Partial Differential Equations
Higher Analysis II, summer term 2012
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Exercise sheet 10

Exercise 10.1: Weak derivatives

Let $\Omega = B_1(0) \subset \mathbb{R}^d$ with $d \geq 2$ and $u(x) = 1/|x|$. For which p do we have $u \in L^p(\Omega)$ and for which q does the classical gradient $\nabla_{\text{cl}} u$ lie in $L^q(\Omega)$? Show that for these values the weak gradient coincides with $\nabla_{\text{cl}} u$.

Exercise 10.2

(a) **Fundamental Theorem of Differentiation and Integration.** Let $I \subset \mathbb{R}$ be an open intervall. If $u \in W^{1,1}(I)$, then for almost all $x_1, x_2 \in I$ it holds

$$u(x_2) - u(x_1) = \int_{x_1}^{x_2} u'(x)\,\mathrm{d}x.$$

(b) **An embedding inequality.** Let $(a,b) \subset \mathbb{R}$ be a bounded intervall. Let $p \in (1,\infty)$ and $\alpha = 1-\frac{1}{p}$. Show that there exists a constant $C > 0$ such that for all $u \in C^1([a,b])$ and all $x_0 \in [a,b]$ it holds

$$\|u\|_{C^{0,\alpha}([a,b])} \leq |u(x_0)| + C\,\|Du\|_{L^p(a,b)}$$

and derive the embedding inequality

$$\exists \widetilde{C} > 0 \text{ such that for all } u \in W^{1,p}(a,b)\text{:}\;\; \|u\|_{C^{0,\alpha}([a,b])} \leq \widetilde{C}\,\|u\|_{W^{1,p}(a,b)}\,.$$

Exercise 10.3: Poincaré-Friedrichs' **inequality I**

A domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the Poincaré-Friedrichs' inequality, if there exists a constant $C > 0$ such that it holds

$$\text{for all } u \in C_0^\infty(\Omega)\text{:}\quad C\,\|u\|_{L^2(\Omega)} \leq \|\nabla u\|_{L^2(\Omega)}\,. \tag{1}$$

(a) Show that (1) holds if and only if the same inequality is valid for all $u \in H_0^1(\Omega)$.
(b) Show that every open and bounded domain satisfies a Poincaré-Friedrichs' inequality and that for unbounded domains both cases can occur.

Exercise 10.4: Poincaré-Friedrichs' **inequality II**

Prove the following version of the Poincaré-Friedrichs' inequality:

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and $V \subset H^1(\Omega)$ a closed subspace with the property $\quad u \in V$ and $\|\nabla u\|_{L^2(\Omega)} = 0$ implies $u = 0$.

Then there exists a constant $c_{PF} > 0$ such that for all $u \in V$ it holds

$$\|\nabla u\|_{L^2(\Omega)} \geq c_{PF}\,\|u\|_{H^1(\Omega)}\,.$$

HINT: Prove the inequality by contradiction assuming that for all $n \in \mathbb{N}$ $\exists u_n \in V\backslash\{0\}$ such that $\|\nabla u_n\|_{L^2(\Omega)} \leq \frac{1}{n}\|u_n\|_{W^{1,p}(\Omega)}$. Use the part (c) of the embedding Theorem 6.4 and remember that $H^1(\Omega)$ is reflexive. (See also Exercise 8.2 for a comparable argument).

(please turn)

Exercise 10.5 (written): $W_0^{1,p}(\mathbb{R}^d) = W^{1,p}(\mathbb{R}^d)$

Let $p \in [1, \infty)$.

(a) Prove the product rule: For $\eta \in C_0^\infty(\mathbb{R}^d)$ and $u \in W^{1,p}(\mathbb{R}^d)$ we have $\eta u \in W^{1,p}(\mathbb{R}^d)$ and $D^\alpha(\eta u) = u D^\alpha \eta + \eta D^\alpha u$ for $|\alpha| = 1$, $\alpha \in \mathbb{N}_0^d$ multi-index.

(b) Let $\varrho \in C_0^\infty(\mathbb{R}^d)$ with $\operatorname{supp} \varrho \subset B_1(0)$, $0 \leq \varrho(x) \leq 1$ and $\int_{\mathbb{R}^d} \varrho(x)\,\mathrm{d}x = 1$. For $\varepsilon > 0$ we set $\varrho_\varepsilon(x) := \frac{1}{\varepsilon^d} \varrho(\frac{x}{\varepsilon})$. For $v \in W^{1,p}(\mathbb{R}^d)$ with compact support we define

$$v_\varepsilon(x) := (\varrho_\varepsilon * v)(x) = \int_{\mathbb{R}^d} \varrho_\varepsilon(x - y) v(y)\, dy, \ x \in \mathbb{R}^d.$$

Prove: For all $\varepsilon > 0$ we have $v_\varepsilon \in C_0^\infty(\mathbb{R}^d)$ and $\lim_{\varepsilon \to 0} \|v_\varepsilon - v\|_{W^{1,p}(\Omega)} = 0$. This implies in particular that $v \in W_0^{1,p}(\mathbb{R}^d)$.

Hint: Without proof one may use that for all $f \in L^p(\Omega)$: $\lim_{\varepsilon \to 0} \|\varrho_\varepsilon * f - f\|_{L^p(\mathbb{R}^d)} = 0$.

(c) Prove that $W^{1,p}(\mathbb{R}^d) = W_0^{1,p}(\mathbb{R}^d)$.

Hint: Let $u \in W^{1,p}(\mathbb{R}^d)$. Choose a suitable sequence of cut-off functions $(\eta_n)_{n \in \mathbb{N}}$ with increasing supports such that $\eta_n u$ converges to u in $W^{1,p}(\mathbb{R}^d)$.

Ex. 10.5 is to be delivered in written form by teams of two persons each in the exercise lesson on 25/06/2012. It will be discussed in the subsequent week.