UNI KASSEL
VERSITÄT

Evolutionary variational inequalities in the context of inelastic solids

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Plan for the lectures

1. Modeling of elasto-plastic phenomena
2. Existence of solutions, nonconvex case
3. Existence of solutions, uniformly convex case
4. Further solution concepts in the nonconvex case

Tools from convex analysis

$\mathcal{Q}$ real, reflexive Banach space,

$\mathcal{R} : \mathcal{Q} \to \mathbb{R}_\infty$ proper, convex, lower semicontinuous functional

Subdifferential

$$\text{For } u \in \mathcal{Q}: \quad \partial\mathcal{R}(u) := \{\, \xi \in \mathcal{Q}^* \,;\, \mathcal{R}(v) \geq \mathcal{R}(u) + \langle \xi, v - u \rangle \;\; \forall v \in \mathcal{Q}\,\}.$$

Legendre/Fenchel transformation

$$\text{For } \xi \in \mathcal{Q}^*: \quad \mathcal{R}^*(\xi) := \sup_{u \in \mathcal{Q}} \langle \xi, u \rangle - \mathcal{R}(u).$$

Fenchel inequality and identity

$$\mathcal{R}(u) + \mathcal{R}^*(\xi) \geq \langle \xi, u \rangle \quad \forall u \in \mathcal{Q}, \xi \in \mathcal{Q}^*$$

$$\mathcal{R}(u) + \mathcal{R}^*(\xi) = \langle \xi, u \rangle \;\; \Leftrightarrow \;\; \xi \in \partial\mathcal{R}(u) \;\; \Leftrightarrow u \in \partial\mathcal{R}^*(\xi).$$

Characteristic functions and 1-homogeneous functionals

$\mathcal{R} : \mathcal{Q} \to \mathbb{R}_\infty$ positively homogeneous of degree one

$$\Leftrightarrow \mathcal{R}(0) = 0 \text{ and } \mathcal{R}(\lambda u) = \lambda \mathcal{R}(u) \text{ for all } \lambda > 0.$$

Subdifferential of positively 1-homogeneous functionals:

$\mathcal{R} : \mathcal{Q} \to \mathbb{R}_\infty$ convex,lower semi-continuous, pos. 1-homog. Then

$$\partial \mathcal{R}(0) = \{ \xi \in \mathcal{Q}' \,;\, \mathcal{R}(v) \geq \langle \xi, v \rangle \forall v \in \mathcal{Q} \} \quad \text{is closed, convex, } 0 \in \partial \mathcal{R}(0).$$

$$\partial \mathcal{R}(u) = \{ \xi \in \partial \mathcal{R}(0) \,;\, \mathcal{R}(u) = \langle \xi, u \rangle \}$$

Conjugate functionals of positively 1-homogeneous functionals:

$$\mathcal{R}^* = \chi_{\mathcal{K}} \quad \text{with} \quad \mathcal{K} = \partial \mathcal{R}(0)$$

Conjugate functionals of characteristic functionals

$\mathcal{K} \subset \mathcal{Q}$ closed, convex, $0 \in \mathcal{K}$

$$\Rightarrow \mathcal{R} := \chi_{\mathcal{K}}^* \text{ is positively 1-homogeneous with } \partial \mathcal{R}(0) = \mathcal{K}.$$

Example: $\mathcal{Q} = \mathbb{R}$, $\mathcal{R}(u) = \mu\,|u| \quad \Rightarrow \quad \partial \mathcal{R}(0) = [-\mu, \mu]$, $\mathcal{R}^*(\xi) = \chi_{[-\mu,\mu]}(\xi)$

Equivalent formulations of the evolution laws

Assumptions

- $\mathcal{E} \in C^1([0,T] \times \mathcal{Q}; \mathbb{R})$ and for all t: $q \mapsto \mathcal{E}(t,q)$ is convex
- $\mathcal{R} : \mathcal{Q} \to [0,\infty]$ is convex, lower semicont., pos. homog. of degree one.

For $z \in AC([0,T];\mathcal{Q})$ the following statements are equivalent:

(GEF) Global energetic formulation: for all $t \in [0,T]$, z satisfies (S)&(E):

$$\text{(S)} \quad \forall v \in \mathcal{Q} : \mathcal{E}(t,z(t)) \leq \mathcal{E}(t,v) + \mathcal{R}(v - z(t))$$

$$\text{(E)} \quad \mathcal{E}(t,z(t)) + \int_0^t \mathcal{R}(\dot{z}(r))\,\mathrm{d}r = \mathcal{E}(0,z(0)) + \int_0^t \partial_t\mathcal{E}(r,z(r))\,\mathrm{d}r$$

(EVI) Evolutionary variational inequality: for a.e. $t \in [0,T]$, z satisfies

$$\forall v \in \mathcal{Q} : \quad \mathcal{R}(v) - \mathcal{R}(\dot{z}(t)) + \langle \mathrm{D}_z\mathcal{E}(t,z(t)), v - \dot{z}(t)\rangle \geq 0.$$

(FB) Force balance relation: for a.e. $t \in [0,T]$, z satisfies

$$0 \in \partial\mathcal{R}(\dot{z}(t)) + \mathrm{D}\mathcal{E}(t,z(t))$$

(DI) Differential inclusion: for a.e. $t \in [0,T]$, z satisfies

$$\dot{z}(t) \in \partial\chi_{\mathcal{K}}(-\mathrm{D}\mathcal{E}(t,z(t))) \quad \text{with } \mathcal{K} = \partial\mathcal{R}(0).$$

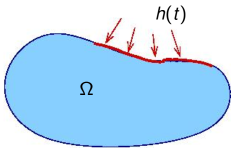
Modeling frame-work in this lecture series

- small-strain framework
  (mathematical results are very rare for finite-strain elasto-plasticity)
- phenomenological description of plasticity
  (no microscopic modeling, no explicit modeling of dislocations)
- main focus on rate-independent models
- concept applicable to other rate-independent phenomena
  (crack propagation, damage models, shape memory alloys,...)

