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Mehrfeldprobleme in der Kontinuumsmechanik

Dorothee Knees

Regularity results for transmission problems for the Laplace and Lamé operators on polygonal or polyhedral domains

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Abstract

Boundary value problems for the Laplace and Lamé operators with piecewise constant material coefficients are investigated on polygonal or polyhedral domains. Because of geometric peculiarities and non-smooth material constants the solutions and especially the derivatives have a singular behavior in a neighborhood of corners, edges and crossing points. For 3D problems for the Lamé operator it is not clear if the displacement fields are bounded. In this paper we derive sufficient conditions on the material constants and geometry which guarantee that weak solutions of the BVPs are bounded and piecewise continuous. We further give a short overview over known results.

1 Introduction

In this paper we consider boundary transmission problems for the Laplace and Lamé operators on polygonal or polyhedral domains. It is well known that harmonic and linear elastic fields have a singular behavior near geometrical peculiarities such as corners, edges, crossing points or crossing edges. The singular behavior can be characterized by an asymptotic expansion for weak solutions u in a neighborhood of a corner point S. For 3D polyhedral domains the expansion has the following form:
Let $u \in \mathcal{H}^1(\Omega) := \{u \in L^2(\Omega) : u\big|_{\Omega_i} \in H^1(\Omega_i)\}$ be a weak solution for the Laplace or Lamé equations with piecewise constant coefficients on $\Omega_i \subset \mathbb{R}^3$; $\overline{\Omega} = \cup_i \overline{\Omega_i}$, Ω_i polyhedral. Then u can be decomposed in the following way in a neighborhood of a corner point S [3]:

$$\eta^S u = u_{\text{reg}} + \eta^S u_{\text{edge}} + \eta^S u_{\text{corner}}.$$

Here, $u_{\text{reg}}\big|_{\Omega_i} \in H^{2-\varepsilon}(\Omega_i)$ for a small $\varepsilon > 0$, η^S is a cut-off function. Further

$$u_{\text{corner}} = \sum_{-\frac{1}{2} < \text{Re } \beta_j < \frac{1}{2} - \varepsilon} c_j \, \rho^{\beta_j} W_j(\ln \rho, \theta, \varphi), \tag{1.1}$$

where (ρ, θ, φ) are spherical coordinates. Finally

$$u_{\text{edge}} = \sum_{\text{edges e}} \; \sum_{0 < \text{Re } \alpha_{j,e} < 1 - \varepsilon} d_{\alpha_e}(z_e, \rho) \, r_e^{\alpha_{j,e}} V_{j,e}(\ln r_e, \varphi).$$

Here we sum over all edges e which contain S, r_e is the distance to edge e. The regularity of a weak solution is determined by the smallest real parts of the singular exponents $\beta_j, \alpha_{j,e}$. If there are no edge

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exponents $\alpha_{j,e}$ in the strip Re $\alpha \in]0, \frac{1}{2}]$ and no corner exponents β_j in the strip Re $\beta \in] -\frac{1}{2}, 0[$ then we have the following regularity result for weak solutions: $\eta^S u\big|_{\Omega_i} \in H^{\frac{3}{2}+\varepsilon}(\Omega_i)$ which is embedded in $\mathcal{C}(\overline{\Omega_i})$.

The main goal of this paper is to describe classes of transmission problems for which weak solutions admit an asymptotic expansion as in (1.1) where no edge exponents are situated in the strip Re $\alpha \in]0, \frac{1}{2}]$ and no corner exponents in the strip Re $\beta \in] -\frac{1}{2}, 0]$. These classes consist of transmission problems with an arbitrary number of subdomains where the material parameters are distributed quasi-monotonely and where some additional geometric conditions are satisfied. In the two dimensional case there are examples which show that if these conditions are violated there can be stronger singularities, [8], whereas in the three dimensional case such examples are unknown for the Lamé operator.

The paper is organized as follows: In sections 2 and 3 we give the basic definitions and recall asymptotic expansions for weak solutions, [10, 3, 24]. In section 4, a homotopy argument based on Rouché's Theorem for operator-valued functions is presented which we will use in section 5 to prove the main result formulated in Theorem 5.1. In this Theorem we describe in detail the assumptions on the material parameters (quasi-monotonicity) and the geometry which guarantee that there are no corner exponents in the strip Re $\beta \in] -\frac{1}{2}, 0[$. For boundary value problems which satisfy these conditions we then get the regularity $u\big|_{\Omega_i} \in \mathcal{C}(\overline{\Omega_i})$. The main idea of the proof is to carry over known estimates of the exponents for problems with constant parameters to problems with piecewise constant parameters by a homotopy argument.
A short overview of known estimates of the singular exponents will also be given in section 5. There is a variety of estimates for the Laplace transmission operator on 2D domains [2, 14, 24, 25, 19] which we summarize in table 1. In contrast to the Laplace operator, there are only few results for transmission problems of the Lamé system in the literature. Estimates for one subdomain were derived in [7, 22, 12, 26]. The results presented in this paper are a generalization of those in [24], where estimates for boundary transmission problems on two subdomains with a plane interface were developed.

2 Formulation of the Problem

2.1 Domains

In this paper we will consider polygonal or polyhedral domains $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, which are divided into polygonal or polyhedral subdomains. In order to include domains with cracks and other non-Lipschitz domains, we first introduce the notion of generalized polyhedrons and composites. In section 2.2 the corresponding Sobolev-spaces and needed trace theorems will be specified.

2.1.1 Generalized polyhedrons

Let $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, be bounded and let the cone property be satisfied:

Definition 2.1. *[29, Def. 2.2] $\Omega \subset \mathbb{R}^N$ satisfies the cone property if for every $x \in \overline{\Omega}$ there exists an open spherical cone $C(x)$ with vertex in x which is congruent to a fixed cone C_0 and $C(x) \subset \Omega$.*

We further assume that $\partial\Omega$ is the union of oriented $N-1$ dimensional plane surfaces, that means: There is a finite number of pairs $(\Gamma_i, \vec{n}_i)$ with $\Gamma_i \subset \partial\Omega$ such that

1. $\partial\Omega = \cup_i \overline{\Gamma}_i$,

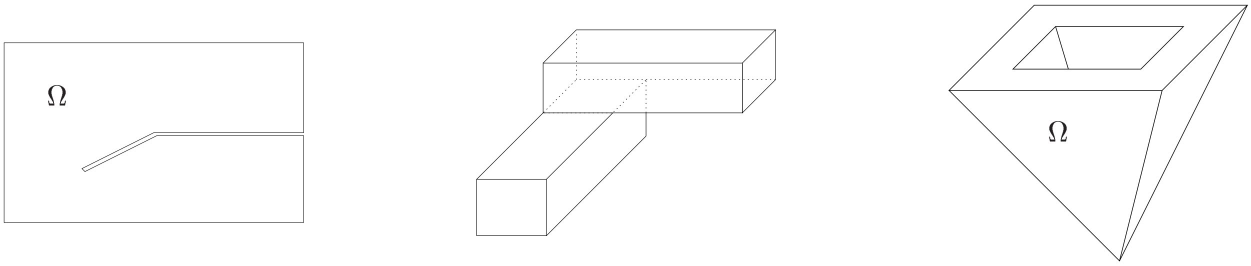


Figure 1: Generalized polyhedrons

2. Every Γ_i is an open connected polygonal subset of a $N-1$ dimensional hyperplane, Γ_i has a Lipschitz boundary. $\vec{n}_i$ is a unit normal vector on Γ_i for which we assume: $\forall\, x_0 \in \Gamma_i \;\exists\, \delta(x_0) > 0$ such that $\forall\, 0 < \delta < \delta(x_0)$ there holds: $x_0 - \delta\vec{n}_i \in \Omega$.

3. $\forall i, j$: if $\Gamma_i \cap \Gamma_j \neq \emptyset$ and $\vec{n}_i = \vec{n}_j$, then $\Gamma_i = \Gamma_j$.

4. If $S := (\operatorname{int}\overline{\Omega})\backslash\Omega \neq \emptyset$ (i. e. if Ω has cracks) then there exist $(\Gamma_{i_1}, \vec{n}_{i_1}), \ldots, (\Gamma_{i_l}, \vec{n}_{i_l})$ where the Γ_{i_j} are pairwise disjoint and $\overline{S} = \cup_{1 \leqslant j \leqslant l} \overline{\Gamma_{i_j}}$.
   Further there exist $(\Gamma_{k_1}, \vec{n}_{k_1}), \ldots, (\Gamma_{k_l}, \vec{n}_{k_l})$ with $\Gamma_{k_j} = \Gamma_{i_j}$ and $\vec{n}_{k_j} = -\vec{n}_{i_j}$ for $1 \leqslant j \leqslant l$.

Domains which satisfy these conditions will be called *generalized polyhedrons*.
These conditions can be interpreted as follows: $\partial\Omega$ is divided into plane faces Γ_i. To each Γ_i is associated a normal vector $\vec{n}_i$ which is directed to the exterior of Ω if a part of Γ_i is contained in the exterior boundary of Ω. If Ω has a crack S then S shall be covered twice by the $(\Gamma_i, \vec{n}_i)$ such that one can identify left and right crack sides (with the corresponding normal vectors). Two different parts Γ_i and Γ_j may only intersect if $\vec{n}_i = -\vec{n}_j$ (this can happen at a crack only).

