Exercise sheet 8

Exercise 8.1: Distributional derivatives

(a) For $a \in C^\infty(\mathbb{R})$ and $T \in \mathcal{D}'(\mathbb{R})$ define $aT : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ via

$$\forall \varphi \in \mathcal{D}(\mathbb{R}) : \qquad (aT)[\varphi] := T[a\varphi].$$

Verify that $aT \in \mathcal{D}'(\mathbb{R})$ and prove the product rule $D_x(aT) = (D_x a)T + aD_x T$.

(b) Calculate the distributional derivative of $f(x) = \ln|x|$, $x \in \mathbb{R}$.

(c) Calculate the first derivative of the regular distribution induced by

$$f_\lambda(x) := \begin{cases} 0 & \text{for } x \leq 0 \\ x^\lambda & \text{for } x > 0 \end{cases}, \qquad \lambda \in (-1, 0).$$

"Hint": $\forall c \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$: $\varphi'(x) = (\varphi(x) + c)'$.

(d) For $\Omega = \mathbb{R}^2$ calculate $\triangle_x \ln|x|$ in the distributional sense.

Exercise 8.2: Distributions

Prove that for every $T \in \mathcal{D}'(\Omega)$ the following estimate is valid:

$$\forall K \subset \Omega, K \text{ compact}, \exists C = C(K,T) > 0, \exists k = k(K,T) \in \mathbb{N} \text{ such that}$$
$$\forall \varphi \in \mathcal{D}_K(\Omega) : \ \left|T[\varphi]\right| \leq C \sum_{|\alpha| \leq k} \sup_K |D^\alpha \varphi(x)| .$$

Here, $\mathcal{D}_K(\Omega) = \{\, \varphi \in \mathcal{D}(\Omega) \, ; \, \operatorname{supp} \varphi \subset K \,\}$. HINT: Prove the estimate by contradiction and consider a sequence $\{\varphi_j, j \in \mathbb{N}\} \subset \mathcal{D}(\Omega)$, which satisfies

$$\left|T[\varphi_j]\right| > j \sum_{|\alpha| \leq j} \sup_K |D^\alpha \varphi_j| .$$

Exercise 8.3: Fundamental solutions for linear ODEs

For constant coefficients $a_i \in \mathbb{R}$, $1 \leq i \leq m-1$, consider the scalar, one-dimensional differential operator

$$L(D) := D_x^m + a_{m-1} D_x^{m-1} + \ldots + a_1 D_x + a_0.$$

Prove: If u_0 is a solution of the initial value problem

$$L(D)u_0(x) = 0, \quad x > 0,$$
$$u_0(0) = 0, \cdots, D_x^{m-2} u_0(0) = 0, \ \ D_x^{m-1} u_0(0) = 1,$$

then the distribution generated by the function $g(x) := H(x)u_0(x)$, where H denotes the Heaviside-function, is a fundamental solution (in the distributional sense) for the equation $L(D)u(x) = f(x)$, $x \in \mathbb{R}$.

(please turn)

Exercise 8.4: Green's function for the strip $\Omega = \mathbb{R} \times (0, \pi)$ (written)

(a) For $\alpha > 0$ consider the ODE $-u'' + \alpha^2 u = f$ on $\mathbb{R}$ (one-dimensional elliptic problem). Show that for $f \in BC^0(\mathbb{R})$ (bounded and continuous functions on $\mathbb{R}$) the unique bounded classical solution is given by

$$u(x) = \int_{y\in\mathbb{R}} G_\alpha(x-y) f(y)\,\mathrm{d}y \quad \text{with} \quad G_\alpha(z) = \frac{1}{2\alpha}\exp(-\alpha\,|z|).$$

(b) To solve the Dirichlet-Problem $\triangle u = f$ in $\Omega = \mathbb{R} \times (0, \pi)$ decompose u and f in Fourier series with respect to the x_2 coordinate ($u(x_1, x_2) = \sum_{k=1}^{\infty} u_k(x_1)\sin(kx_2)$). Derive solution formulas for the coefficients u_k and construct a series representation of the Green's function G. (Formal calculations are sufficient, no justification of the interchange of limits is required).

(c) Find an explicit formula for the Green's function. Hint: Use trigonometric identities and the identity $\sum_{n=1}^{\infty} \frac{p^n}{n}\cos(n\alpha) = -\ln\sqrt{1 - 2p\cos\alpha + p^2}$.

Ex. 8.4 is to be delivered in written form by teams of two persons each in the exercise lesson on 11/06/2012. It will be discussed in the subsequent week.