Dissipative, inelastic solids

Variables

displacements	$u : [0,T] \times \Omega \to \mathbb{R}^d$
linearized strain	$e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$
internal variables	$z : [0,T] \times \Omega \to \mathbb{R}^N$



Constitutive functions

stored energy density $W : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^N \to \mathbb{R}$

set-valued function $g : \mathbb{R}^N \to \mathcal{P}(\mathbb{R}^N)$

$0 \in g(0), \quad \mu \cdot \xi \geq 0$ for all $\xi \in \mathbb{R}^N, \mu \in g(\xi)$.

Constitutive laws/evolution laws:

$$\sigma(t,x) = \mathrm{D}_e W(e(u(t,x)), z(t,x))$$

$$\rho_0 \ddot{u}(t,x) = \operatorname{div} \sigma(t) + f(t),$$

$$\dot{z}(t) \in g\big(-\mathrm{D}_z W(e(u(t)), z(t))\big),$$

$$u(t)\big|_{\Gamma_D} = u_D(t) \ \text{ Dirichlet cond.}; \quad \sigma(t)\big|_{\Gamma_N} n = h(t) \ \text{ Neumann cond.}$$

Initial conditions: $u(0) = u_0$, $\partial_t u(0) = u_1$, $z(0) = z_0$.

Variables

displacements $u : [0,T] \times \Omega \to \mathbb{R}^d$

linearized strain $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$

internal variables $z = (e_p, \alpha) : [0,T] \times \Omega \to \mathbb{R}^N \equiv \mathbb{R}^{d\times d}_{\text{sym}} \times \mathbb{R}^{\widetilde{N}}$

Energy and dissipation

energy functional: $\mathcal{E}(t,u,z) = \int_\Omega \frac{1}{2}\mathbb{A}\big(e(u)-e_p\big) : \big(e(u)-e_p\big) + H(e_p,\alpha)\,\mathrm{d}x - \langle \ell(t), u\rangle$

dissipation pot.: $\mathcal{R}(\dot z) := \int_\Omega \chi^*_{\mathbb{K}}(\dot z)\,\mathrm{d}x$, with $\mathbb{K} \subset \mathbb{R}^N$ closed, convex and $0 \in \mathbb{K}$.

Evolution law (quasi-static case):

$$0 = \operatorname{div} A(e(u) - e_p) + f(t),$$

$$\dot z(t,x) = \begin{pmatrix} \dot e_p(t) \\ \dot\alpha(t) \end{pmatrix} \in \partial\chi_{\mathbb{K}}\left(- \begin{pmatrix} -\mathbb{A}(e(u(t,x)) - e_p(t,x)) + \partial_{e_p} H(e_p,\alpha) \\ \partial_\alpha H(e_p,\alpha) \end{pmatrix} \right),$$

$u(t)\big|_{\Gamma_D} = u_D(t)$ Dirichlet cond.; $\quad A(e(u) - e_p)\big|_{\Gamma_N} n = h(t)$ Neumann cond.

Initial condition: $z(0) = z_0$.

Small-strain elasto-plasticity with hardening

Variables

displacements $u : [0,T] \times \Omega \to \mathbb{R}^d$

linearized strain $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$

internal variables $z = (e_p, \alpha) : [0,T] \times \Omega \to \mathbb{R}^N \equiv \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{\widetilde{N}}$

Energy and dissipation

energy functional: $\mathcal{E}(t,u,z) = \int_\Omega \frac{1}{2}\mathbb{A}\big(e(u) - e_p\big) : \big(e(u) - e_p\big) + H(e_p,\alpha)\,\mathrm{d}x - \langle \ell(t), u\rangle$

dissipation pot.: $\mathcal{R}(\dot z) := \int_\Omega \chi^*_{\mathbb{K}}(\dot z)\,\mathrm{d}x$, with $\mathbb{K} \subset \mathbb{R}^N$ closed, convex and $0 \in \mathbb{K}$.

Evolution law (quasi-static case):

Find $(u(t), z(t)) \in \mathcal{Q} := H^1_{\Gamma_D}(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{R}^N)$ with

$$0 = \partial_u \mathcal{E}(t, u(t), z(t)) \quad \text{in } \mathcal{U}^*\,,$$
$$\dot z(t,x) \in \partial \chi_{\mathbb{K}}\big(-\partial_z \mathcal{E}(t, u(t), z(t))\big) \quad \text{in } \mathcal{Z}^*\,.$$

Initial condition: $z(0) = z_0$.

Questions

- In which sense are the introduced models thermodynamically consistent? ($\rightarrow$ Exercises)
- Existence of solutions (depending on convexity of $\mathcal{E}$) in the global energetic framework:

  for all $t \in [0, T]$, z satisfies (S)&(E):

  $$\text{(S)} \quad \forall v \in \mathcal{Q} : \mathcal{E}(t, z(t)) \leq \mathcal{E}(t, v) + \mathcal{R}(v - z(t))$$

  $$\text{(E)} \quad \mathcal{E}(t, z(t)) + \int_0^t \mathcal{R}(\dot{z}(r))\,\mathrm{d}r = \mathcal{E}(0, z(0)) + \int_0^t \partial_t \mathcal{E}(r, z(r))\,\mathrm{d}r$$

- Further solution concepts in the nonconvex case ($\rightarrow$ on Friday)

2 The nonconvex case

Global energetic formulation, with dissipation distance

Given:

- $\mathcal{Q}$ reflexive Banach space
- Energy functional $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$
- Dissipation distance $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$
  pseudo distance, lower semi cont., positivity property

Dissipation along the path $q : [0, T] \to \mathcal{Q}$

$$\text{diss}_{\mathcal{D}}(q, [0, T]) := \sup_{\text{partitions of } [0,T]} \sum_i \mathcal{D}(q(t_{i-1}), q(t_i)).$$