Example 2.1. Every standard polyhedron and polyhedrons with cracks are generalized polyhedrons. In fig. 1 generalized polyhedrons are plotted which have no Lipschitz boundary.

2.1.2 Composites

We now introduce composed polyhedral domains where we will study transmission problems.
Let $\Omega \subset \mathbb{R}^N$ be a generalized polyhedron. We assume that Ω is divided into a finite number of generalized

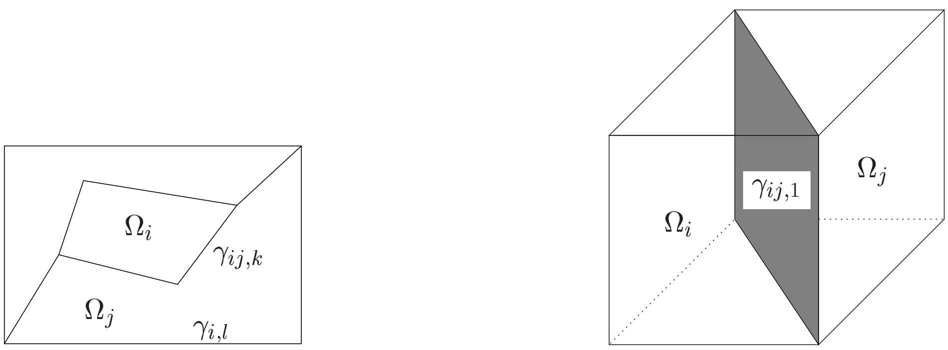
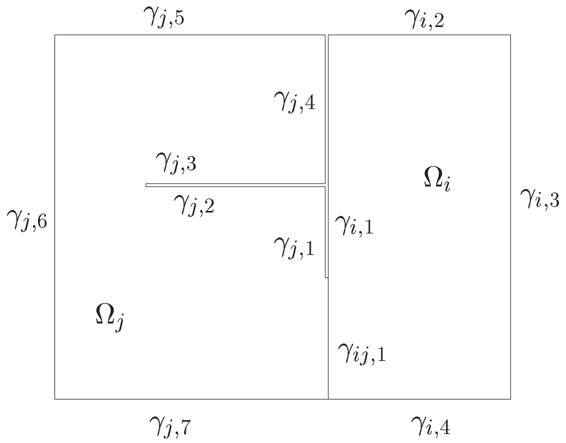


Figure 2: Examples for composites, 2D and 3D



$$\overline{\Gamma_i} = \bigcup_{s=1}^{4} \overline{\gamma_{i,s}},$$
$$\Gamma_{ij} = \gamma_{ij,1},$$
$$\overline{\Gamma_j} = \bigcup_{s=1}^{7} \overline{\gamma_{j,s}}.$$

Figure 3: Example for a composite with crack, 2D

polyhedra Ω_i in the following way:

$$\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega_i},$$

where $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, $\Omega_i \subset \Omega$ and if $S_i := \mathrm{int}(\overline{\Omega}_i)\backslash\Omega_i \neq \emptyset$ then $S_i \subset \partial\Omega$. Such domains will be called *composites*, fig. 2.

To describe the boundary $\partial\Omega$ we introduce the notation, fig. 3:
If $\mathrm{mes}_{N-1}(\partial\Omega \cap \partial\Omega_i) \neq 0$ then we divide the common boundary into oriented parts $(\gamma_{i,l}, \vec{n}_{i,l})$, $1 \leqslant l \leqslant n(i)$, such that:
$\gamma_{i,l}$ is an open subset of a $N-1$ dimensional hyperplane, $\gamma_{i,l}$ has a polygonal Lipschitz boundary and $\gamma_{i,l} \subset \partial\Omega_i \cap \partial\Omega$; $\vec{n}_{i,l}$ is normal to $\gamma_{i,l}$ and for $\overline{\Gamma}_i := \bigcup_{l=1}^{n(i)} \overline{\gamma_{i,l}}$ there holds: $\mathrm{mes}_{N-1}(\Gamma_i) = \mathrm{mes}_{N-1}(\partial\Omega_i \cap \partial\Omega)$. Further the pairs $(\gamma_{i,l}, \vec{n}_{i,l})$ satisfy 2. and 3. in section 2.1.1 and in addition:

4'. If $S_i := (\mathrm{int}\,\overline{\Omega}_i)\backslash\Omega_i \neq \emptyset$ (i. e. if Ω_i has cracks) then there exist $(\gamma_{i,l_1}, \vec{n}_{i,l_1}), \ldots, (\gamma_{i,l_k}, \vec{n}_{i,l_k})$ where the γ_{i,l_j} are pairwise disjoint and $\overline{S_i} = \cup_{1 \leqslant j \leqslant k} \overline{\gamma_{i,l_j}}$.
Further there exist $(\gamma_{i,s_1}, \vec{n}_{i,s_1}), \ldots, (\gamma_{i,s_k}, \vec{n}_{i,s_k})$ with $\gamma_{i,s_j} = \gamma_{i,l_j}$ and $\vec{n}_{i,s_j} = -\vec{n}_{i,l_j}$ for $1 \leqslant j \leqslant k$.

For the interfaces we use the following notation:
If $\mathrm{mes}_{N-1}(\partial\Omega_i \cap \partial\Omega_j \backslash \partial\Omega) \neq 0$ then we divide the interface of Ω_i, Ω_j into plane faces $\gamma_{ij,l}$, $1 \leqslant l \leqslant n(ij)$: $\gamma_{ij,l}$ is an open subset of a $N-1$ dimensional hyperplane and has a polygonal Lipschitz boundary; $\gamma_{ij,l} \cap \gamma_{ij,k} = \emptyset$ for $l \neq k$ and for $\overline{\Gamma_{ij}} := \bigcup_{l=1}^{n(ij)} \overline{\gamma_{ij,l}}$ there holds: $\mathrm{mes}_{N-1}(\Gamma_{ij}) = \mathrm{mes}_{N-1}(\partial\Omega_i \cap \partial\Omega_j \backslash \partial\Omega)$. For $\gamma_{ij,k} \in \mathcal{G}$ we denote by $\vec{n}_{ij,k}$ the exterior normal vector of $\gamma_{ij,k}$ with respect to Ω_i, by $\vec{n}_{ji,k}$ the exterior normal vector of $\gamma_{ij,k}$ with respect to Ω_j.
Finally we collect the parts of $\partial\Omega$ in the set $\mathcal{F} := \{(\gamma_{i,l}, \vec{n}_{i,l}),\ 1 \leqslant i \leqslant M,\ 1 \leqslant l \leqslant n(i)\} =: \mathcal{D} \cup \mathcal{N}$, where $\mathcal{D}, \mathcal{N}$ are disjoint and characterize the Dirichlet- and the Neumann-boundary respectively. $\mathcal{G} := \{\gamma_{ij,k},\ i, j, k\}$ describes the interface.

By $\mathcal{S}$ we denote the set of geometrical singularities which consist of corners and edges.

2.2 Spaces

The following Sobolev spaces will be used:
Let $l \in \mathbb{N}_0$, $\Omega \subset \mathbb{R}^N$ be an open, connected domain.

$$H^l(\Omega) := \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega),\, 0 \leqslant |\alpha| \leqslant l\}.$$

Here, $D^\alpha u$ is the distributional derivative of u , α is a multi-index. $(H^l(\Omega), \|\cdot\|_{H^l(\Omega)})$ is a separable Hilbert space with the usual norm and inner product, [29]. If Ω is a composite, we set

$$\mathcal{H}^l(\Omega) := \{u \in L^2(\Omega) : u\big|_{\Omega_i} \in H^l(\Omega_i)\}.$$

We shortly write u_i for $u\big|_{\Omega_i}$. For $\Omega \subset \mathbb{R}^N$, open, $\mathcal{D}(\Omega)$ is the set of infinitely differentiable functions in $\mathbb{R}^N$ with compact support in Ω, $\mathcal{D}(\overline{\Omega}) = \{u\big|_{\overline{\Omega}} : u \in \mathcal{D}(\mathbb{R}^N)\}$.
We further need the following trace spaces for $l = 1, 2, \ldots$: Let Ω be a composite, $\gamma \in \mathcal{G} \cup \mathcal{F}$.

$$H^{l-\frac{1}{2}}(\gamma) := \overline{\mathcal{D}(\overline{\gamma})}^{\|\cdot\|_{H^{l-\frac{1}{2}}(\gamma)}},$$

where the norm is defined by the Sobolev-Slobodetskij norm

$$\|u\|^2_{H^{l-\frac{1}{2}}(\gamma)} = \|u\|^2_{H^{l-1}(\gamma)} + \sum_{|\alpha| \leqslant l-1} \int_{\gamma\times\gamma} \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{N+1}}\, dx\, dy.$$

