Exercise sheet 4

Exercise 4.1:

a) Let $[a,b] \subset \mathbb{R}$ and $f \in \mathrm{C}([a,b])$. Show that $\int_a^b f v' \,\mathrm{d}x = 0$ for all $v \in \mathrm{C}^1([a,b])$ with $v(a) = v(b) = 0$ implies that f is constant on $[a,b]$.
Hint: construct a test function v with $v'(x) = f(x) - \gamma$ for some constant γ.

b) Let $\Omega \subset \mathbb{R}^d$ be a bounded, open domain. Let $\mathbf{k} \in \mathbb{R}^d$. Consider the functions

$$\Phi_{\mathbf{k}}(x) := \begin{cases} \cos(\frac{\mathbf{k}\cdot x}{\varepsilon}) c_\varepsilon \mathrm{e}^{-\frac{\varepsilon^2}{\varepsilon^2 - |x-x_0|^2}} & \text{if } x \in B_\varepsilon(x_0), \\ 0 & \text{otherwise,} \end{cases}$$

$$\Psi_{\mathbf{k}}(x) := \begin{cases} \sin(\frac{\mathbf{k}\cdot x}{\varepsilon}) c_\varepsilon \mathrm{e}^{-\frac{\varepsilon^2}{\varepsilon^2 - |x-x_0|^2}} & \text{if } x \in B_\varepsilon(x_0), \\ 0 & \text{otherwise,} \end{cases}$$

where $B_\varepsilon(x_0) := \{x \subset \mathbb{R}^d,\ |x - x_0| < \varepsilon\}$. Verify that $\Phi_{\mathbf{k}}, \Psi_{\mathbf{k}} \in \mathrm{C}_0^\infty(\Omega)$ for $x_0 \in \Omega$ and $\overline{B_\varepsilon(x_0)} \subset \Omega$.

c) Prove the **Lemma of Du Bois-Reymond**: Let $\Omega \subset \mathbb{R}^d$ be a bounded, open domain. Let $f \in L^1_{loc}(\Omega)$, i.e. $f \in L^1(C)$ for every bounded interior subset $C \subset \Omega$. Then,

$$f = 0 \text{ a.e. in } \Omega \quad \Leftrightarrow \quad \int_\Omega f v \,\mathrm{d}x = 0 \ \text{ for all } v \in \mathrm{C}_0^\infty(\Omega)\,.$$

Hint: Argue via a Fourier expansion.

Exercise 4.2: We consider the following Cauchy problem of the Burgers' equation:

$$u_t + u u_x = 0 \quad \text{in } (0,\infty) \times \mathbb{R}, \tag{1a}$$

$$u(0,x) = \begin{cases} u_l & \text{if } x < 0, \\ u_r & \text{if } x > 0, \end{cases} \tag{1b}$$

where $u_l > u_r$.

a) Construct a weak solution $u : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ of (1) which satisfies the Rankine-Hugoniot condition.

b) Verify that u obtained in a) indeed is a weak solution of (1).

Exercise 4.3: Consider the Burgers' equation (1a) together with the initial condition

$$u(0,x) = u_0(x) = \begin{cases} 0 & \text{if } x < 0\,, \\ 1 & \text{if } 0 \le x \le 2\,, \\ 2 & \text{if } x > 2 \end{cases}$$

and construct a solution which satisfies the Rankine-Hugoniot condition.

(please turn)

Exercise 4.4 Vanishing viscosity solutions (written): We consider the following Cauchy problem:

$$u_t + a u_x = \varepsilon u_{xx} \text{ in } (0,\infty)\times\mathbb{R}\,, \tag{2a}$$

$$u(0,x) = u_0(x) = \begin{cases} u_l & \text{if } x \le 0, \\ u_r & \text{if } x > 0. \end{cases} \tag{2b}$$

a) For $\varepsilon = 0$, equation (2a) is the linear advection equation; determine a weak solution that satisfies the Rankine-Hugoniot condition.

b) For each $\varepsilon > 0$, assume that $u^\varepsilon : (0,\infty)\times\mathbb{R} \to \mathbb{R}$ satisfies (2). Perform the change of variables $\xi = x - at$, $v^\varepsilon(t,\xi) = u^\varepsilon(t,x)$ and determine the corresponding Cauchy problem for $v^\varepsilon : (0,\infty)\times\mathbb{R}\to\mathbb{R}$.

c) Show that

$$v^\varepsilon(t,\xi) := (4\pi\varepsilon t)^{-1/2} \int_{\mathbb{R}} \mathrm{e}^{-\frac{(\xi-y)^2}{4\varepsilon t}} u_0(y)\,\mathrm{d}y \tag{3}$$

indeed solves the Cauchy problem derived in b) and reconstruct the solution u^ε of (2). Note that $\int_0^\infty \mathrm{e}^{-a^2z^2}\mathrm{d}z = \sqrt{\pi}/(2a)$ for $a > 0$.

d) For $\varepsilon \to 0$, show that the solutions u^ε converge pointwise to $u_0(x - at)$ in points $(t,x) \in (0,\infty)\times\mathbb{R}$ where the solution of a) is continuous.

Ex. 3.5 is to be delivered in written form by teams of two persons each in the exercise lesson on 07/05/2012. It will be discussed in the subsequent week.

Exam dates: July 24+25, 2012, September 27+28, 2012.