Definition: Global energetic formulation

$q : [0, T] \to \mathcal{Q}$ global energetic solution w.r.to $(\mathcal{E}, \mathcal{D})$ and initial condition $q_0 \in Q$, if

- $q(0) = q_0$,
- $t \mapsto \partial_t \mathcal{E}(t, q(t)) \in L^1((0, T))$, for all t: $\mathcal{E}(t, q(t)) < \infty$,
- for all $t \in [0, T]$:

(S) $\forall v \in \mathcal{Q} : \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, v) + \mathcal{D}(q(t), v)$

(E) $\mathcal{E}(t, q(t)) + \text{diss}_{\mathcal{D}}(q, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_t \mathcal{E}(r, q(r))\, \mathrm{d}r$

Assumptions for the existence theorem of GES (nonconvex case)

Assumptions on $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$

(A1) coercivity and weak lower semicontiunuity

$\mathcal{E}(t, q) = \mathcal{W}(u) - \langle \ell(t), u \rangle$ with

- $\ell \in C^1([0, T], \mathcal{Q}^*)$
- $\mathcal{W} : \mathcal{Q} \to [0, \infty]$ is (seq.) weakly lower semicontinuous
- $\mathcal{W}$ is coercive in the following sense: $\exists p > 1$ such that $\lim_{n\to\infty} \|q_n\|_{\mathcal{Q}} = \infty \Rightarrow \frac{\mathcal{W}(q_n)}{\|q_n\|^p} \to \infty$

Assumptions for the existence theorem of GES (nonconvex case)

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Assumptions on $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to [0, \infty]$

(A2.1) $\mathcal{D}$ is a pseudo distance, i.e

$\forall z_1, z_2, z_3 \in \mathcal{Q}$: $\mathcal{D}(z_1, z_1) = 0$, $\mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3)$

(A2.2) $\mathcal{D}$ is weakly lower semicontinuous

(A2.3) positivity of $\mathcal{D}$: For all weakly (seq.) compact $C \subset \mathcal{Q}$ it holds

$(q_k)_{k\in\mathbb{N}} \subset C$ and $\min\left\{\mathcal{D}(q_k, q), \mathcal{D}(q, q_k)\right\} \to 0 \implies q_k \rightharpoonup q$.

Remark: (A1) does not require convexity of $\mathcal{E}$! Existence theorem valid with more general assumptions on $\mathcal{E}$.

Compatibility condition

Set of stable states:

$$\mathcal{S}(t) := \{\, q \in \mathcal{Q}\,;\ \forall v \in \mathcal{Q} \qquad \mathcal{E}(t,q) \leq \mathcal{E}(t,v) + \mathcal{D}(q,v)\,\}$$

Compatibility condition **(A3)**

For every sequence $(t_n, q_n)_{n\in\mathbb{N}}$ it holds

$$\left.\begin{array}{c} q_n \in \mathcal{S}(t_n), \\ \sup_{n\in N} \mathcal{E}(t_n, q_n) < \infty, \\ t_n \to t, \quad q_n \rightharpoonup q \end{array}\right\} \implies q \in \mathcal{S}(t),\ \partial_t\mathcal{E}(t_n,q_n) \to \partial_t\mathcal{E}(t,q).$$

Assumptions

- $\mathcal{Q}$ reflexive, separable Banach space
- $\mathcal{E} : [0, T] \times \mathcal{Q} \to \mathbb{R}_\infty$ with (A1)
- $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \to \mathbb{R}_\infty$ with (A2.1)–(A2.3)
- compatibility condition (A3)

Theorem

Then for every $q_0 \in \mathcal{S}(0)$ there exists a global energetic solution associated with $(\mathcal{E}, \mathcal{D})$, and $q(0) = q_0$.

References:

Mielke/Theil/Levitas ($\leq$ '01), Mainik/Mielke '05 (generalized Helly selection principle), Francfort/Mielke '06, Mielke/Roubicek/Stefanelli '08 (evolutionary Γ-convergence for rate-independent systems)

adapted to perfect plasticity: Dal Maso/DeSimone/Mora '06

adapted to general crack propagation: Dal Maso/Toader '02, Dal Maso/Francfort/Toader '06

Structure of the existence proof

Step 1: time-discretization

- time-incremental minimization
- a-priori estimates
- discrete energy inequality (E_N, $\leq$)
- discrete stability (S_N)

Step 2: selection of weakly convergent subsequences
relies on generalized version of Helly selection principle

Step 3: stability of the limit function

- here the compatibility condition on $(\mathcal{E}, \mathcal{D})$ is used

Step 4: upper energy estimate (E, $\leq$)

- lower-semicontinuity arguments

Step 5: lower energy estimate (E, $\geq$)

- based on stability,

An example (inspired by damage models)

Variables

internal variable $z : [0, T] \times \Omega \to \mathbb{R}$, $\mathcal{Q} = W_0^{1,p}(\Omega)$

Energy and dissipation

energy functional: $\mathcal{E}(t, z) = \int_\Omega \frac{1}{p} |\nabla z|^p \,\mathrm{d}x - \langle \ell(t), z \rangle$

dissipation pot.: $\mathcal{R}_1(\dot{z}) := \int_\Omega \mu \, |\dot{z}| \,\mathrm{d}x$

$$\mathcal{R}_2(\dot{z}) := \begin{cases} \int_\Omega \mu \, |\dot{z}| \,\mathrm{d}x & \text{if } \dot{z} \le 0 \text{ a.e. on } \Omega \\ \infty & \text{otherwise} \end{cases} .$$

$\mathcal{D}_i(z_1, z_z) := \mathcal{R}(z_2 - z_1)$.

Theorem

For every $p \in (1, \infty)$, every $\ell \in C^1([0, T], \mathcal{Q}^*)$ and every $z_0 \in \mathcal{S}(0)$ there exists a global energetic solution associated with $(\mathcal{E}, \mathcal{D}_1)$ and a global energetic solution associated with $(\mathcal{E}, \mathcal{D}_2)$.