Finally

$$\tilde{H}^{l-\frac{1}{2}}(\gamma) = \{u \in H^{l-\frac{1}{2}}(h(\gamma)) : \operatorname{supp} u \subset \overline{\gamma}\},$$

where $h(\gamma)$ is that $N-1$ dimensional hyperplane which contains γ. Since γ has a Lipschitz boundary $H^{l-\frac{1}{2}}(\gamma) = \{u\big|_\gamma : u \in H^{l-\frac{1}{2}}(h(\gamma))\}$, [29, Thm. 3.6].
For $l = 1$ we define the following dual spaces:

$$H^{-\frac{1}{2}}(\gamma) := \big(\tilde{H}^{\frac{1}{2}}(\gamma)\big)', \qquad \tilde{H}^{-\frac{1}{2}}(\gamma) := \big(H^{\frac{1}{2}}(\gamma)\big)'.$$

For $v \in \tilde{H}^{\frac{1}{2}}(\gamma), h \in H^{-\frac{1}{2}}(\gamma)$ we denote by $\langle h, v, \rangle_\gamma := \langle h, v\rangle_{(H^{-\frac{1}{2}}(\gamma), \tilde{H}^{\frac{1}{2}}(\gamma))}$ the dual pairing.
For generalized polyhedrons we have the following trace theorems:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a generalized polyhedron , $(\Gamma_i, \vec{n}_i)$ as in sect. 2.1.1, $m \in \mathbb{N}$.*

1. *Let $0 \leqslant l \leqslant m-1$. There exists a unique linear and continuous mapping*

$$\gamma^l_{(\Gamma_i, \vec{n}_i)} : H^m(\Omega) \to \prod_{j=0}^{l} H^{m-j-\frac{1}{2}}(\Gamma_i)$$

with the following property:
If $\tilde{\Omega} \subset \mathbb{R}^N$ is an open Lipschitz-domain with $\tilde{\Omega} \subset \Omega$, $\Gamma_i \subset \partial\tilde{\Omega}$ and $\vec{n}_i$ as exterior normal vector, then for all $u \in H^m(\Omega)$ with $u\big|_{\tilde{\Omega}} \in \mathcal{D}(\overline{\tilde{\Omega}})$:

$$\gamma^l_{(\Gamma_i, \vec{n}_i)}(u) = \left\{ u\big|_{\Gamma_i}, \frac{\partial u}{\partial \vec{n}_i}\big|_{\Gamma_i}, \ldots, \frac{\partial^l u}{\partial \vec{n}_i^l}\big|_{\Gamma_i} \right\}.$$

2. There exists a linear, continuous extension operator

$$F_{(\Gamma_i,\vec{n}_i)} : \tilde{H}^{m-\frac{1}{2}}(\Gamma_i) \longrightarrow \{v \in H^m(\Omega) : \operatorname{supp}(v\big|_{\partial\Omega}) \subset \overline{\Gamma_i}\}$$

such that $\gamma^0_{(\Gamma_i,\vec{n}_i)} \circ F_{(\Gamma_i,\vec{n}_i)}(u) = u$ *for all* $u \in \tilde{H}^{m-\frac{1}{2}}(\Gamma_i)$.

This Theorem is proved in [6] for generalized polyhedrons with Lipschitz boundaries and can easily be extended to those without Lipschitz boundaries.
For the definition of the normal derivative for H^1-functions we introduce analogous to [6] the space $E(\Omega) := \{u \in H^1(\Omega) : \triangle u \in L^2(\Omega)\}$. This space is a Banach space regarding the norm $\|u\|_E := \|u\|_{H^1(\Omega)} + \|\triangle u\|_{L^2(\Omega)}$. If Ω is a bounded Lipschitz domain, then $\mathcal{D}(\overline{\Omega})$ is dense in $E(\Omega)$. For composites Ω we set

$$\mathcal{E}(\Omega) := \left\{u \in \mathcal{H}^1(\Omega) : \ u\big|_{\Omega_i} \in E(\Omega_i)\right\}.$$

Theorem 2.2 (Normal derivative). *[6] Let $\Omega \subset \mathbb{R}^n$ be a generalized polyhedron, $(\Gamma_i, \vec{n}_i)$ as in sect. 2.1.1. Then there exists a unique linear, continuous operator*

$$\frac{\partial}{\partial \vec{n}_i} : E(\Omega) \longrightarrow H^{-\frac{1}{2}}(\Gamma_i)$$

with the following property:
If $\tilde{\Omega} \subset \mathbb{R}^N$ is an open Lipschitz-domain with $\tilde{\Omega} \subset \Omega$, $\Gamma_i \subset \partial\tilde{\Omega}$ and $\vec{n}_i$ the exterior normal vector, then for all $u \in H^1(\Omega)$ with $u\big|_{\tilde{\Omega}} \in \mathcal{D}(\overline{\tilde{\Omega}})$ the classical normal derivative and $\frac{\partial}{\partial \vec{n}_i}$ coincide. Furthermore the following Green's formula is valid for all $u \in E(\Omega)$, $v \in H^1(\Omega)$ with $v\big|_{(\Gamma_i,\vec{n}_i)} \in \tilde{H}^{\frac{1}{2}}(\Gamma_i)$ for all i:

$$\int_\Omega \triangle u v \ \mathrm{d}x + \int_\Omega \nabla u \nabla v \ \mathrm{d}x = \sum_{\Gamma_i} \langle \frac{\partial u}{\partial \vec{n}_i}, v\rangle_{\Gamma_i}, \tag{2.1}$$

where $\langle\cdot,\cdot\rangle_{\Gamma_i}$ is the dual pairing $\langle\cdot,\cdot\rangle_{(H^{-\frac{1}{2}}(\Gamma_i),\tilde{H}^{\frac{1}{2}}(\Gamma_i))}$.

Remark 2.1. An analogous Green's formula holds for composites.

2.3 Boundary transmission problems

We now introduce the variational formulation of boundary transmission problems for the Laplace and Lamé operators. Let $\Omega \subset \mathbb{R}^N$ be a bounded composite. In order to describe admissible Dirichlet data we define the following space for $l = 1, 2, \ldots$:

$$\begin{aligned} B^{l-\frac{1}{2}} := \Big\{ F_D : \ F_D = \big(&u_1\big|_{\gamma^D_{1,1}}, \ldots, u_1\big|_{\gamma^D_{1,n(1)}}, u_2\big|_{\gamma^D_{2,1}}, \ldots, u_M\big|_{\gamma^D_{M,n(M)}}, \ldots, \\ &\ldots, u_i\big|_{\gamma_{ij,k}} - u_j\big|_{\gamma_{ij,k}}\big) : \ u_i \in H^l(\Omega_i) \cap E(\Omega_i) \Big\}, \end{aligned}$$

which is a subspace of $\prod_{(\gamma_{i,l},\vec{n}_{i,l})\in\mathcal{D}} H^{l-\frac{1}{2}}(\gamma_{i,k}) \times \prod_{\gamma_{ij,k}\in\mathcal{G}} H^{l-\frac{1}{2}}(\gamma_{ij,k})$. By $u_i\big|_{\gamma^D_{i,j}}$ we mean the restriction of u_i to $\gamma^D_{i,j}$ if $\gamma^D_{i,j}$ is part of the Dirichlet boundary.

Remark 2.2. If $F \in B^{\frac{1}{2}}$ then there are satisfied some compatibility conditions between the data on the Dirichlet boundaries and the Dirichlet data on the interfaces. These conditions were studied in [6, 24].

For the right hand sides we assume condition **(D)**

(D)
$$f_i \in \big(H^1(\Omega_i)\big)'; \; g_{i,k}^D \in H^{\frac{1}{2}}(\gamma_{i,k}) \text{ for } \gamma_{i,k} \in \mathcal{D}; \; h_{ij,k}^D \in H^{\frac{1}{2}}(\gamma_{ij,k}) \text{ for } \gamma_{ij,k} \in \mathcal{G};$$
$$g_{i,k}^N \in H^{-\frac{1}{2}}(\gamma_{i,k}) \text{ for } \gamma_{i,k} \in \mathcal{N}; \; h_{ij,k}^N \in H^{-\frac{1}{2}}(\gamma_{ij,k}) \text{ for } \gamma_{ij,k} \in \mathcal{G}.$$
For the Dirichlet data $g_{i,k}^D$ and $h_{ij,k}^D$ we further assume:
$$F_D := (\dots, g_{i,k}^D, \dots, h_{ij,k}^D, \dots) \in B^{\frac{1}{2}}.$$

Finally
$$V := \big\{u \in H^1(\Omega) : \; \forall \gamma \in \mathcal{D} : u\big|_\gamma = 0; \forall \gamma \in \mathcal{N} : u\big|_\gamma \in \tilde{H}^{\frac{1}{2}}(\gamma); \forall \gamma_{ij} \in \mathcal{G} : \; u_i\big|_{\gamma_{ij}}, u_j\big|_{\gamma_{ij}} \in \tilde{H}^{\frac{1}{2}}(\gamma_{ij})\big\}.$$

2.3.1 Laplace operator

We are now ready to define the boundary transmission problem for the Laplace operator. Thereby we reduce problems with nonhomogeneous Dirichlet data by a standard procedure to problems with homogeneous Dirichlet data.

