Exercise sheet 12

Exercise 12.1: Spaces involving time

Let X be a Banach space.

(a) Let $u : [0,T] \to X$ be Bochner integrable.

(i) Prove that $\left\| \int_0^T u(t)\,\mathrm{d}t \right\|_X \le \int_0^T \|u(t)\|_X\,\mathrm{d}t$.

(ii) Prove that for all $\ell \in X'$ it holds: $\langle \ell, \int_0^T u(t)\,\mathrm{d}t\rangle_{X',X} = \int_0^T \langle \ell, u(t)\rangle_{X',X}\,\mathrm{d}t$.

(b) We recall for $p \in [1,\infty)$ that

$L^p(0,T;X) = \{\, u : [0,T] \to X\,;\, u$ Bochner measurable and $\int_0^T \|u(t)\|_X^p\,\mathrm{d}t < \infty \,\}$.

Further, $v \in L^1(0,T;X)$ is the weak derivative of $u \in L^1(0,T;X)$, $v = u'$, if

$$\forall \varphi \in C_0^\infty(0,T;\mathbb{R}) : \quad \int_0^T \varphi'(t)u(t)\,\mathrm{d}t = -\int_0^T \varphi(t)v(t)\,\mathrm{d}t.$$

Let $V := H_0^1(\Omega)$ and $H := L^2(\Omega)$. Let $(u_k)_{k\in\mathbb{N}} \subset L^2(0,T;V)$ such that for all $k \in \mathbb{N}$ the weak derivative u_k' exists and belongs to $L^2(0,T;V')$. Let further $u \in L^2(0,T;V)$ and $v \in L^2(0,T;V')$ such that

$$u_k \rightharpoonup u \quad \text{weakly in } L^2(0,T;V), \qquad u_k' \rightharpoonup v \quad \text{weakly in } L^2(0,T;V').$$

Show that $u' = v$.

HINT: Use and verify that for all $k \in \mathbb{N}$, $\varphi \in C_0^\infty(0,T;\mathbb{R})$ and $w \in V$ it holds

$$\int_0^T \langle u_k'(t), \varphi(t)w\rangle_{V',V}\,\mathrm{d}t = -\int_0^T \langle u_k(t), \varphi'(t)w\rangle_H\,\mathrm{d}t,$$

with $\langle w_1, w_2\rangle_H = \int_\Omega w_1(x)w_2(x)\,\mathrm{d}x$ for $w_1, w_2 \in H$.

Exercise 12.2: Fourier series

Let $\mathbb{S} = (0,2\pi)$ and consider for $k \in \mathbb{N}$ the Hilbert space

$$W^{k,2}_{\mathrm{per}}(\mathbb{S}) := \{\, f \in W^{k,2}(\mathbb{S})\,;\, D^j f(0) = D^j f(2\pi) \text{ for } j = 0,\ldots,k-1 \,\}.$$

For $n \in \mathbb{N}$ let $S_n(\varphi) = s_n \sin(n\varphi)$ and for $m \in \mathbb{N}_0$ let $C_m(\varphi) = c_m \cos(m\varphi)$. The constants s_n, c_m are chosen such that $\|S_n\|_{L^2(\mathbb{S})} = 1 = \|C_m\|_{L^2(\mathbb{S})}$ for all n, m. From *Functional Analysis* we know that the set

$$\mathcal{O} := \{\, S_n, C_m\,;\, n \in \mathbb{N},\, m \in \mathbb{N}_0 \,\}$$

is a complete orthonormal system in $L^2(\mathbb{S})$. Hence, for all $f \in L^2(\mathbb{S})$ we have $\|f\|^2_{L^2(\mathbb{S})} = \sum_{n=0}^\infty \big(\langle f, S_n\rangle^2 + \langle f, C_n\rangle^2\big)$ and $f = \sum_{n=0}^\infty \big(\langle f, S_n\rangle S_n + \langle f, C_n\rangle C_n\big)$, where $\langle\cdot,\cdot\rangle$ is the scalar product in $L^2(\mathbb{S})$.

(continued on next page)

Let $f \in L^2(\mathbb{S})$ with $f = \sum_{n=0}^{\infty} a_n S_n + b_n C_n$. Show that

$$f \in W^{k,2}_{\text{per}}(\mathbb{S}) \qquad \Longleftrightarrow \qquad \sum_{n=0}^{\infty} (1+n^2)^k (a_n^2 + b_n^2) < \infty$$

and that in this case we may differentiate the series representation of f term by term.

Hint: Compare the series differentiated term by term with a suitable new expansion of the derivative.

Exercise 12.3: Neumann problem on a disc

Let $\Omega = B_1(0) \subset \mathbb{R}^2$ and consider the Neumann-problem

$$\triangle u = 0 \qquad \text{for } x \in \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} = g \qquad \text{for } x \in \partial\Omega. \tag{2}$$

(a) For $g_N(\varphi) := \sum_{n=1}^{N} a_n S_n(\varphi) + b_n C_n(\varphi)$, $\varphi \in \mathbb{S} = (0, 2\pi)$ and S_n, C_n as in Exercise 12.2, construct the solution u_N of the Neumann-problem (1)–(2) and calculate the norm of u_N in $L^2(\Omega)$ and $H^1(\Omega)$.

Hint: Use separation ansatzes of the form $R(r)v(\varphi)$ and superposition. Observe that $\|\nabla_x u\|^2_{L^2(\Omega)} = \int_0^1 \int_0^{2\pi} r\,|\partial_r u|^2 + r^{-1}\,|\partial_\varphi u|^2 \,\mathrm{d}\varphi \,\mathrm{d}r$.