References: simplified version of results for damage models.

$p > d$: Mielke/Roubicek '06,

$p \le d$: Marita Thomas (PhD thesis '10),

Example (continued)

Proof: We only verify compatibility condition (A3).
Let $(t_n, z_n)_n$ be a stable sequence, i,e, $t_n \to t$, $z_n \rightharpoonup z$ weakly in $\mathcal{Q}$ and $z_n \in \mathcal{S}(t_n)$.

Case 1, $\mathcal{R}_1$: Since $W^{1,p}(\Omega) \Subset L(\Omega)$ (compact embedding), we have for all $\xi \in \mathcal{Q}$: $\mathcal{D}_1(z_n, \xi) = \mu \left\|z_n - \xi\right\|_{L^1} \to \mathcal{D}(z, \xi)$. Hence one may pass to the limit for $n \to \infty$ in the stability condition (exploit weak lower semicontinuity of $\mathcal{E}$ on left hand side)

$$\mathcal{E}(t_n, z_n) \leq \mathcal{E}(t_n, \xi) + \mathcal{D}_1(z_n, \xi) \tag{1}$$

and finds $z \in \mathcal{S}(t)$, i.e. $\forall \xi \in \mathcal{Q}$: $\quad \mathcal{E}(t, z) \leq \mathcal{E}(t, \xi) + \mathcal{D}_1(z, \xi)$.

Case 2, $\mathcal{R}_2$: If $\xi \in \mathcal{Q}$ with $\mathcal{D}_2(z, \xi) = \infty \Rightarrow$ stability estimate OK.
Let now $\xi \in \mathcal{Q}$ with $\mathcal{D}_2(z, \xi) < \infty$, i.e. $\xi \leq z$ a.e. on Ω.
Let $p > d$, hence $W^{1,p}(\Omega) \Subset C(\overline{\Omega})$.
Choose $\xi_n := \xi - \left\|z - z_n\right\|_{C^0}$. Then $\xi_n \leq \min\{z, z_n\}$ a.e. in Ω and $\xi_n \to \xi$ strongly in $W^{1,p}$. Taking the limit in

$$\mathcal{E}(t_n, z_n) \leq \mathcal{E}(t_n, \xi_n) + \mathcal{D}_2(z_n, \xi_n)$$

finally implies $z \in \mathcal{S}(t)$.
Case $p \leq d$: long and technical, see PhD thesis by Marita Thomas (WIAS)

3 Existence of solutions in the uniformly convex case

Assumptions for the existence theorem, uniformly convex case

(A0) $\mathcal{Q}$ reflexive, real, separable Banach space

(A1) Energy functional and external forces: $\mathcal{E} : [0, T] \times \mathcal{Q}$ is of the form

$$\mathcal{E}(t, q) = \mathcal{W}(q) - \langle \ell(t), q \rangle$$

where $\quad \ell \in C^1([0, T], \mathcal{Q}^*) \quad$ and $\quad \mathcal{W} : \mathcal{Q} \to [0, \infty]$ satisfies

- $\mathcal{W}$ is coercive on $\mathcal{Q}$,
- $\mathcal{W}$ is seq. lower semicontinuous
- $\mathcal{W}$ is uniformly convex, i.e. $\quad \exists \alpha > 0 \; \forall q_1, q_2 \in \mathcal{Q} \; \forall \theta \in [0, 1]$

$$\mathcal{W}(\theta q_1 + (1 - \theta) q_2) + \frac{\alpha}{2} \theta (1 - \theta) \|q_1 - q_2\|^2 \leq \theta \mathcal{W}(q_1) + (1 - \theta) \mathcal{W}(q_2).$$

(A2) dissipation potential
$\mathcal{R} : \mathcal{Q} \to [0, \infty]$ is
convex, seq. lower semicontinuous, pos. homog. of degree one.

Theorem (Existence of solutions)

Assume **A0 – A2** *and compatibility condition* **A3**.

For every $\ell \in C^1([0,T];\mathcal{Q}^*)$ *and every initial datum* $q_0 \in S(0)$ *there exists* $q \in C^{Lip}([0,T];\mathcal{Q})$ *with* $q(0) = q_0$ *and*

(S) $\forall t$: $\quad q(t) \in S(t)$,

(E) $\forall t_1 \le t_2$: $\quad \mathcal{E}(t_2, q(t_2)) + \int_{t_1}^{t_2} \mathcal{R}(\dot q(s))\,\mathrm{d}s = \mathcal{E}(t_1, q(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(s, q(s))\,\mathrm{d}s.$

Idea of the proof: improved stability estimate due to uniform convexity: $\forall t, \forall \widetilde q \in \mathcal{Q}$

$$\frac{\alpha}{2}\,\|q(t) - \widetilde q\|^2 + \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \widetilde q) + \mathcal{R}(\widetilde q - q(t)).$$

With $\widetilde q = q(s)$ and using (E), after some calculations:

$$\|q(s) - q(t)\| \le c\,|s - t|\,.$$

Example: elasto-plasticity with linear kinematic hardening

Variables

displacements	$u : [0, T] \times \Omega \to \mathbb{R}^d$
linearized strain	$e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$
plastic strain	$e_p \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ with $\operatorname{tr} e_p = 0$

Energy and dissipation

energy functional: $\mathcal{E}(t, u, z) = \int_\Omega \frac{1}{2}\mathbb{A}\big(e(u) - e_p\big) : \big(e(u) - e_p\big) + \frac{1}{2}\,|e_p|^2 \,\mathrm{d}x - \langle \ell(t), u\rangle$

dissipation pot.: $\mathcal{R}(\dot{e}_p) := \int_\Omega \sigma_0\,|\dot{e}_p|\,\mathrm{d}x.$

Evolution law (quasi-static case):

$$0 = \operatorname{div} A(e(u) - e_p) + f(t),$$

$$\dot{e}_p(t) \in \partial\chi_{\mathbb{K}}\big(\mathbb{A}(e(u(t,x)) - e_p(t,x)) - e_p\big)$$

$$\mathbb{K} = \{\, \tau \in \mathbb{R}^{d\times d}_{\mathrm{sym}}\ ;\ |\mathrm{dev}\,\tau| \le \sigma_0 \,\} \quad \text{von Mises flow rule}$$

Initial condition: $e_p(0) = e_{p_0}$.

