Partial Differential Equations
Higher Analysis II, summer term 2012
Dr. Dorothee Knees, Dr. Marita Thomas
11/06/2012

Exercise sheet 9

Exercise 9.1: Convolution of distributions

Let $S, T \in \mathcal{D}'(\mathbb{R})$ and assume that the convolution $S * T$ exists. Prove that for all $k \in \mathbb{N}$ it holds

$$x^k(S * T) = \sum_{j=0}^{k} \binom{k}{j} (x^j S) * (x^{k-j} T).$$

Exercise 9.2: Harmonic oscillator

The Cauchy problem for the harmonic oscillator is

$$\partial_t^2 u(t) + a^2 u(t) = f(t) \quad \text{for } t > 0, \tag{1}$$
$$u(0) = u_0, \tag{2}$$
$$\partial_t u(0) = u_1. \tag{3}$$

Construct solutions using fundamental solutions via the following steps:

(a) Use Exercise 8.3 in order to construct a fundamental solution for the operator L with $L(u) = \partial_t^2 u + a^2 u$ on the whole R.

(b) Assume that u is a solution of (1)–(3). For $t \in \mathbb{R}$ define $\widetilde{u}(t) := H(t)u(t)$, $\widetilde{f}(t) := H(t)f(t)$, where $H : \mathbb{R} \to \{0, 1\}$ with $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ for $t > 0$ is the Heaviside-function. Derive the ordinary differential equation on $\mathbb{R}$, which is solved by $\widetilde{u}$ in the distributional sense.

(c) Use (a) to construct a solution for the differential equation derived in (b) and reduce it to a solution formula for the original problem (1)–(3). Provide sufficient conditions on f such that everything in (c) is rigorous.

Exercise 9.3: Some properties of the fundamental solution of the heat equation

We recall that the fundamental solution for the heat equation operator on $\mathbb{R} \times \mathbb{R}^d$ is

$$\Phi_d(t, x) = \frac{H(t)}{(2\sqrt{\pi t})^d} \exp\left(-\frac{|x|^2}{4t}\right), \tag{4}$$

which means that $(\partial_t - \triangle_x)\Phi_d = \delta_0$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$. Here, $H : \mathbb{R} \to \{0, 1\}$ with $H(t) = 0$ if $t \leq 0$ and $H(t) = 1$ for $t > 0$ is the Heaviside-function.

(a) Prove that for all $t > 0$ it holds $\int_{\mathbb{R}^d} \Phi_d(t, x)\, \mathrm{d}x = 1$,

(b) Sketch the functions $x \mapsto \Phi_1(t, x)$ for different (small/large) values of $t > 0$. Prove that for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$ it holds

$$\lim_{t \searrow 0} T_{\Phi_d(t,\cdot)}[\varphi] = \lim_{t \searrow 0} \int_{\mathbb{R}^d} \Phi_d(t, x)\varphi(x)\, \mathrm{d}x = \varphi(0) = \delta_0[\varphi].$$

Exercise 9.4: $W^{k,p}(\Omega)$ is a Banach space

Let $\Omega \subset \mathbb{R}^d$ be an open domain. Prove that for $k \in \mathbb{N}$ and $p \in [1, \infty]$ the space $W^{k,p}(\Omega)$ as defined in the lecture is a Banach space.

Exercise 9.5: Initial value problem for the heat equation (written)

Consider the initial value problem for the heat equation on $\mathbb{R}^d$:

$$\partial_t u(t,x) - \triangle_x u(t,x) = f(t,x) \qquad x \in \mathbb{R}^d, t > 0, \tag{5}$$

$$u(0+,x) = g(x) \qquad x \in \mathbb{R}^d. \tag{6}$$

The goal is to construct an explicit solution formula for (5)–(6) and to investigate the properties of the solution.

(a) Assume that u is a (classical) solution of (5)–(6). Extend u and f to values $t < 0$ by defining $\widetilde{u}(t,x) := H(t)u(t,x)$, $\widetilde{f}(t,x) = H(t)f(t,x)$. Derive the PDE on $\mathbb{R} \times \mathbb{R}^d$, which is satisfied by $\widetilde{u}$ in the distributional sense, and write down an explicit formula for its solution. Formulate sufficient conditions on f, g such that the solution formula indeed defines an element from $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$.

(b) For $f \in C^0([0,\infty) \times \mathbb{R}^d)$ and $g \in C^0(\mathbb{R}^d)$, both with compact support, let

$$u_0 := T_{\Phi_d} * \left(g\delta_{t=0} + T_{\widetilde{f}}\right),$$

where $\widetilde{f}(t,x) = H(t)f(t,x)$ and $g\delta_{t=0} \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$ is defined as $g\delta_{t=0}[\varphi] := \int_{\mathbb{R}^d} g(x)\varphi(0,x)\,\mathrm{d}x$ for $\varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$. Show that $u_0 \in \mathcal{D}'(\mathbb{R} \times \mathbb{R}^d)$.
Prove that there exists $w \in L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R}^d)$ such that for all $\varphi \in \mathcal{D}((0,\infty) \times \mathbb{R}^d)$ it holds

$$\widetilde{u}[\varphi] = T_w[\varphi],$$

and derive an explicit formula for w.

(c) For $g \in C^0(\mathbb{R}^d)$ with compact support, $t > 0$ and $x \in \mathbb{R}^d$ define with Φ_d from (4):

$$w(t,x) := \int_{\mathbb{R}^d} \Phi_d(t, x-y) g(y)\,\mathrm{d}y.$$

Prove that $w \in C^\infty((0,\infty) \times \mathbb{R}^d)$, that w satisfies the heat equation (5) with $f = 0$ and that for all $x \in \mathbb{R}^d$ it holds $\lim_{t \searrow 0} |w(t,x) - g(x)| = 0$.

Hints: $\int_{\mathbb{R}} e^{-t^2}\,\mathrm{d}t = \sqrt{\pi}$; without proof one may use that Φ_d from (4) belongs to $C^\infty((0,\infty) \times \mathbb{R}^d)$. As always, T_f denotes the regular distribution induced by $f \in L^1_{\mathrm{loc}}$.

Ex. 9.5 is to be delivered in written form by teams of two persons each in the exercise lesson on 18/06/2012. It will be discussed in the subsequent week.