Exercise sheet 7

Exercise 7.1: For $u = u(x, y)$ consider the PDE

$$\tfrac{4(y-x)^2}{(x+y)^2+(y-x)^2}u_{xx} + \tfrac{8(y-x)(x+y)}{(x+y)^2+(y-x)^2}u_{xy} + \tfrac{4(y+x)^2}{(x+y)^2+(y-x)^2}u_{yy} + u_x + 3u_y = 0 \quad \text{in } \mathbb{R}^2\backslash\{(0,0)\}\,.$$

Analyze the type of the PDE, transform the main part into its canonical form and determine the transformation to use.

Exercise 7.2 Polygons: Consider the Dirichlet problem

$$\Delta u = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = g \text{ on } \partial\Omega \tag{1}$$

with $f \in \mathrm{C}(\overline{\Omega})$ and $g \in \mathrm{C}(\partial\Omega)$.

a) Let $\Omega \subset \mathbb{R}^2$ be a polygon with the k corner points $y_1, \ldots, y_k$. Let α_j denote the interior angle enclosed by the two edges meeting in y_j, $j \in \{1, \ldots, k\}$. For $u \in \mathrm{C}^2(\overline{\Omega})$ verify the following representation formula

$$\sigma(x)u(x) = \int_\Omega K_2(y-x)\Delta u(y)\,\mathrm{d}y - \int_{\partial\Omega}\big(K_2(y-x)\tfrac{\partial u}{\partial n} - u(y)\tfrac{\partial K_2(y-x)}{\partial n}\big)\,\mathrm{d}n_y \tag{2a}$$

$$\text{with} \quad \sigma(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 1/2 & \text{if } x \in \partial\Omega\backslash\{y_1, \ldots, y_k\} \\ \alpha_j/(2\pi) & \text{if } x \in \{y_1, \ldots, y_k\} \end{cases} \tag{2b}$$

and $K_2(x) = \frac{1}{2\pi}\ln|x|$. Hint: To check for y_j, start from 2nd Green's formula (Formula (4.2) in the lecture) on $\Omega_\varepsilon = \Omega\backslash S_\varepsilon(y_j)$ with $S_\varepsilon(y_j)$ being the segment of the ball of radius ε around y_j.

b) Show that, for a polygon with a reentrant corner, there does not always exist a solution $u \in \mathrm{C}^2(\overline{\Omega})$ of (1). Hint: You may consider the open polygon $\Omega \subset \mathbb{R}^2$ with the corner points $\{(-1,-1), (0,-1), (0,0), (1,0), (1,1), (-1,1)\}$ and u of the form $u(r,\varphi) = r^a \sin(a\varphi)$; then f and g have to be determined suitably.

Exercise 7.3 Poisson's formula for a disc: Let $\Omega = B_R(0) \subset \mathbb{R}^2$, $g \in \mathrm{C}(\partial\Omega)$ and

$$u(x) = \int_{|y|=R} P(x,y)g(y)\,\mathrm{d}a_y \quad \text{with } P(x,y) = \tfrac{R^2-|x|^2}{2\pi R|x-y|^2}\,. \tag{3}$$

a) Show that (3) defines a function $u \in \mathrm{C}^2(\Omega)$ satisfying $\Delta u = 0$ in Ω.

b) Show that $u \in \mathrm{C}(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. Hint: Verify $\int_{\partial\Omega} P(x,y)\,\mathrm{d}a_y = 1$ for all $x \in \Omega$.

(please turn)

Exercise 7.4 (written) Step 6 in the proof of the Cauchy Kovalevskaya Theorem: Let $\boldsymbol{p} = (p_0, p_1, \ldots, p_d)$ be an analytical solution of the first order system

$$
\begin{aligned}
\partial_{x_d}\boldsymbol{p}(x) &= \begin{pmatrix} p_d(x) \\ 0 \\ \vdots \\ 0 \\ -\frac{b(x,\boldsymbol{p}(x))}{A_{dd}(x,\boldsymbol{p}(x))} \end{pmatrix} + \begin{pmatrix} 0 \\ \partial_{x_1} p_d \\ \vdots \\ \partial_{x_{d-1}} p_d \\ -\frac{1}{A_{dd}(x,\boldsymbol{p}(x))} \sum_{i,j\neq d} A_{ij}(x,\boldsymbol{p}(x))\partial_{x_i} p_j(x) \end{pmatrix}, \\
\boldsymbol{p}(x',0) &= (g_0(x'), \partial_{x_1} g_0, \ldots, \partial_{x_{d-1}} g_0, g_1(x'))^\top, \quad x' \in \mathbb{R}^{d-1}.
\end{aligned}
$$

Show that $u(x) = p_0(x)$ is a solution of the second order Cauchy problem

$$
\begin{aligned}
&A_{dd}(x,u,Du)\partial^2_{x_{dd}} u + \sum_{i,j\neq d} A_{ij}(x,u,Du)\partial_{x_i}\partial_{x_j} u + b(x,u,Du) = 0 \quad \text{in } \mathbb{R}^d \backslash C, \\
&u = g_0 \quad \text{on } C, \qquad \frac{\partial u}{\partial n} = g_1 \quad \text{on } C.
\end{aligned}
$$

Exercise 7.5 (written) Maximum principle for functions with mean value property: Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain. Let $u \in \mathrm{C}(\overline{\Omega})$ such that for all $x_0 \in \Omega$, for all $r > 0$ with $B_r(x_0) \subset \Omega$:

$$
u(x_0) = \frac{1}{\omega_d r^{d-1}} \int_{\partial B_r(x_0)} u(z)\,\mathrm{d}a_z, \tag{4}
$$

where $\omega_d = \int_{\partial B_1(0)} 1\,\mathrm{d}a$ and $B_r(x_0)$ denotes the open ball of radius r around the point $x_0 \in \Omega$. Moreover, let $M = \sup\{u(x),\, x \in \Omega\}$.

a) Assume that $u(x_0) = M$ for a particular $x_0 \in \Omega$. Conclude that $u(y) = M$ for every $y \in B_r(x_0) \subset \Omega$.

b) Conclude that u is constant in Ω if u attains its maximum in some $x_0 \in \Omega$. Hint: argue by contradiction. For this, you may use a continuous path that connects x_0 with a point $x_1 \in \Omega$ with $u(x_1) < M$.

c) Let u be non-constant. Conclude that u attains its maximum and minimum on $\partial\Omega$.

d) Let $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}(\overline{\Omega})$ such that

$$
\Delta u = f \text{ in } \Omega, \tag{5}
$$

$$
u = 0 \text{ on } \partial\Omega. \tag{6}
$$

Conclude from the maximum principle that this Dirichlet problem admits at most one solution.

e) Let $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}(\overline{\Omega})$ solve (5) with $f = 0$ together with the boundary condition

$$
\frac{\partial u}{\partial n} + hu = 0 \text{ on } \partial\Omega, \tag{7}
$$

where n is the outer unit normal vector to Ω. Assume that $h \in \mathrm{C}(\partial\Omega)$ satisfies $h \geq \kappa > 0$ on $\partial\Omega$. Conclude that $u = 0$.

Ex. 7.4 and 7.5 are to be delivered in written form by teams of two persons each in the exercise lesson on 04/06/2012. They will be discussed in the subsequent week.