Example: elasto-plasticity with linear kinematic hardening

Variables

displacements $u : [0, T] \times \Omega \to \mathbb{R}^d$

linearized strain $e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$

plastic strain $e_p \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ with $\mathrm{tr}\, e_p = 0$

Energy and dissipation

energy functional: $\mathcal{E}(t, u, z) = \int_\Omega \frac{1}{2}\mathbb{A}\big(e(u) - e_p\big) : \big(e(u) - e_p\big) + \frac{1}{2}\,|e_p|^2 \,\mathrm{d}x - \langle \ell(t), u\rangle$

dissipation pot.: $\mathcal{R}(\dot e_p) := \int_\Omega \sigma_0\, |\dot e_p| \,\mathrm{d}x$.

Evolution law (quasi-static case):

Find $(u(t), z(t)) \in \mathcal{Q} := H^1_{\Gamma_D}(\Omega, \mathbb{R}^d) \times L^2(\Omega, \mathbb{R}^{d\times d}_{\mathrm{sym,dev}})$ with

$$0 = \partial_u \mathcal{E}(t, u(t), z(t)) \qquad \text{in } \mathcal{U}^*,$$

$$\dot z(t, x) \in \partial \chi_{\mathcal{K}}\big(-\partial_z \mathcal{E}(t, u(t), z(t))\big) \qquad \text{in } \mathcal{Z}^*.$$

Initial condition: $e_p(0) = e_{p_0}$.

Example: elasto-plasticity with linear kinematic hardening

Variables

displacements	$u : [0,T] \times \Omega \to \mathbb{R}^d$
linearized strain	$e(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$
plastic strain	$e_p \in \mathbb{R}^{d\times d}_{\text{sym}}$ with $\operatorname{tr} e_p = 0$

Energy and dissipation

energy functional: $\mathcal{E}(t,u,z) = \int_\Omega \frac{1}{2}\mathbb{A}(e(u)-e_p) : (e(u)-e_p) + \frac{1}{2}|e_p|^2 \,\mathrm{d}x - \langle \ell(t), u\rangle$

dissipation pot.: $\mathcal{R}(\dot{e}_p) := \int_\Omega \sigma_0 |\dot{e}_p| \,\mathrm{d}x.$

Theorem

$\forall e_{p,0} \in \mathbb{R}^{d\times d}_{sym}$, $\ell \in C^1([0,T];\mathcal{U}^*)$ there exists a unique pair

$$(u, e_p) \in C^{Lip}([0,T]; H^1_{\Gamma_D}(\Omega,\mathbb{R}^d) \times L^2(\Omega, \mathbb{R}^{d\times d}_{sym,dev}))$$

that solves the model of elasto-plasticity with linear kinematic hardening.

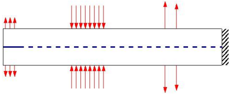
Initial condition: $e_p(0) = e_{p_0}$.

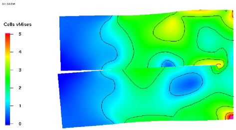
4 Further solution concepts in the nonconvex case: vanishing viscosity solutions

Explained at the example of a single propagating crack

Joint work with Chiara Zanini (Politecnico di Torino)
Andres Schröder (University of Salzburg)
Alexander Mielke (WIAS, Berlin)

Introduction





Goal: Existence of solutions for a rate-independent model for crack propagation that is based on the Griffith criterion.

$$\mathrm{D}_u\mathcal{E}(u(t), s(t)) = \ell(t), \quad 0 \in \partial\mathcal{R}(\dot{s}(t)) + \mathrm{D}_s\mathcal{E}(u(t), s(t)). \tag{1}$$

Questions: What is $\mathrm{D}_s\mathcal{E}$? (energy release rate)

The energy $\mathcal{E}$ is not convex in s

$\Rightarrow$ The evolution might be dicontinuous, as an example will show.

Does the global energetic formulation predict physically reasonable jumps?

Is there an alternative way to interpret (1) for discontinuous evolutions?

$\rightarrow$ vanishing viscosity solutions

Notation, linear elasticity with contact conditions (2D)

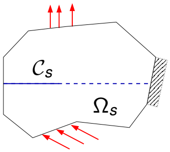
- Linear elasticity:

  $u : [0,T] \times \Omega_s \to \mathbb{R}^2$ displacements,

  $\mathbf{e}(u) = \frac{1}{2}(\nabla u + \nabla u^\top)$ linearized strain tensor

- Elastic energy density:

  $W(\nabla u) = \frac{1}{2}\mathbb{C}\mathbf{e}(u) : \mathbf{e}(u)$, $\quad \mathbb{C}$ elasticity tensor.



- Non-interpenetration condition (Signorini conditions) on the crack $\mathcal{C}_s$:

$$K(\Omega_s) = \{\, v \in H^1(\Omega_s)\,;\ v\big|_{\Gamma_D} = 0,\ \ [\![v]\!] \cdot \vec{n} \geq 0 \text{ on } \mathcal{C}_s \,\}$$

Stored energy: $\quad \mathcal{E}(t,s,v) = \int_{\Omega_s} W(\nabla v)\,\mathrm{d}x - \int_{\Gamma_N} \ell(t) \cdot v\,\mathrm{d}a \quad$ for $v \in K(\Omega_s)$.

Reduced energy: $\quad \mathcal{I}(t,s) = \min_{v \in K(\Omega_s)} \mathcal{E}(t,s,v)$.

