Exercise sheet 6

Exercise 6.1:

a) For $u = u(x, y)$ consider the PDE

$$u_{xy} = 0 \quad \text{in } \mathbb{R}^2. \tag{1}$$

Analyze the type of the PDE, transform the PDE into its canonical form and determine the transformation matrix.

b) For $u = u(x, y)$ consider the PDE

$$u_{xx} + 2xu_{xy} + yu_{yy} + u_x^2 - uu_y = 0\,. \tag{2}$$

Determine the sectors in the xy-plane where the PDE is elliptic, parabolic or hyperbolic and sketch them.

Exercise 6.2: Consider the PDE of second order

$$\sum_{i,j=1}^{d} A_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) + b(x, u(x), \nabla u(x)) = 0 \tag{3}$$

for $A_{i,j} \in \mathrm{C}(\mathbb{R}^d, \mathbb{R})$ with $A_{i,j}(x) = A_{j,i}(x)$ for all $x \in \mathbb{R}^d$ and $b : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$. Let $T : \mathbb{R}^d \to \mathbb{R}^d$, $T(x) = y$ be an admissible transformation, i.e. T is a C^2-diffeomorphism and $\det \nabla T(x) \neq 0$ for all $x \in \mathbb{R}^d$. Transform the PDE into the new coordinates $y = T(x)$ and show that the type of the PDE does not change under admissible transformations.

Exercise 6.3 (Hadamard's example):

a) Solve the Laplace equation in $\mathbb{R}^2$ for following Cauchy data:

$$u_{xx} + u_{yy} = 0\,, \qquad u = 0\,, \qquad u_y = \tfrac{1}{n}\sin(nx) \quad \text{on } C = \{(x, 0), x \in \mathbb{R}\}. \tag{4}$$

Hint: Separation of variables.

b) What happens to the solution u as $n \to \infty$? Is this Cauchy problem well-posed?

c) Check the assumptions of the Theorem of Cauchy-Kowalevskaya (Thm. 3.4 in the lecture) for $n \in \mathbb{N}$.

d) The Theorem of Cauchy-Kowalevskaya states the existence of a real analytic solution which can be written in terms of a power series. Determine the power series for the solution u of (4).

(please turn)

Exercise 6.4 (written): Consider the following Cauchy problem for the PDE of second order:

$$a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u_x,u_y), \tag{5a}$$
$$u(x,0) = g_0(x), \quad u_y(x,0) = g_1(x) \text{ on } C = \{(x,0), x \in \mathbb{R}\} \tag{5b}$$

with $a \neq 0$.

a) Reduce the PDE to a system of first order.

b) Determine the first order systems of the Laplace equation and the wave equation.

c) Characterize the type of the PDE (5a) in dependence of the coefficients.

d) Let $a(x,y) = (x-1)$, $b(x,y) = xy$ and $c(x,y) = -y^2$ in (5a). Determine the sectors in the xy-plane where this PDE is elliptic, parabolic or hyperbolic and sketch them.

Ex. 6.4 is to be delivered in written form by teams of two persons each in the exercise lesson on 21/05/2012. It will be discussed in the subsequent week.

Exam dates: July 24+25, 2012, September 27+28, 2012.