Exercise sheet 11

Exercise 11.1: Compact operators on Banach spaces

Let X, Y be Banach spaces.

(a) Let X be reflexive and $A : X \to Y$ be strongly continuous (i.e. $u_n \rightharpoonup u$ weakly in X implies that $A(u_n) \to A(u)$ strongly in Y). Show that A is a compact operator, meaning that A maps bounded sets on relatively compact sets.

(b) Show that the inverse assertion is valid if A is linear: Let $A : X \to Y$ be linear and compact. Then for all $u_n \rightharpoonup u$ weakly in X it follows that $A(u_n) \to A(u)$ strongly in Y.

Exercise 11.2: A collection of questions

Let $\Omega = B_1(0) \subset \mathbb{R}^d$.

(a) Let $p \in [1, \infty)$. For $v \in L^r(\Omega)$ define $\ell_v : W^{1,p}(\Omega) \to \mathbb{R}$ via $\ell_v(u) := \int_\Omega v(x)u(x)\,\mathrm{d}x$, $u \in W^{1,p}(\Omega)$. Determine the smallest $r \geq 1$ such that for all $v \in L^r(\Omega)$ it holds $\ell_v \in (W^{1,p}(\Omega))'$ and justify your answer.

(b) Determine (and justify your answer) $k \in \mathbb{N}_0, p \in [1, \infty)$ such that the operator $\delta_0 : C^\infty(\overline{\Omega}) \to \mathbb{R}$, defined via $\delta_0[u] := u(0)$, has a (unique) extension to an element from $(W^{1,p}(\Omega))'$.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $p > d$. Show: if the sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p}(\Omega)$ converges weakly in $W^{1,p}(\Omega)$ to $u \in W^{1,p}(\Omega)$ for $n \to \infty$, then all the intgrals $\int_\Omega f(u_n(x))\,\mathrm{d}x$ are finite and $\lim_{n\to\infty} \int_\Omega f(u_n(x))\,\mathrm{d}x = \int_\Omega f(u(x))\,\mathrm{d}x$.

Remark: This observation will be useful in the lecture *Calculus of Variations (Variationsrechnung)* which will be held in the winter term 12/13.

(d) An unbounded function in $H^1(B_{\frac{1}{2}}(0))$ in two dimensions: Let $\Omega = B_{\frac{1}{2}}(0) \subset \mathbb{R}^2$. Show that $u(x) := \ln(\ln(\frac{1}{|x|}))$ belongs to $H^1(B_{\frac{1}{2}}(0))$. Connection to Exercise 11.2.(b)?

Exercise 11.3.

Let $\Omega \subset \mathbb{R}^d$ be bounded with Lipschitz boundary. For $g_i \in H^1(\Omega)$, $i \in \{1, 2\}$, with $\gamma_0(g_1) = \gamma_0(g_2)$ (γ_0 trace operator) consider the problems: Find $u_i \in H^1_0(\Omega)$, $i \in \{1, 2\}$, satisfying for all $v \in H^1_0(\Omega)$ the equations $\int_\Omega \nabla u_i \cdot \nabla v\,\mathrm{d}x = -\int_\Omega \nabla g_i \cdot \nabla v\,\mathrm{d}x$.

Prove that for each i this problem has a unique solution $u_i \in H^1_0(\Omega)$ and show that $u_1 + g_1 = u_2 + g_2$ in Ω. Interpretation?

(please turn)

Exercise 11.4: Coercivity of bilinear forms

Let $\Omega \subset \mathbb{R}^d$ be a domain (i.e. open and connected). On $H := H_0^1(\Omega)$ the bilinear form $a : H \times H \to \mathbb{R}$ is defined as

$$a(u,v) = \int_\Omega \nabla u(x) \cdot A(x)\nabla v(x) + c(x)u(x)v(x)\,\mathrm{d}x,$$

where $A \in L^\infty(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}})$ and $c \in L^\infty(\Omega)$. Further, there exists $\alpha > 0$ such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^d$ it holds $\xi \cdot A(x)\xi \geq \alpha\,|\xi|^2$.

(a) Show that the bilinear form a is symmetric and continuous. Give sufficient conditions on c that guarantee that a is also coercive on H. Investigate this question for bounded and for unbounded domains.

(b) Consider the Schrödinger operator $A_\lambda u := -\triangle u + Vu + \lambda u$ with $\Omega = \mathbb{R} \times (0,\pi)$, where $V(x) = 1/(1+x_1^2)$. Show that the bilinear form associated with A_λ is coercive on H for $\lambda > -1$ and that it is not coercive for $\lambda < -1$. What is valid for $\lambda = -1$?

Hint: Without proof (we will discuss this later in the lecture) one may use the following: $\forall u \in H$ with Ω from (b) we have $\|\nabla u\|_{L^2(\Omega)} \geq \|u\|_{L^2(\Omega)}$ and there exists $u_0 \in H$, $u_0 \neq 0$, with $\|\nabla u_0\|_{L^2(\Omega)} = \|u_0\|_{L^2(\Omega)}$.

Exercise 11.5: (written) Robin boundary conditions

Consider the domain $\Omega = (0,\pi)^2$ with Dirichlet-boundary $\Gamma_D := \{\,(x_1,x_2) \in \mathbb{R}^2\,;\, x_1 \in \{0,\pi\},\, x_2 \in (0,\pi)\,\}$ and the bilinear form

$$a(u,v) = \int_\Omega \nabla u(x) \cdot \nabla v(x) + \big(\mu \cdot \nabla u(x)\big)v(x) + \alpha u(x)v(x)\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_D} \rho u(x)v(x)\,\mathrm{d}a_x,$$

which is defined on the Hilbert space $H^1_{\Gamma_D}(\Omega) := \{\, v \in H^1(\Omega)\,;\, v\big|_{\Gamma_D} = 0 \text{ in the trace sense}\,\}$. Further, $\mu \in \mathbb{R}^d, \alpha, \rho \in \mathbb{R}$ are some given constants.

(a) Show that $H^1_{\Gamma_D}(\Omega)$ is a closed subspace of $H^1(\Omega)$, that $a : H^1_{\Gamma_D}(\Omega) \times H^1_{\Gamma_D}(\Omega) \to \mathbb{R}$ is continuous and discuss its corecivity. Is it possible to have coercivity also for negative values of α or ρ?

(b) Let $f \in L^2(\Omega)$ and $h \in L^2(\partial\Omega\setminus\Gamma_D)$. Investigate the existence and uniqueness of weak solutions of the equation

$$a(u,v) = \int_\Omega fv\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma_D} h(x)v(x)\,\mathrm{d}a_x \quad \forall v \in H^1_{\Gamma_D}(\Omega). \tag{1}$$

(c) Derive the strong form of the boundary value problem that is associated to the weak equation defined by (1).

Ex. 11.5 is to be delivered in written form by teams of two persons each in the exercise lesson on 02/07/2012. It will be discussed in the subsequent week.