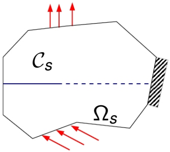
Energy release rate: $\quad \mathcal{G}(t,s) = -\partial_s \mathcal{I}(t,s) = -\partial_s\big(\min_{\varphi \in K(\Omega_s)} \mathcal{E}(t,s,\varphi)\big)$

Theorem

$\mathcal{I} \in C^{1,1}_{loc}([0,T] \times (0,L))$; $\quad \partial_s \mathcal{I}(t,s) = \int_{\Omega_s} (W(\nabla u_s)\mathbb{I} - \nabla u_s^T DW(\nabla u_s)) : (\mathbf{e}_1 \otimes \nabla \theta_s)\,\mathrm{d}x$

Crack evolution based on the Griffith fracture criterion

Griffith criterion (1921)



The crack is stationary, if the
(locally) released elastic energy is less than
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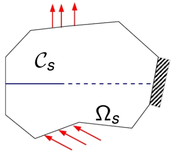
Crack evolution based on the Griffith fracture criterion



Griffith criterion (1921)

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Dissipated energy:

$$\mathcal{R}(s_{\text{new}} - s_{\text{old}}) = \begin{cases} \kappa(s_{\text{new}} - s_{\text{old}}) & \text{if } s_{\text{new}} \geq s_{\text{old}} \\ \infty & \text{else} \end{cases} .$$

$\kappa = G_c > 0$ fracture toughness

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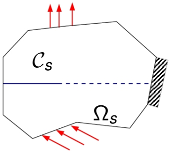

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Evolution criterion:

$$\left.\begin{array}{ll} \text{local stability:} & \kappa \geq -\partial_s \mathcal{I}(t, s(t)), \\ \text{complementarity:} & \dot{s}(t)\big(\kappa + \partial_s \mathcal{I}(t, s(t))\big) = 0 \end{array}\right\} \Leftrightarrow \boxed{0 \in \partial\mathcal{R}(\dot{s}(t)) + \partial_s \mathcal{I}(t, s(t))}$$

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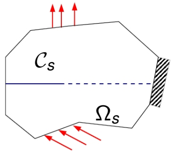

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Problem: Discontinuous solutions may occur!

Why discontinuous solutions?

Evolution law:

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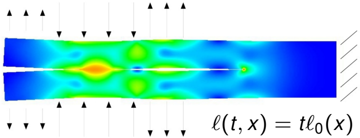

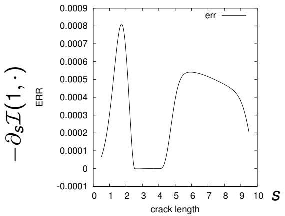

Observe:

$s \mapsto \partial_s \mathcal{I}(t, s)$ is not monotone.

$\Rightarrow s \mapsto \mathcal{I}(t, s)$ is NOT convex!

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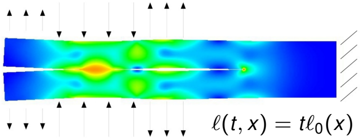

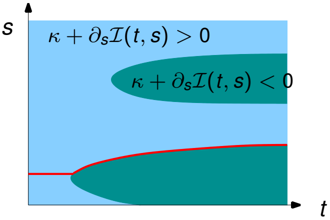

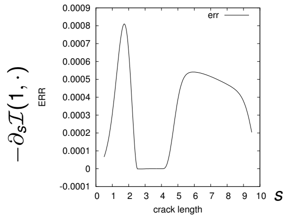

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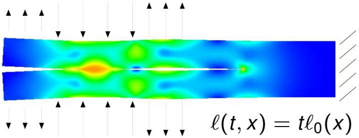
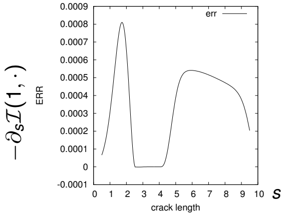
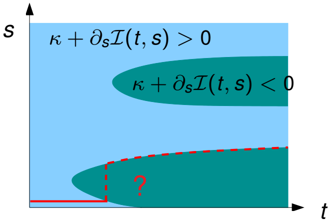






Theorem

For every initial datum $s_0 \in [0, L]$ there exists at least one global energetic solution.

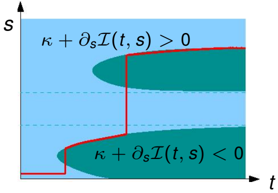
Global energetic solutions

(S) **Global** stability

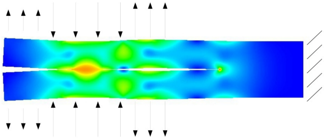
$$\mathcal{I}(t, s(t)) \leq \mathcal{I}(t, \tilde{s}) + \int_{s(t)}^{\tilde{s}} \kappa \mathrm{d}\sigma \qquad \forall \tilde{s} \geq s(t),$$

(E) Energy equality

$$\mathcal{I}(t, s(t)) + \int_{s(0)}^{s(t)} \kappa \mathrm{d}\sigma = \mathcal{I}(0, s(0)) + \int_0^t \partial_t \mathcal{I}(\tau, s(\tau)) \mathrm{d}\tau.$$



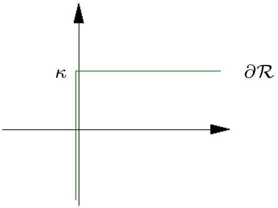
Jump condition: $\int_{s_-}^{s_+} \left(\partial_s \mathcal{I}(t, \sigma) + \kappa\right) \mathrm{d}\sigma = 0$



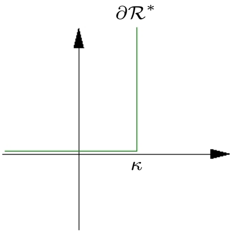
Alternative approach: Vanishing viscosity solutions

