Exercise sheet 1

Exercise 1.1: Determine general solutions $u = u(x, y)$ of the following PDEs:

$$\text{a)}\ \frac{\partial u}{\partial x} = 1, \qquad \text{b)}\ \frac{\partial^2 u}{\partial y^2} = 1, \qquad \text{c)}\ \frac{\partial^2 u}{\partial x \partial y} = 1, \qquad \text{d)}\ 3\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0\,.$$

Exercise 1.2: Check whether the following PDEs are semilinear, quasilinear or fully nonlinear and determine their order:

a) Nonlinear Poisson equation: $-\Delta u = f(u)$,
b) p-Laplace equation: $\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0$ with $p \in (1, \infty)$,
c) Navier-Stokes system:

$$u_t + u \cdot \nabla u - \nu \Delta u = -\nabla p, \quad \operatorname{div} u = 0\,,$$

where $u = (u_1, u_2, u_3)$ is a velocity field, p is a scalar pressure field, and ν is a material constant.

Exercise 1.3: Separation of variables. Consider the following PDEs:

$$\text{(i)}\ u_{tt} - a^2 u_{xx} = 0, \qquad \text{(ii)}\ u_{tt} + u_{xx} = 0, \qquad \text{(iii)}\ u_t = u_{xx}.$$

a) Construct solutions for the above PDEs via separation of variables, i.e. solutions have the form $u(t, x) = T(t)X(x)$ for real valued twice differentiable functions T, X. (Hint: Insert this ansatz into the PDE, separate the variables on each side of the '=' and argue that both sides have to be constant.)
b) For the heat equation (iii) determine all the solutions of a), which additionally satisfy $X(0) = X(1) = 0$. Then, construct a solution of (iii), which satisfies the following boundary and initial conditions:

$$u(t, 0) = u(t, 1) = 0, \ u(0, x) = 33.3 \sin(15\pi x)\,.$$

c) For the wave equation (i) determine all the solutions of (a), which additionally satisfy $X(0) = X(l) = 0$ with given $l > 0$. Show that $u(t + \frac{2nl}{a}, x) = u(t, x)$ with $n \in \mathbb{Z}$.

(please turn)

Exercise 1.4 (written): One dimensional wave equation.

a) Determine solutions $u = u(t, x)$ of the spatially one dimensional wave equation using the coordinate transformation $\xi_1 = x - at$, $\xi_2 = x + at$:

$$u_{tt} - a^2 u_{xx} = 0\,. \tag{1}$$

Solutions u describe the oscillation of an infinitely long string.

b) Under which conditions is $u(t, x) := f(x + at) + g(x - at)$ a general solution of (1)? Determine the functions f, g in such a way that the initial conditions

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x) \tag{2}$$

are satisfied. This representation of the solution is the so-called d'Alembert's formula.

c) Plucked string: For the initial deflection

$$u_0(x) := \begin{cases} 2 - 2|x| & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

and the initial velocity $u_1(x) = 0$ the oscillation $u(t, x)$ is to be visualized graphically at time $t = 0$, $t = 1/2$, $t = 1$ and $t = 3/2$ for two different velocities a.

First exercise lesson: Monday, April 16, 2012, 11-13h, room RUD25, 4.007.

Ex. 1.4 is to be delivered in written form by teams of two persons each in the exercise lesson on 16/04/2012. It will be discussed in the subsequent week.