Definition 2.2 (Variational solution). *Let $\Omega \subset \mathbb{R}^N$ be a composite, $\mu_1, \dots, \mu_M \in \mathbb{R}$, let the data satisfy* **(D)**. *$u \in \mathcal{H}^1(\Omega)$ is a variational solution of the boundary transmission problem for the Laplace operator if there exists $w \in V$ such that $u = w + \hat{g}$ where $\hat{g} \in \mathcal{E}(\Omega)$ satisfies the Dirichlet conditions (i.e. $\forall \gamma_{i,l} \in \mathcal{D} : \hat{g}\big|_{\gamma_{i,l}} = g_{i,l}^D$, $\forall \gamma_{ij,k} \in \mathcal{G} : \hat{g}_i\big|_{\gamma_{ij,k}} - \hat{g}_j\big|_{\gamma_{ij,k}} = h_{ij,k}^D$) and w is a solution of*

$$\begin{aligned} a(w,v) \;\; = \;\; & \sum_{i=1}^M \langle f_i, v\rangle_{((H^1(\Omega_i))', H^1(\Omega_i))} + \sum_{\gamma_{i,j}^N \in \mathcal{N}} \langle g_{i,j}^N, v\rangle_{\gamma_{i,j}^N} + \sum_{\gamma_{ij,k} \in \mathcal{G}} \langle h_{ij,k}^N, v\rangle_{\gamma_{ij,k}} \\ & + \sum_{i=1}^M \mu_i \int_{\Omega_i} \triangle \hat{g}_i \overline{v}\, dx - \sum_{i=1}^M \mu_i \langle \frac{\partial \hat{g}_i}{\partial \vec{n}_i}, v\rangle_{\partial\Omega_i} \qquad \textit{for all } v \in V. \end{aligned} \tag{2.2}$$

Here,
$$\langle \frac{\partial \hat{g}_i}{\partial \vec{n}_i}, v\rangle_{\partial\Omega_i} := \sum_{\substack{(\gamma_{i,k}, \vec{n}_{i,k}) \in \mathcal{N}, \\ \gamma_{i,k} \subset \partial\Omega_i}} \langle \frac{\partial \hat{g}_i}{\partial \vec{n}_{i,k}}, v\rangle_{\gamma_{i,k}} + \sum_{\substack{\gamma_{ij,k} \in \mathcal{G}, \\ \gamma_{ij,k} \subset \partial\Omega_i}} \langle \frac{\partial \hat{g}_i}{\partial \vec{n}_{ij,k}}, v\rangle_{\gamma_{ij,k}}$$

and
$$a(u,v) = \sum_{i=1}^M \mu_i \int_{\Omega_i} \nabla u_i \, \nabla v_i \, \mathrm{d}x. \tag{2.3}$$

Lemma 2.1. *If condition* **(D)** *holds and if in addition $f_i \in L^2(\Omega_i)$ for all $1 \leqslant i \leqslant M$, then a variational solution u is in $\mathcal{E}(\Omega)$ and solves*

$$-\mu_i \triangle u_i = f_i \qquad \textit{in } \Omega_i, \tag{2.4}$$
$$u_i\big|_{\gamma_{i,k}} = g_{i,k}^D, \qquad \gamma_{i,k} \in \mathcal{D}, \tag{2.5}$$
$$\frac{\partial u_i}{\partial \vec{n}_{i,k}}\Big|_{\gamma_{i,k}} = g_{i,k}^N, \qquad \gamma_{i,k} \in \mathcal{N}, \tag{2.6}$$
$$u_i\big|_{\gamma_{ij,k}} - u_j\big|_{\gamma_{ij,k}} = h_{ij,k}^D, \qquad \gamma_{ij,k} \in \mathcal{G}, \tag{2.7}$$
$$\mu_i \frac{\partial u_i}{\partial \vec{n}_{ij,k}}\Big|_{\gamma_{ij,k}} + \mu_j \frac{\partial u_j}{\partial \vec{n}_{ji,k}}\Big|_{\gamma_{ij,k}} = h_{ij,k}^N, \qquad \gamma_{ij,k} \in \mathcal{G}, \tag{2.8}$$

Proof. Let u be a variational solution. Then $w := u - \hat{g} \in V$ and satisfies (2.2) for all $v \in V$, in particular for all $v_i \in \mathcal{D}(\mathbb{R}^N)$ with $\operatorname{supp} v_i \subset \Omega_i$. Using Green's formula (2.1) we obtain:

$$-\int_{\Omega_i} \mu_i(u_i - \hat{g}_i)\triangle v_i \,\mathrm{d}x = \int_{\Omega_i} (f_i + \mu_i \triangle \hat{g}_i) v_i \,\mathrm{d}x$$

for all $v_i \in \mathcal{D}(\Omega_i)$. Thus $-\mu_i\triangle(u_i - \hat{g}_i) = f_i + \mu_i\triangle\hat{g}_i$ in the distributional sense and due to the assumptions on $f_i, \hat{g}_i$ we may conclude that $u_i \in E(\Omega_i)$ with $-\mu_i\triangle u_i = f_i$, finally $u \in \mathcal{E}(\Omega)$.
Now let $(\gamma_{i,l}, \vec{n}_{i,l}) \in \mathcal{N}$. For all $v_i \in W(\Omega_i, \gamma_{i,l}) := \{v \in H^1(\Omega_i) : \operatorname{supp}(v\big|_{\partial\Omega_i}) \subset \overline{\gamma_{i,l}}\}$ there holds (Green's formula):

$$\int_{\Omega_i} \nabla(u_i - \hat{g}_i)\nabla v_i \ \mathrm{d}x = -\int_{\Omega_i} \triangle(u_i - \hat{g}_i) v_i \ \mathrm{d}x + \langle \frac{\partial(u_i - \hat{g}_i)}{\partial \vec{n}_{i,l}}, v_i\rangle_{\gamma_{i,l}},$$

which leads to

$$\forall\, v_i \in W(\Omega_i, \gamma_{i,l}) : \qquad \langle \frac{\partial(u_i - \hat{g}_i)}{\partial \vec{n}_{i,l}}, v_i\rangle_{\gamma_{i,l}} = \langle g_{i,l}^N - \frac{\partial \hat{g}_i}{\partial \vec{n}_{i,l}}, v_i\rangle_{\gamma_{i,l}}$$

and finally (with Thm. 2.1)

$$\frac{\partial(u_i - \hat{g}_i)}{\partial \vec{n}_{i,l}} = g_{i,l}^N - \frac{\partial \hat{g}_i}{\partial \vec{n}_{i,l}} \ \text{ in } H^{-\frac{1}{2}}(\gamma_{i,l}).$$

Analogous considerations show the validity of (2.7) and (2.8). □

In the sequel we assume $\mu_i > 0$ for all i. Problem (2.4)-(2.8) then describes an elliptic boundary transmission problem (Def. see [23, 15]).
One can prove existence and uniqueness of variational solutions in the usual way using the Lemma of Lax/Milgram. Note that the Poincaré/Friedrichs inequality is valid on generalized polyhedrons (One can prove this inequality using embedding theorems for Sobolev spaces. These theorems are true for generalized polyhedrons, [17]).

2.3.2 Lamé operator

Before we formulate the boundary value problem for the Lamé operator we introduce some notation:
By $u : \Omega \to \mathbb{R}^N$ we denote the displacement field, $\lambda, \mu \in \mathbb{R}$ are the Lamé constants. The stress tensor for linear elastic isotropic and homogeneous materials is given via Hooke's law by $\sigma(u) = \lambda \operatorname{tr}\varepsilon(u) + 2\mu\varepsilon(u)$, where $\varepsilon(u) = \frac{1}{2}\big(\nabla u + (\nabla u)^T\big)$ is the linearized strain tensor. For quadratic matrices A, B we denote by $A : B = \operatorname{tr}(A^T B)$ the inner product.
In order to define the normal stresses on the boundary we introduce for $\Omega \subset \mathbb{R}^N$ (open domain)

$$E_{\text{Lamé}}(\Omega) := \{u \in H^1(\Omega) : \operatorname{div}\sigma(u) \in L^2(\Omega)\},$$
$$\mathcal{E}_{\text{Lamé}}(\Omega) := \{u \in \mathcal{H}^1(\Omega) : u\big|_{\Omega_i} \in E_{\text{Lamé}}(\Omega_i)\} \quad \text{if } \Omega \text{ is a composite.}$$