Effect of additional viscosity

$$0 \in \partial\big(\mathcal{R}(\dot{s}(t)) \qquad\qquad \big) + \partial_s \mathcal{I}(t, s(t)).$$



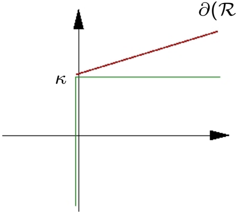
$$\dot{s}(t) \in \underbrace{\partial\big(\mathcal{R}(\cdot) \qquad\qquad \big)^*}\big(- \partial_s \mathcal{I}(t, s(t))\big).$$



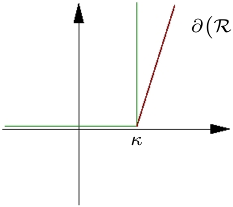
Effect of additional viscosity

$$0 \in \partial\Big(\mathcal{R}(\dot s(t)) + \frac{\nu}{2}\,|\dot s(t)|^2\Big) + \partial_s \mathcal{I}(t, s(t)).$$

$\nu > 0$ viscosity parameter



$$\dot s(t) \in \underbrace{\partial\Big(\mathcal{R}(\cdot) + \frac{\nu}{2}\,|\cdot|^2\Big)^*}_{\text{Lipschitz continuous}}\Big(-\partial_s \mathcal{I}(t, s(t))\Big).$$



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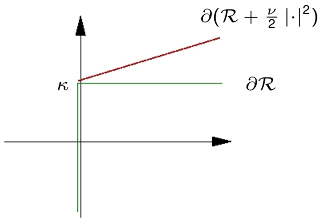

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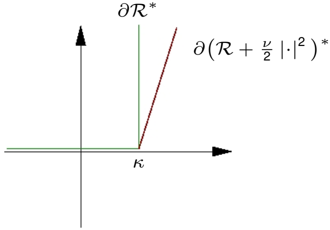

Lemma

For every $\nu > 0$, there exists a unique solution $s_\nu \in H^1(0, T; [s_0, L])$.

Theorem (K./Mielke/Zanini)

There exists $s \in BV([0,T],\mathbb{R})$ *and a subsequence* $\nu \searrow 0$ *with*

$$s_\nu \overset{*}{\rightharpoonup} s \quad \text{in } BV([0,T],\mathbb{R}) \quad \text{and} \quad s_\nu(t) \to s(t) \ \text{ for every } t \in [0,T].$$

Moreover,

- **(a)** s *is non-decreasing,*
- **(b)** $\kappa + \partial_s \mathcal{I}(t, s(t)) \geq 0$ *for every* $t \in [0,T] \backslash J(s)$*,*
- **(c)** *if* $\kappa + \partial_s \mathcal{I}(t, s(t)) > 0$*, then* $t \in D(s)$ *and* $\dot{s}(t) = 0$*,*
- **(d)** $\forall\, t \in J(s), \forall s_* \in [s(t_-), s(t_+)]$ *we have* $\kappa + \partial_s \mathcal{I}(t, s_*) \leq 0$.

$J(s)$ jump set of s; $D(s)$ set of differentiable points of s.

Proof: A-priori estimates, Helley selection principle, continuity property of $\partial_s \mathcal{I}$, change of variables ($s \leftrightarrow t$) to obtain (d).

Proof of the jump condition (d)

Let $t_* \in J(s)$, $\quad s_0, s_1 \in (s(t_*-), s(t_*+))$ arbitrary

Continuity of $\hat{s}_\tau$: $\quad \exists\ t_\tau^0 < t_\tau^1$ with $t_\tau^{0,1} \overset{\tau\to 0}{\to} t_*$ and $\hat{s}_\tau(t_\tau^{0,1}) = s_{0,1}$.

Complementarity condition: $\quad \forall \psi \in L^2([s_0, s_1])$, $\psi \geq 0$ we have

$$0 \geq \int_{t_\tau^0}^{t_\tau^1} \Big(\kappa + \partial_s \mathcal{I}(\bar{t}_\tau(t), \bar{s}_\tau(t))\Big)\, \psi(\hat{s}_\tau(t))\, \hat{s}_\tau'(t)\, \mathrm{d}t.$$

Change of variables: $\quad \sigma = \hat{s}_\tau(t), \quad \tilde{t}_\tau(\sigma) = \min\{\, t \in [t_\tau^0, t_\tau^1]\,;\, \hat{s}_\tau(t) = \sigma \,\}$

$$0 \geq \int_{s_0}^{s_1} \Big(\kappa + \partial_s \mathcal{I}(\bar{t}_\tau(\tilde{t}_\tau(\sigma)), \bar{s}_\tau(\tilde{t}_\tau(\sigma)))\Big)\, \psi(\sigma)\, \mathrm{d}\sigma.$$

Note: $\quad \bar{t}_\tau(\tilde{t}_\tau(\sigma)) \to t_*, \quad \left|\bar{s}_\tau(\tilde{t}_\tau(\sigma)) - \sigma\right| = \left|\bar{s}_\tau(\tilde{t}_\tau(\sigma)) - \hat{s}_\tau(\tilde{t}_\tau(\sigma))\right| \leq c\sqrt{\tau/\nu} \to 0.$

By lsc. of $\partial_s \mathcal{I}$ we conclude that $\quad \boxed{\forall \sigma \in [s_0, s_1]: \quad 0 \geq \kappa + \partial_s \mathcal{I}(t_*, \sigma).}$

Energy balance

Theorem (K./Mielke/Zanini)

Every vanishing viscosity solution satisfies for every $t_0 < t_1$, $t_i \notin J(s)$:

$$\mathcal{I}(t_1, s(t_1)) + \int_{s(t_0)}^{s(t_1)} \kappa \mathrm{d}\sigma + \sum_{\hat{t} \in J(s) \cap (t_0, t_1)} \int_{s(\hat{t}-)}^{s(\hat{t}+)} -(\kappa + \partial_s \mathcal{I}(\hat{t}, \sigma)) \mathrm{d}\sigma$$
$$= \mathcal{I}(t_0, s(t_0)) + \int_{t_0}^{t_1} \partial_t \mathcal{I}(t, s(t)) \mathrm{d}\tau.$$

Proof: Switch to a parameterized formulation of the evolution problem, use a chain rule and (a), (b)–(d).