(b) Investigate via the limit $N \to \infty$ for which functions $g : \partial\Omega \simeq \mathbb{T} \to \mathbb{R}$ solutions of (1)–(2) exist and belong to $H^1(\Omega)$. For $s \in \mathbb{R}$ use the spaces

$$H^s(\mathbb{T}) := \{\, v \in \mathcal{D}'(\mathbb{T})\,;\ \sum_{n \in \mathbb{N}_0} (1+n^2)^s \big(\langle v, S_n\rangle^2 + \langle v, C_n\rangle^2\big) < \infty \,\}$$

Interpret your result in view of the statements of Exercise 12.5 and discuss the connection with weak solutions.

Exercise 12.4: Wave equation and Fourier method

OPTIONAL!

Let $\Omega = (0, \pi)^2$. We consider the initial value problem for the wave equation:

$$\partial_t^2 u(t,x) - \triangle_x u(t,x) = 0, \qquad t \in (0,T),\ x \in \Omega, \tag{3}$$

$$u(t,x) = 0, \qquad x \in \partial\Omega, \tag{4}$$

$$u(0,x) = u_0(x), \qquad x \in \Omega, \tag{5}$$

$$\partial_t u(0,x) = u_1(x), \qquad x \in \Omega. \tag{6}$$

(a) Use a separation ansatz ($\widetilde{u}(t,x) = T_\omega(t) u_n(x_1) v_m(x_2)$) to construct a solution with initial data $u_0(x) = \frac{2}{\pi} \sum_{n,m=1}^{N} b_{n,m} \sin(nx_1)\sin(mx_2)$ and $u_1(x) = \frac{2}{\pi} \sum_{n,m=1}^{N} a_{n,m} \sin(nx_1)\sin(mx_2)$.

(b) By studying the limit $N \to \infty$ derive conditions on the coefficients $(a_{n,m})_{n,m\in\mathbb{N}}$ and $(b_{n,m})_{n,m\in\mathbb{N}}$ such that the corresponding solution satisfies $u \in L^2(0,T; H^1_0(\Omega))$ and $\partial_t u \in L^2(0,T; L^2(\Omega))$.

(please turn)

Exercise 12.5: Fourier series and $H^s(\mathbb{T})$ (extension of Exercise 12.2 to $s \in \mathbb{R}$)
OPTIONAL!

Let $\mathbb{T} = \partial B_1(0) \subset \mathbb{R}^2$ be the one-dimensional torus. The set $\mathcal{D}(\mathbb{T}) \equiv C_0^\infty(\mathbb{T})$ can be identified with the set $C^\infty_{\text{per}}([0,2\pi]) := \{\, v \in C^\infty([0,2\pi])\,;\, \forall k \in \mathbb{N}_0 : \; D^k v(0) = D^k v(2\pi) \,\}$.

For $n \in \mathbb{N}_0$ let $S_n, C_n \in \mathcal{D}(\mathbb{T})$ be the functions from Exercise 12.2.

For $s \in \mathbb{R}$ we define

$$H^s(\mathbb{T}) := \{\, v \in \mathcal{D}'(\mathbb{T})\,;\, \sum_{n\in\mathbb{N}_0} (1+n^2)^s \big(\langle v, S_n\rangle^2 + \langle v, C_n\rangle^2\big) < \infty \,\},$$

$$\|v\|_s := \left(\sum_{n\in\mathbb{N}_0} (1+n^2)^s \big(\langle v, S_n\rangle^2 + \langle v, C_n\rangle^2\big) \right)^{\frac{1}{2}}.$$

Then it holds

(i) For all $s \in \mathbb{R}$ the spaces $H^s(\mathbb{T})$ are Hilbert spaces with scalar product $(u,v)_s = \sum_{n\in\mathbb{N}_0} (1+n^2)^s \big(u_{1,n}v_{1,n} + u_{2,n}v_{2,n}\big)$, where $u_{1,n} = \langle u, S_n\rangle$ and $u_{2,n} = \langle u, C_n\rangle$.

(ii) Dual spaces: For all $s \in \mathbb{R}$ we have $(H^s(\mathbb{T}))' = H^{-s}(\mathbb{T})$, where for $u \in H^s(\mathbb{T})$ and $v \in H^{-s}(\mathbb{T})$ the following pairing is used

$$\langle v, u\rangle_{-s,s} := \sum_{n\in\mathbb{N}_0} \langle v, S_n\rangle\langle u, S_n\rangle + \langle v, C_n\rangle\langle u, C_n\rangle.$$

(iii) For all $s \geq 0$ the spaces $H^s(\mathbb{T})$ and $W^{s,2}(\mathbb{T})$ are isomorphic, where, for $k \in \mathbb{N}_0$ and $\sigma \in (0,1)$,

$$W^{k,2}(\mathbb{T}) = \{\, v \in L^2(\mathbb{T})\,;\, \text{the weak derivatives order} \leq k \text{ exist and belong to } L^2(\mathbb{T}) \,\},$$

$$W^{k+\sigma,2}(\mathbb{T}) = \{\, v \in W^{k,2}(\mathbb{T})\,;\, \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\big|D^k v(x) - D^k v(y)\big|^2}{|x-y|^{1+2\sigma}} \, \mathrm{d}a_x \, \mathrm{d}a_y < \infty \,\}.$$

References:

Hans-Jürgen Schmeisser, Hans Triebel: Topics in Fourier Analysis and Function Spaces. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig 1987. Chapter 3: Periodic Spaces.

H.W. Alt, Lineare Funktionalanalysis, Springer (2006)

D. Werner, Funktionalanalysis, Springer (2005)

The last tutorial will be on WEDNESDAY, July 11.
Exam dates: July 24+25, 2012, September 27+28, 2012.