The same arguments as for the Laplace operator show

Theorem 2.3 (Normal stresses). *Let $\Omega \subset \mathbb{R}^n$ be a generalized polyhedron, $(\Gamma_i, \vec{n}_i)$ as in sect. 2.1.1. Then there exists a unique linear, continuous operator*

$$T_i : E_{\textit{Lamé}}(\Omega) \longrightarrow H^{-\frac{1}{2}}(\Gamma_i)$$

with the following property:
If $\tilde{\Omega} \subset \mathbb{R}^N$ is an open Lipschitz-domain with $\tilde{\Omega} \subset \Omega$, $\Gamma_i \subset \partial\tilde{\Omega}$ and $\vec{n}_i$ the exterior normal vector, then for

all $u \in H^1(\Omega)$ with $u\big|_{\tilde\Omega} \in \mathcal{D}(\overline{\tilde\Omega})$ the classical normal stress $\sigma(u)\vec{n}_i\big|_{\Gamma_i}$ and $T_i(u)$ coincide. Furthermore the following Green's formula is valid for all $u \in E_{Lamé}(\Omega)$, $v \in H^1(\Omega)$ with $v\big|_{(\Gamma_i,\vec{n}_i)} \in \tilde{H}^{\frac{1}{2}}(\Gamma_i)$ for all i:

$$\int_\Omega \operatorname{div}(\sigma(u))v \ \mathrm{d}x + \int_\Omega \sigma(u) : \varepsilon(v) \ \mathrm{d}x = \sum_{\Gamma_i} \langle \sigma(u)\vec{n}_i, v\rangle_{\Gamma_i}.$$

Definition 2.3 (Variational solution). *Let $\Omega \subset \mathbb{R}^N$ be a composite and the data satisfy* **(D)**. *$u \in \mathcal{H}^1(\Omega)$ is a variational solution of the boundary transmission problem for the Lamé operator if there exists $w \in V$ such that $u = w + \hat{g}$ where $\hat{g} \in \mathcal{E}_{Lamé}(\Omega)$ satisfies the Dirichlet conditions and w is a solution of*

$$\begin{aligned} a(w,v) \ = \ & \sum_{i=1}^{M} \langle f_i, v\rangle_{((H^1(\Omega_i))',H^1(\Omega_i))} + \sum_{\gamma_{i,l}\in\mathcal{N}} \langle g^N_{i,k}, v\rangle_{\gamma_{i,l}} + \sum_{\gamma_{ij,k}\in\mathcal{G}} \langle h^N_{ij,k}, v\rangle_{\gamma_{ij,k}} \\ & + \sum_{i=1}^{M} \int_{\Omega_i} \big(\mu_i \triangle \hat{g}_i + (\lambda_i+\mu_i)\operatorname{grad}\operatorname{div}\hat{g}_i\big)v \,\mathrm{d}x \\ & - \sum_{i=1}^{M} \langle \sigma_i(\hat{g}_i)\vec{n}_i, v\rangle_{\partial\Omega_i} \qquad\qquad \textit{for all } v \in V. \end{aligned}$$

Here, $a(\cdot,\cdot)$ is the bilinear form

$$a(u,v) = \sum_{i=1}^{M} \int_{\Omega_i} \sigma_i(u_i) : \varepsilon(v_i)\,\mathrm{d}x, \tag{2.9}$$

the pairings $\langle\cdot,\cdot\rangle_{\gamma_i}$ have the same meaning as in Definition 2.2.

If condition **(D)** holds and if in addition $f_i \in L^2(\Omega_i)$ for all i then a variational solution u is in $\mathcal{E}_{\text{Lamé}}(\Omega)$ and solves

$$-\big(\mu_i \triangle u_i + (\lambda_i+\mu_i)\operatorname{grad}\operatorname{div} u_i\big) = f_i \qquad \text{in } \Omega_i, \tag{2.10}$$

$$u_i\big|_{\gamma_{i,k}} = g^D_{i,k}, \qquad \gamma_{i,k} \in \mathcal{D}, \tag{2.11}$$

$$\sigma_i(u_i)\vec{n}_{i,k} = g^N_{i,k}, \qquad \gamma_{i,k} \in \mathcal{N}, \tag{2.12}$$

$$u_i\big|_{\gamma_{ij,k}} - u_j\big|_{\gamma_{ij,k}} = h^D_{ij,k}, \qquad \gamma_{ij,k} \in \mathcal{G}, \tag{2.13}$$

$$\sigma_i(u_i)\vec{n}_{ij,k} + \sigma_j(u_j)\vec{n}_{ji,k} = h^N_{ij,k}, \qquad \gamma_{ij,k} \in \mathcal{G}, \tag{2.14}$$

A sufficient condition for the ellipticity of this boundary transmission problem is $\mu_i > 0$, $\lambda_i + \mu_i > 0$ for all i, [27], which we assume in the sequel.
Existence and uniqueness of variational solutions can be proved using the Lax/Milgram Lemma. Note that Korn's inequality is valid on generalized polyhedrons which are the union of a finite number of disjoint generalized polyhedrons with Lipschitz boundary.

3 Regularity and asymptotic expansion of weak solutions

The regularity of weak solutions is mainly influenced by the presence of geometric singularities such as edges, corners, crossing points. The asymptotic expansion of a solution in a neighborhood of these geometric singularities can be described with the help of eigenvalues and eigenfunctions of operator bundles which are related to model problems for edges or corners.

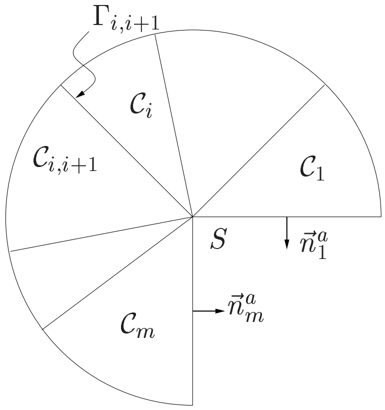


Figure 4: Model problem, 2D

3.1 Two dimensional domains

Let $\Omega \subset \mathbb{R}^2$ be a composite. For a corner point S let $\Omega_1, \dots, \Omega_m$ be those subdomains of Ω which meet in S. By $\mathcal{C}_i$ we denote the infinite cone with tip in S which coincides with Ω_i in a neighborhood of S. Let the numbering be such that we have in polar coordinates: $\mathcal{C}_i = \{x : 0 < r, \phi_{i-1} < \varphi < \phi_i\}$ where $\phi_0 < \ldots < \phi_m \leqslant \phi_0 + 2\pi$, $\mathcal{C}^S := \{x : \phi_0 < \varphi < \phi_m\}$, fig. 4. The model problem for the Laplace or Lamé operator in the cone $\mathcal{C}^S$ reads now:

$$\mathcal{A}u = f, \quad x \in \mathcal{C}^S,$$

where $\mathcal{A}$ is given by (2.4)–(2.8) resp. (2.10)–(2.14). Rewriting the model problem in polar coordinates and applying the Mellin transform, $\mathcal{M}[g](\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^\infty r^{-\alpha-1} g(r)\, dr$, $(r\partial_r) \to \alpha$, we get the following nonlinear eigenvalue problem: Find $v \neq 0$ and $\alpha \in \mathbb{C}$ such that

$$\mathcal{A}(\alpha) v(\alpha, \varphi) = 0, \quad \varphi \in (\phi_0, \phi_m).$$

Example 3.1. For the transmission problem of the Laplace operator the corresponding eigenvalue problem reads: Find $\alpha \in \mathbb{C}, v \neq 0$ such that:

$$\begin{aligned}
-\mu_i\left(\alpha^2 v_i + v_i''\right) &= 0 \quad \phi_{i-1} < \varphi < \phi_i,\ 1 \leqslant i \leqslant m, && (3.1)\\
v_{i+1}(\phi_i) - v_i(\phi_i) &= 0 \quad 1 \leqslant i \leqslant m-1 && (3.2)\\
\mu_{i+1} v_{i+1}'(\phi_i) - \mu_i v_i'(\phi_i) &= 0 \quad 1 \leqslant i \leqslant m-1, && (3.3)\\
v_1(0) = v_m(\phi_m) &= 0 \quad \text{for Dirichlet conditions}, && (3.4)\\
v_1'(0) = v_m'(\phi_m) &= 0 \quad \text{for Neumann conditions}, && (3.5)\\
v_1(0) = v_m'(\phi_m) &= 0 \quad \text{for mixed conditons}. && (3.6)
\end{aligned}$$

In the case of an interior crossing point S we have to replace the boundary conditions by transmission conditions for $\varphi = \phi_m$. Since the parameters μ_i are supposed to be positive, the operator corresponding to the eigenvalue problem is elliptic with parameter, for the definition see for example [1, 21]. The eigenvalue problem for the Lamé operator is given in [13]. The corresponding operator is elliptic with parameter as well.

Example 3.2. In figure 5 are plotted the positive eigenvalues for the Neumann problem for the Laplace operator on a domain with a crack ($\phi_1 = \frac{\pi}{4}$, $\phi_2 = \frac{\pi}{4} + \varphi$, $\phi_3 = 2\pi$) for $0 < \varphi < \frac{7\pi}{4}$.