Vanishing viscosity solutions

(a) s nondecreasing,

(b) $\kappa + \partial_s\mathcal{I}(t, s(t)) \geq 0$ for all $t \in [0, T]\backslash J(s)$,

(c) if $\kappa + \partial_s\mathcal{I}(t, s(t)) > 0$, then $t \in D(s)$ and $\dot{s}(t) = 0$,

(d) $\forall\, t \in J(s)$, $\forall s_* \in [s(t_-), s(t_+)]$ it holds
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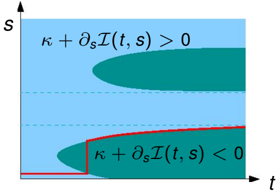

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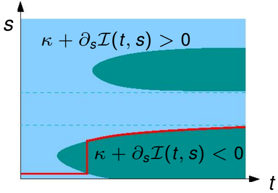

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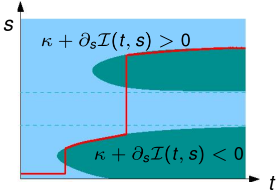
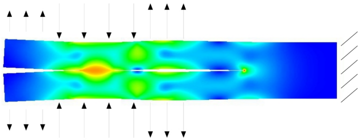


Jump condition: $\int_{s_-}^{s_+} \big(\partial_s \mathcal{I}(t, \sigma) + \kappa \big) \, \mathrm{d}\sigma = 0$

A numerical example

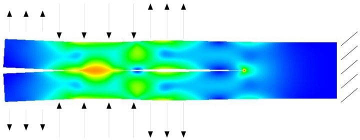


Computations: A. Schröder

monotone loading: $\ell_1 \in L^2(\Gamma_N), \quad \ell(t) = t\,\ell_1.$

Deformation energy: $\mathcal{E}(t,s,v) = \int_{\Omega_s} \frac{1}{2}\mathbb{C}\varepsilon(v) : \varepsilon(v)\,\mathrm{d}x - \int_{\Gamma_N} t\,\ell_1 \cdot v\,\mathrm{d}a.$

density quadratic $\Rightarrow$ $\mathcal{I}(t,s) = t^2\mathcal{I}(1,s), \qquad \partial_s\mathcal{I}(t,s) = t^2\partial_s\mathcal{I}(1,s).$



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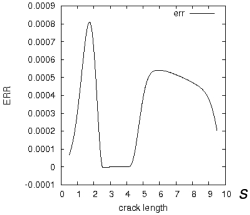
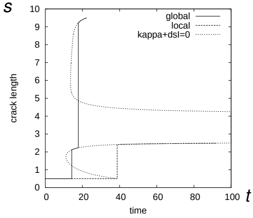
$-\partial_s\mathcal{I}(1,\cdot)$:

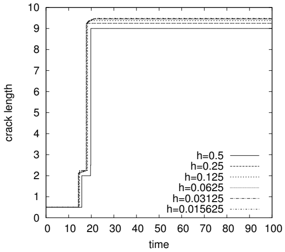


Solutions:



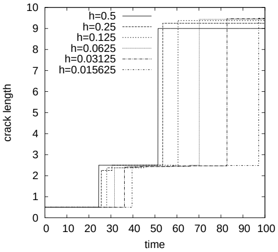
A numerical example

- Finite element discretization with continuous, piecewise bilinear functions on quadrilaterals, CG-like contact solver (Braess et al. '04), SOFAR (Scientific Object Oriented Finite Element Library for Application and Research)



global energetic solution

$\rho = h, \nu = 0, \tau = 0.01$



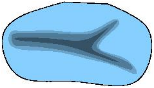
viscosity solution

$\rho = h, \nu = 0.013h^{0.5}, \tau = 0.1h$

$$s_N^k \in \text{Argmin}\{\mathcal{I}_N(t_N^k, \tilde{s}) + \frac{\nu_N}{2\tau_N}(\tilde{s} - s_N^{k-1})^2 + \kappa(\tilde{s} - s_N^{k-1})\,;\, \tilde{s} \in Z^N,\ \tilde{s} \geq s_N^{k-1}\}.$$

- Dynamic model instead of viscous regularization?
- More reliable method to compute the vanishing viscosity solution?

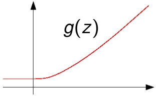
Extension to damage models (with R. Rossi, C. Zanini)



Damage variable $z : [0, T] \times \Omega \to [0, 1]$

Energy: $\mathcal{E}(u, z) = \int_\Omega \frac{g(z)}{2} \mathbb{C}\mathbf{e}(u) : \mathbf{e}(u) + \frac{\gamma}{2} |\nabla z|^2 \, \mathrm{d}x$

Dissipation: $\mathcal{R}(\dot z) = \begin{cases} \int_\Omega \kappa |\dot z| \, \mathrm{d}x & \text{if } \dot z \le 0 \\ \infty & \text{otherwise} \end{cases}$



Evolution law: $0 \in \partial \mathcal{R}(\dot z) + \nu \dot z + \frac{g'(z)}{2} \mathbb{C}\mathbf{e}(u) : \mathbf{e}(u) - \gamma \triangle z$

Further results on rate-independent models based on viscous approximations:

general theory: Mielke/Efendiev ’06, Rossi/Mielke/Savaré ’08-’12, Mielke/Zelik ’10
cracks: K./Mielke/Zanini ’08/’10, K./Schröder ’11, Lazzaroni/Toader ’11
damage: K./Rossi/Zanini ’11-today

plasticity: DalMaso, De Simone, Mora, Morini, Solombrino 08/11

Thank you for your attention!

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References (related to vanishing viscosity approach)

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