Partial Differential Equations
Higher Analysis II, summer term 2012
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Exercise sheet 5

Exercise 5.1: Consider the following Cauchy problem:

$$u_t + f(u)_x = 0 \quad \text{in } [0,\infty)\times\mathbb{R}\,, \quad u(0,x) = u_0(x) \tag{1}$$

with a strictly convex flux function $f \in \mathrm{C}^2(\mathbb{R})$ satisfying $f''(y) > \kappa > 0$ for all $y \in \mathbb{R}$. Show that the entropy conditions I (Lax) and II (Oleinik) coincide for a solution $u \in \mathrm{PC}^1([0,\infty)\times\mathbb{R})$ of (1). What happens, if f is strictly concave?

Exercise 5.2: Consider the Cauchy problem (1) with the flux function $f(u) = u^2(1-u)^2$.

a) Let $u_0(x) := \begin{cases} 2 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$
Construct a weak solution which satisfies the Rankine-Hugoniot condition. Check whether the entropy conditions I and II are satisfied.

b) Let $u_0(x) := \begin{cases} (1+\sqrt{2})/2 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$
Construct a weak solution which satisfies the Rankine-Hugoniot condition. Check whether the entropy conditions I and II are satisfied. Are there several solutions which satisfy I but not II? If $(f')^{-1}$ is needed, explain why it exists.

c) Describe a weak solution to problem b) which satisfies entropy condition II.

Exercise 5.3: Show that for a Riemann problem (1) with strictly convex $f \in \mathrm{C}^2(\mathbb{R})$, such that $f''(y) > \kappa > 0$ for all $y \in \mathbb{R}$, and initial datum

$$u_0(x) = \begin{cases} u_- & \text{if } x < 0, \\ u_+ & \text{if } x > 0 \end{cases} \qquad \text{with} \quad u_- < u_+$$

the rarefaction wave solution is given by

$$u(t,x) = \begin{cases} u_- & \text{if } x < f'(u_-)t, \\ (f')^{-1}(x/t) & \text{if } f'(u_-)t \le x \le f'(u_+)t, \\ u_+ & \text{if } x > f'(u_+)t \end{cases}$$

and that it is continuous.

(please turn)

Exercise 5.4 Traffic flow (written): Consider the traffic flow on a road. Let $\rho \in [0, \rho_{max})$ denote the density of cars, where $\rho = 0$ means an empty highway and $\rho = \rho_{max}$ means that the cars are bumper-to-bumper. Moreover, u denotes the velocity. The cars are conserved and hence ρ and u are related by the continuity equation $\rho_t + (\rho u)_x = 0$. In order to write this as a conservation law for ρ alone, we assume that u is a function of ρ: On the free highway we would like to drive at speed limit u_{max}, but in heavy traffic we slow down, with velocity decreasing as density increases; hence $u(\rho) = u_{max}(1 - \rho/\rho_{max})$. Thus we have the traffic model

$$\rho_t + (f(\rho))_x = 0, \qquad \text{with } f(\rho) = \rho\, u_{max}(1 - \rho/\rho_{max}). \tag{2}$$

a) Growing traffic jam: Let

$$\rho_0(x) = \begin{cases} \rho_- & \text{if } x < 0, \\ \rho_+ & \text{if } x > 0, \end{cases} \tag{3}$$

with $0 < \rho_- < \rho_+ = \rho_{max}$. Determine a a weak solution which satisfies the entropy condition I. Sketch the characteristics and vehicle paths for $\rho_- = \rho_{max}/2$ and interpret your solution.

b) Startup of cars at green light: Determine an entropy solution to the problem (2), (3) with $0 < \rho_+ < \rho_- \leq \rho_{max}$. Sketch the characteristics and vehicle paths for $\rho_- = \rho_{max}$ and $\rho_+ = \rho_{max}/2$ and interpret your solution.

Ex. 5.4 is to be delivered in written form by teams of two persons each in the exercise lesson on 14/05/2012. It will be discussed in the subsequent week.

Exam dates: July 24+25, 2012, September 27+28, 2012.