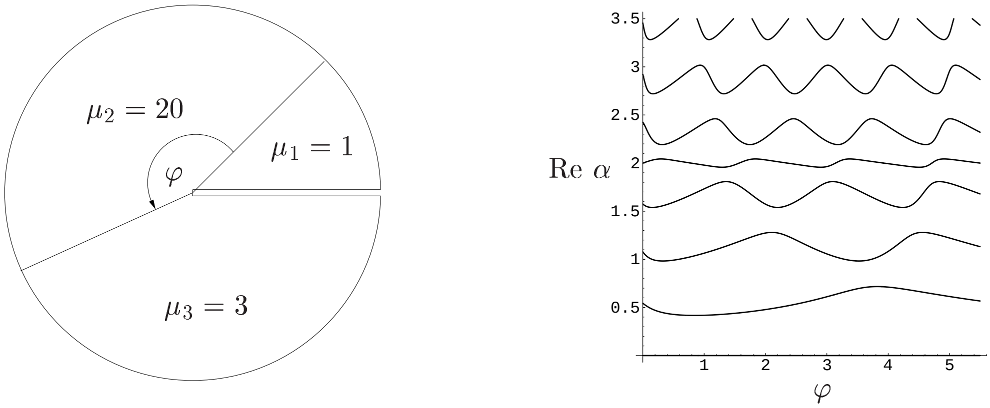


Figure 5: Positive eigenvalues for a Neumann problem

The operators $\mathcal{A}(\alpha)$ have the following property:

Lemma 3.1. *[1, 5] For every $\alpha \in \mathbb{C}$ the operator*

$$\mathcal{A}(\alpha) : \prod_{i=1}^{m} H^2(\phi_{i-1}, \phi_i) \to L^2(\phi_0, \phi_m) \times \mathbb{C}^{2dm},$$

(Laplace: $d = 1$, Lamé: $d = 2$) is Fredholm, the pencil $\{\mathcal{A}(\alpha), \alpha \in \mathbb{C}\}$ is a Fredholm operator pencil (i.e. Fredholm for every $\alpha \in \mathbb{C}$ and invertible for at least one α). The spectrum of $\mathcal{A}(\cdot)$ consists only of eigenvalues which are isolated points in $\mathbb{C}$ and which have no accumulation points in $\mathbb{C}$. Further there exist $\rho, \delta > 0$ such that there are no eigenvalues in the domain $\{\alpha \in \mathbb{C} : \ |\alpha| > \rho, |\mathrm{Re}\ \alpha| < \delta |\mathrm{Im}\ \alpha|\}$, fig. 6.

Let α be an eigenvalue of $\mathcal{A}(\cdot)$. By $\{\Phi_{\alpha,\mu,\kappa}, 1 \leqslant \mu \leqslant I(\alpha), 0 \leqslant \kappa \leqslant M_{\alpha,\mu} - 1\}$ we denote a canonical system of Jordan chains where $\Phi_{\alpha,\mu,0}$ are the eigenfunctions and $\Phi_{\alpha,\mu,\kappa}$, $\kappa > 0$, the associated eigenfunctions; $I(\alpha)$ is the geometric multiplicity of α, $\sum_\mu M_{\alpha,\mu}$ the algebraic multiplicity of α (see e.g. [12]). For $\delta := (\alpha, \mu, \kappa)$ we finally set

$$v_\delta(r, \varphi) = \sum_{q=0}^{\kappa} \frac{(\ln r)^q}{q!} \Phi_{\alpha,\mu,\kappa-q}(\varphi), \quad r > 0,\ \varphi \in (\phi_0, \phi_m). \tag{3.7}$$

With these notations we are ready to describe an asymptotic expansion for a weak solution:

Theorem 3.1. *[3] Let the right hand sides of* (2.4)-(2.8) *resp.* (2.10)-(2.14) *be such that*

$$F_D \in B^{\frac{3}{2}},\ f_k \in L^2(\Omega_k),$$
$$g^N_{i,k} \in H^{\frac{1}{2}}(\gamma_{i,k}) \ \text{for}\ \gamma_{i,k} \in \mathcal{N},$$
$$h^N_{ij,k} \in H^{\frac{1}{2}}(\gamma_{ij,k}) \ \text{for}\ \gamma_{ij,k} \in \mathcal{G}.$$

If for a corner or crossing point S the corresponding operator pencil $\mathcal{A}_S(\alpha)$ has no eigenvalues on the line Re $\alpha = 1$ *(except for $\alpha = 1$ where the geometric and algebraic multiplicities have to coincide), then a weak solution $u \in \mathcal{H}^1(\Omega)$ admits the following asymptotic expansion in a neighborhood of S:*

$$\eta^S u = u_{reg} + \eta^S \sum_{\delta \in \Lambda_S} c_\delta r^\alpha v_\delta(r, \varphi).$$

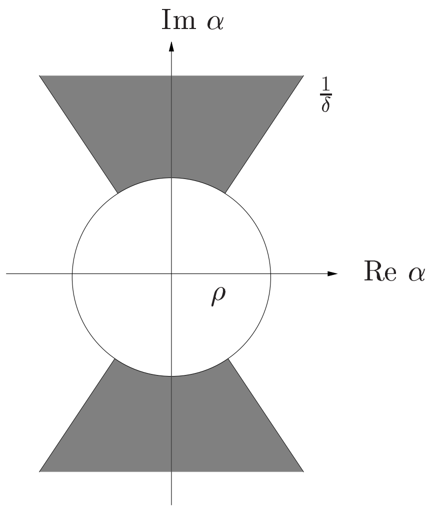


Figure 6: Regions without eigenvalues (grey)

Here $u_{reg}\big|_{\Omega_i} \in H^2(\Omega_i)$. *Further* η^S *is a cut-off function with* $\eta^S \equiv 1$ *in a neighborhood of* S, $\Lambda_S = \{\delta = (\alpha, \mu, \kappa) : \alpha$ *eigenvalue of* $\mathcal{A}(\alpha)$, $0 < \text{Re } \alpha < 1$, $1 \leqslant \mu \leqslant I(\alpha), 0 \leqslant \kappa \leqslant M_{\alpha,\mu}\}$. c_δ *is a constant (stress intensity factor) and* v_δ *are the singular functions given by* (3.7).

3.2 Three dimensional domains

In three dimensional polyhedral domains singularities can arise because of corners and edges. Correspondingly we have to investigate model problems which are defined in a neighborhood of corners and crossing points and model problems which are related to the edges.

3.2.1 Corner singularities

The eigenvalue problem for corners or crossing points can be deduced analogous to the two dimensional case:

Let S be a crossing point, $\Omega_1, \dots, \Omega_m$ the subdomains of Ω which contain S. Let $\mathcal{K}_i$ be the infinite cone with tip in S which coincides with Ω_i in a neighborhood of S; $\mathcal{K}^S$ is the cone which coincides with Ω in a neighborhood of S. Note, that if S is an interior crossing point then $\mathcal{K}^S = \mathbb{R}^3$. We further denote by $G_i := \mathcal{K}_i \cap S^2$, $G := \mathcal{K}^S \cap S^2$ the intersections of $\mathcal{K}_i$ resp. $\mathcal{K}$ with the unit sphere S^2; $\gamma'_{i,l} := \gamma_{i,l} \cap S^2$ for $\gamma_{i,l} \in \mathcal{F}$, resp. $\gamma'_{ij,l} := \gamma_{ij,l} \cap S^2$ for $\gamma_{ij,l} \in \mathcal{G}$. The exterior parts of the boundary of G are divided in Dirichlet (Γ_{Dir}) and Neumann boundaries (Γ_{Neu}) in the same way as the exterior parts of the boundary of $\mathcal{K}^S$. Further we introduce spherical coordinates (ρ, θ, φ) with respect to S. We denote by $\tilde{G}_i$ and $\tilde{G}$, both $\subset [0, \pi] \times [0, 2\pi[$, the regions of the parameters (θ, φ) such that $G_i = \{x \in S^2 : r = 1, (\theta, \varphi) \in \tilde{G}_i\}$ and $G = \{x \in S^2 : r = 1, (\theta, \varphi) \in \tilde{G}\}$. For the definition of the eigenvalue problem which corresponds to corner S we set

$$\tilde{V} := \{u \in H^1(G) : u\big|_{\Gamma_{\text{Dir}}} = 0\}.$$

This space is equipped with the norm

$$\|u\|_{\tilde{V}}^2 := \int_{\tilde{G}} |u|^2 \, \mathrm{d}\omega + \int_{\tilde{G}} \left| \frac{1}{\sin\theta} \partial_\varphi u \right|^2 + |\partial_\theta u|^2 \, \mathrm{d}\omega, \qquad \mathrm{d}\omega = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi.$$

For the Laplace operator the eigenvalue problem reads: Find $\alpha \in \mathbb{C}$, $u \in \tilde{V}\backslash\{0\}$ such that

$$\begin{aligned} a_c(\alpha, u, v) \quad &:= \quad (\alpha^2 + \alpha) \sum_i \int_{\tilde{G}_i} \mu_i u_i \overline{v_i} \mathrm{d}\omega \\ &\quad - \sum_i \mu_i \int_{\tilde{G}_i} \big(\frac{1}{\sin^2 \theta} \partial_\varphi u_i \partial_\varphi \overline{v_i} + \partial_\theta u_i \partial_\theta \overline{v_i} \big) \mathrm{d}\omega = 0 \qquad \text{for all } v \in \tilde{V}. \end{aligned} \tag{3.8}$$

