant $\alpha > 0$ such that for all $q \in X$:

$$\langle Aq, q\rangle_{X',X} \geq \alpha \|q\|^2 .$$

Given $\ell \in C^1([0,T]; V')$ let the energy and the dissipation potential be defined as follows for $q = (v, z) \in V \times Z = X$:

$$\begin{aligned} \mathcal{E}(t,q) &= \tfrac{1}{2}\langle Aq, q\rangle_{X',X} - \langle \ell(t), v\rangle_{V',V} \\ \mathcal{R}(q) &= \widetilde{\mathcal{R}}(z), \end{aligned}$$

where $\widetilde{\mathcal{R}} : z \to [0, \infty]$ is assumed to be convex, strongly lower semicontinuous and positively homogeneous of degree 1.

(a) Verify that conditions (A0)–(A3) are satisfied.
(Hint concerning (A3) (compatibility condition): given sequences $q_n \rightharpoonup q_*$, $t_n \to t_*$ with $q_n \in S(t_n)$ it could be quite interesting to investigate the test-sequences $\xi_n = \xi + q_n - q_*$ for arbitrary $\xi \in X$ and to take into account the quadratic structure of $\mathcal{E}$.)

(b) Assume that in addition to the above A has the particular structure

$$A = \begin{pmatrix} C & -CB \\ -B^*C & B^*CB + L \end{pmatrix}, \tag{1}$$

where $C \in \mathrm{Lin}(V, V')$, $B \in \mathrm{Lin}(Z, V')$ and $L \in \mathrm{Lin}(Z, Z)$.
Show that if C is V-coercive and L is Z-coercive, then A is X-coercive.

Derive the differential inclusion formulation for the rate independent process associated with $\mathcal{E}$ and $\mathcal{R}$ and the operator given in (1).

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Exercise 4: An application of Exercise 3

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, $V = W_0^{1,2}(\Omega, \mathbb{R})$ and $Z = L^2(\Omega, \mathbb{R}^d)$. We define the energy and dissipation potential as

$$\mathcal{E}(t,u,z) = \int_\Omega \frac{1}{2} |\nabla u(x) - z(x)|^2 + \frac{\kappa}{2} |z(x)|^2 \,\mathrm{d}x - \int_\Omega f(t,x)u(x)\,\mathrm{d}x,$$
$$\mathcal{R}(u,z) = \int_\Omega \mu\, |z(x)| \,\mathrm{d}x$$

for given $f \in C^1([0,T]; L^2(\Omega))$ and constants $\kappa, \mu > 0$.

Show with the help of Exercise 3 (in particular 3(b)) that the rate independent process defined via $\mathcal{E}$ and $\mathcal{R}$ has a solution and derive the corresponding formulation as a differential inclusion.

This is the typical structure of elasto-plastic models if one interprets u as a kind of displacements and z as a vector of internal variables, here the plastic strains. Of course, in true elasto-plasticity, one has to deal with vector valued functions u.