Before we introduce the eigenvalue problem of the Lamé operator we first give some abbreviations([16]):

$$A := \begin{pmatrix} \cos\varphi \sin\theta \\ \sin\varphi \sin\theta \\ \cos\theta \end{pmatrix} B := \begin{pmatrix} \cos\varphi \cos\theta \\ \sin\varphi \cos\theta \\ -\sin\theta \end{pmatrix} C := \begin{pmatrix} -\sin\varphi / \sin\theta \\ \cos\varphi / \sin\theta \\ 0 \end{pmatrix}$$

Further we define the following bilinear forms on $\tilde{V} \times \tilde{V} \to C$:

$$a(u, v) := \sum_{s=1}^{m} \sum_{i,j,k,h=1}^{3} \int_{\tilde{G}_s} a^{(s)}_{ijkh} A_j A_h u_i \overline{v}_k \mathrm{d}\omega \qquad \text{for all } u, v \in \tilde{V}, \tag{3.9}$$

$$b(u, v) := \sum_{s=1}^{m} \sum_{i,j,k,h=1}^{3} \int_{\tilde{G}_s} a^{(s)}_{ijkh} (B_j A_h \frac{\partial u_i}{\partial \theta} \overline{v}_k + C_j A_h \frac{\partial u_i}{\partial \varphi} \overline{v}_k) \mathrm{d}\omega \qquad \text{for all } u, v \in \tilde{V}, \tag{3.10}$$

$$c(u, v) := \sum_{s=1}^{m} \sum_{i,j,k,h=1}^{3} \int_{\tilde{G}_s} a^{(s)}_{ijkh} (B_j B_h \frac{\partial u_i}{\partial \theta} \frac{\partial \overline{v}_k}{\partial \theta} + B_j C_h \frac{\partial u_i}{\partial \theta} \frac{\partial \overline{v}_k}{\partial \varphi} + \tag{3.11}$$

$$+ C_j B_h \frac{\partial u_i}{\partial \varphi} \frac{\partial \overline{v}_k}{\partial \theta} + C_j C_h \frac{\partial u_i}{\partial \varphi} \frac{\partial \overline{v}_k}{\partial \varphi}) \mathrm{d}\omega \qquad \text{for all } u, v \in \tilde{V}. \tag{3.12}$$

Here, u_i denotes the i-th component of the vector u (do not confuse with $u\big|_{\Omega_i}$). Further $a^{(s)}_{ijkh} = 2\mu_s \delta_{ki} \delta_{jh} + \lambda_s \delta_{kh} \delta_{ii}$ are the elastic stiffness coefficients for the Lamé operator on subdomain G_s.
The eigenvalue problem for the Lamé operator reads: Find $\alpha \in \mathbb{C}$, $u \in \tilde{V}\backslash\{0\}$ such that

$$a_c(\alpha; u, v) := -\alpha(\alpha + 1) a(u, v) - (\alpha + 1) b(u, v) + \alpha \overline{b(v, u)} + c(u, v) = 0 \quad \text{for all } v \in \tilde{V}. \tag{3.13}$$

The bilinear forms (3.8) and (3.13) can formally be achieved in the following way: Inserting the ansatz $U = \rho^\alpha u(\theta, \varphi)$ into equations (2.4)-(2.8) resp. (2.10)-(2.14) on the cone $\mathcal{K}^s$ with zero right hand sides results in an eigenvalue problem for α and u. After multiplication by $\overline{v}(\theta, \varphi)$ and integration by parts, one obtains equation (3.8) resp. (3.13).
Since the bilinear forms (3.8) and (3.13) are continuous there exists for every $\alpha \in \mathbb{C}$ a unique continuous linear operator $\mathcal{A}_c(\alpha) : \tilde{V} \to \tilde{V}'$ such that

$$\langle \mathcal{A}_c(\alpha) u, v \rangle_{(\tilde{V}', \tilde{V})} = a_c(\alpha, u, v) \quad \text{for all } u, v \in \tilde{V}.$$

Lemma 3.2. *[3, 13] The bilinear form a_c for the Laplace resp. the Lamé operator has the following properties:*

1. *$\forall \beta < \delta \in \mathbb{R}$ $\exists c_{\beta,\delta}, \Lambda_{\beta,\delta} > 0$, such that $\forall \alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \in [\beta, \delta]$ and $|\alpha| > \Lambda_{\beta,\delta}$ there holds:*

$$\|u\|^2_{H^1((G),\alpha)} \leqslant c_{\beta,\delta} \operatorname{Re}\ a_c(\alpha, u, u) \quad \forall u \in \tilde{V}. \tag{3.14}$$

Here we have set $\|u\|^2_{H^1((G),\alpha)} = |\alpha|^2 \|u\|^2_{L^2(G)} + \|u\|^2_{H^1(G)}$.

2. *The operator pencil* $\{\mathcal{A}_c(\alpha), \alpha \in \mathbb{C}\}$ *is a Fredholm operator pencil. The spectrum consists only of eigenvalues of* $\mathcal{A}_c(\cdot)$. *These eigenvalues have finite algebraic multiplicity. There is only a finite number of eigenvalues in any strip of the form* $\mathrm{Re}\ \alpha \in [\beta, \delta]$.

3. *If* α_0 *is an eigenvalue, then the same is true for* $-1-\overline{\alpha_0}$ *and the geometric and algebraic multiplicities of both coincide.*

Proof. Estimate (3.14) can be shown for transmission problems in the same way as in the proof of Prop. 8.4 in [3]. We only need the coercitivity of the bilinear form a in (2.3) resp. (2.9). For the Fredholm property of $\mathcal{A}_c$ we follow the arguments in [3]: According to the first statement of this lemma there exists $\alpha_0 \in \mathbb{C}$ such that $\mathcal{A}_c(\alpha_0) : \tilde{V} \to \tilde{V}'$ is invertible. For any $\alpha \in \mathbb{C}$ there holds: $\mathcal{A}_c(\alpha) = \mathcal{A}_c(\alpha_0) + (\mathcal{A}_c(\alpha) - \mathcal{A}_c(\alpha_0))$. We now prove that $\mathcal{A}_c(\alpha) - \mathcal{A}_c(\alpha_0) : \tilde{V} \to \tilde{V}'$ is compact. As a consequence, $\mathcal{A}_c(\alpha)$ then is Fredholm.
We begin with the Laplace operator: Let $(u_n)_{n\in\mathbb{N}} \subset \tilde{V}$ be bounded. Since the embedding $\tilde{V} \to L^2(G)$ is compact, there exists $u^* \in L^2(G)$ and a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ for which $u_{n_k} \to u^*$ in $L^2(\Omega)$. Therefore

$$\begin{aligned}\|(\mathcal{A}_c(\alpha) - \mathcal{A}_c(\alpha_0))(u_{n_k} - u_{n_l})\|_{\tilde{V}'} &= \sup_{\|v\|_{\tilde{V}}=1} |a_c(\alpha, u_{n_k} - u_{n_l}, v) - a_c(\alpha_0, u_{n_k} - u_{n_l}, v)| \\ &\leqslant c(\alpha, \alpha_0)\, \|u_{n_k} - u_{n_l}\|_{L^2(G)}\,,\end{aligned}$$

thus $(\mathcal{A}_c(\alpha) - \mathcal{A}_c(\alpha_0))(u_{n_k})$ converges in $\tilde{V}'$.
For the Lamé operator we prove with the same arguments as for the Laplace operator that the operators $M : \tilde{V} \to \tilde{V}'$, defined by $\langle Mu, v\rangle_{(\tilde{V}', \tilde{V})} = a(u, v)$ (from equation (3.9)), and $T : \tilde{V} \to \tilde{V}'$, defined by $\langle Tu, v\rangle_{(\tilde{V}', \tilde{V})} = \overline{b(v, u)}$, are compact operators. With similar arguments to those in the proof of Schauder's Theorem (compact operators and adjoint operators, [28, Satz III.4.4]) one then proves, that the operator $B : \tilde{V} \to \tilde{V}'$, which is defined by $\langle Bu, v\rangle_{(\tilde{V}, \tilde{V}')} = b(u, v)$, is also compact. Thus $\mathcal{A}_c(\alpha) - \mathcal{A}_c(\alpha_0)$ is a compact operator.
The properties of the spectrum of Fredholm operator pencils are described in [5].

The eigenvalues of the bundle $\mathcal{A}_c(\alpha)$ which correspond to the Laplace operator are given by $\alpha_{k,\pm} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda_k}$, where $\lambda_k \geqslant 0$ are the eigenvalues of the transmission Laplace-Beltrami operator on $G \subset S^2$. The eigenfunctions of $\alpha_{k,+}$ and $\alpha_{k,-}$ coincide, there are no associated eigenfunctions, see also Lemma 5.1.

For the Lamé operator, assertion 3. is a conclusion of Theorem 1.2.2 in [13] together with the inequality

$$\forall\, u \in \tilde{V}\backslash\{0\},\ \beta \in \mathbb{R} : \quad a_c(-\frac{1}{2} + i\beta, u, u) > 0.$$

This inequality can be shown with similar calculations to those in [13], pp. 108. □

Let α be an eigenvalue of $\mathcal{A}_c$ and $\{\Psi_{\alpha,\mu,\kappa},\ 1 \leqslant \mu \leqslant I(\alpha), 0 \leqslant \kappa \leqslant M_{\alpha,\mu} - 1\}$ a canonical system of Jordan chains. With $\gamma = (\alpha, \mu, \kappa)$ we set

$$w_{\gamma,c}(\rho, \theta, \varphi) = \sum_{q=0}^{\kappa} \frac{(\ln \rho)^q}{q!} \Psi_{\alpha,\mu,\kappa-q}(\theta, \varphi). \tag{3.15}$$

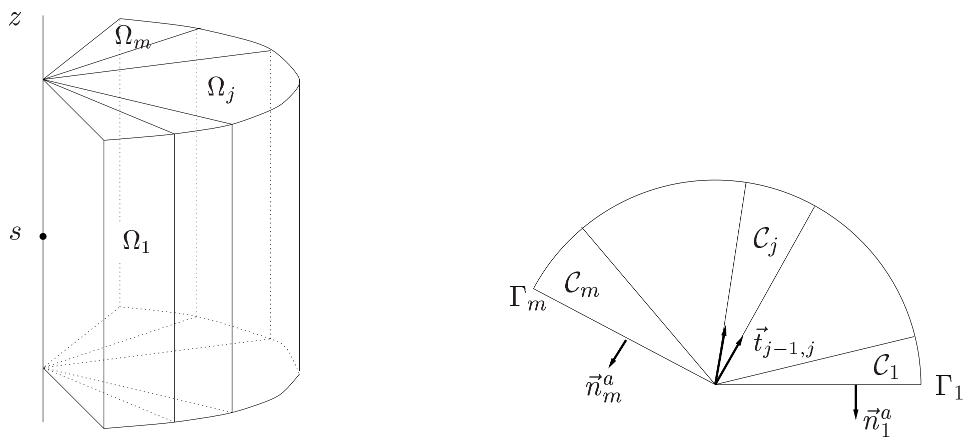


Figure 7: Model problem of a crossing point s on an edge

3.2.2 Edge singularities

For the edge singularities we investigate the following model problem: Let s be a point on an edge far from corners or crossing points. We use local Cartesian coordinates (x, y, z) with origin in s such that the edge is part of the z axis, see fig 7. We denote by $\Omega_1, \ldots, \Omega_m$ those subdomains of Ω which contain s. Let further D_i be the dihedron which coincides with Ω_i in a neighborhood of s. In the (x, y) plane we introduce polar coordinates (r, φ). Then the dihedrons are given by

$$D_i = \{(x, y, z) : (x, y) \in C_{s,i}, z \in \mathbb{R}\},$$

where

$$C_{s,i} = \{(x, y) \in \mathbb{R}^2 : r > 0, \phi_{i-1} < \varphi < \phi_i\}.$$

Here the numbering of the subdomains is such that $\phi_0 < \ldots < \phi_m \leqslant \phi_0 + 2\pi$. Finally $D^s = \{(x, y, z) : r > 0,\ \phi_0 < \varphi < \phi_m,\ z \in \mathbb{R}\}$. Writing the principal part of the operators (2.4)-(2.8) resp. (2.10)-(2.14) in this coordinate system gives the following model problem

$$\mathcal{A}u = f, \quad x \in D^s.$$

Fourier transform of the model problem with respect to z, $\partial_z \to i\xi$, results in a boundary transmission problem with parameter ξ on the two dimensional cone $C_s := \{(x, y) : r > 0,\ \phi_0 < \varphi < \phi_m\}$:

$$\mathcal{A}(\xi, D_x, D_y)\hat{u}(\xi, x, y) = \hat{f}(\xi, x, y), \quad (x, y) \in C_s.$$

This problem can be investigated with the method for two dimensional domains: As a new model problem we consider the principal part $\mathcal{A}^0$ of $\mathcal{A}(\xi, D_x, D_y)$ on the two dimensional cone C_s:

$$\mathcal{A}^0(D_x, D_y)u = f, \quad (x, y) \in C_s.$$

Note that $\mathcal{A}^0$ is independent of ξ.

Example 3.3. We set $\tilde{u}(x, y) := (u_x(x, y), u_y(x, y))$, $(x, y) \in C_{s,i} = \{(x, y) : r > 0,\ \phi_{i-1} < \varphi < \phi_i\}$.

Then for the Lamé operator this model problem reads:

$$-\big(\mu_i\triangle\tilde{u}_i + (\lambda_i+\mu_i)\,\mathrm{grad}\;\mathrm{div}\,(\tilde{u}_i)\big) = 0, \quad (x,y)\in C_{s,i}, \tag{3.16}$$

$$\tilde{u}_i = 0 \quad \text{Dirichlet conditions,} \tag{3.17}$$

$$\sigma_i(\tilde{u}_i)\vec{n}_i = 0 \quad \text{Neumann conditions,} \tag{3.18}$$

$$-\mu_i\triangle u_z = 0, \quad (x,y)\in C_{s,i}, \tag{3.19}$$

$$u_z = 0 \quad \text{Dirichlet conditions,} \tag{3.20}$$

$$\frac{\partial u_z}{\partial \vec{n}_i} = 0 \quad \text{Neumann conditions} \tag{3.21}$$

together with the transmission conditions which are decoupled as well. The equations for $\tilde{u}$ correspond to those for plane strain where $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$.

Polar coordinates (r,φ) for (x,y) and Mellin transform with respect to r lead to an operator bundle $\mathcal{A}_e(\alpha)$ whose eigenvalues and eigenfunctions occur in the asymptotic expansion. The bundle $\mathcal{A}_e$ has the properties given in Lemma 3.1.

Let α be an eigenvalue of $\mathcal{A}_e$, $\{\Phi_{\alpha,\mu,\kappa},\ 1\leqslant\mu\leqslant I(\alpha), 0\leqslant\kappa\leqslant M_{\alpha,\mu}\}$ a canonical system of Jordan chains. For $\delta=(\alpha,\mu,\kappa)$ we set

$$v_{\delta,e}(r,\varphi) = \sum_{q=0}^{\kappa}\frac{(\ln r)^q}{q!}\Phi_{\alpha,\mu,\kappa-q}(\varphi), \quad r>0,\ \varphi\in(\phi_0,\phi_m). \tag{3.22}$$

Remark 3.1. It follows directly from (3.16)-(3.21) that the spectrum of $\mathcal{A}_e$ for the Lamé operator consists of the corner eigenvalues of the two dimensional Lamé operator and those of the two dimensional Laplace operator.

3.2.3 Asymptotic expansion of a weak solution

We are now ready to describe asymptotic expansions of weak solutions.

Theorem 3.2. *[3] Let $\Omega\subset\mathbb{R}^3$ be a composite. For the right hand sides we assume*

$$F_D\in B^{\frac{3}{2}},\ f_k\in L^2(\Omega_k),$$
$$g^N_{i,k}\in H^{\frac{1}{2}}(\gamma_{i,k}) \quad \textit{for } \gamma_{i,k}\in\mathcal{N},$$
$$h^N_{ij,k}\in H^{\frac{1}{2}}(\gamma_{ij,k}) \quad \textit{for } \gamma_{ij,k}\in\mathcal{G}.$$

Let further $u\in\mathcal{H}^1(\Omega)$ be a solution of (2.4)-(2.8), *resp.* (2.10)-(2.14). *For a crossing point S we denote by η^S a cut off function with $\eta\equiv 1$ in a neighborhood of S. $\mathcal{E}_S$ is the set of all edges crossing at S.*
Let $0<s\leqslant 1$ such that there are no eigenvalues of the corner operator pencil $\mathcal{A}_c(\beta)$ on the line $\mathrm{Re}\,\beta = -\frac{1}{2}+s$ *and no eigenvalues of the pencils $\mathcal{A}_e(\alpha)$, $e\in\mathcal{E}_S$, on the line* $\mathrm{Re}\,\alpha = s$.

Then a weak solution $u\in\mathcal{H}^1(\Omega)$ has the following asymptotic expansion:

$$\eta^S u = u_{reg} + \eta^S\sum_{\gamma\in\Lambda_S} c_{\beta,\mu}\rho^\beta w_{\gamma,c} + \eta^S\sum_{e\in\mathcal{E}_S}\sum_{\delta\in\Lambda_e}\mathcal{R}_e(c_e^{\alpha,\mu})r_e^\alpha v_{\delta,e}. \tag{3.23}$$

Here $u_{reg}\big|_{\Omega_i}\in H^{1+s}(\Omega_i)$. Further $\Lambda_S = \{\gamma=(\beta,\mu,\kappa) : \beta =$ eigenvalue of $\mathcal{A}_c(\beta)$, $\mathrm{Re}\,\beta\in]-\frac{1}{2},-\frac{1}{2}+s[$, $1\leqslant\mu\leqslant I(\beta), 0\leqslant\kappa\leqslant M_{\beta,\mu}\}$, $\Lambda_e = \{\delta=(\alpha,\mu,\kappa) : \alpha =$ eigenvalue of the edge pencil $\mathcal{A}_e(\alpha)$, $\mathrm{Re}\,\alpha\in\,]0,s[$, $1\leqslant\mu\leqslant I(\alpha), 0\leqslant\kappa\leqslant M_{\alpha,\mu}\}$, $e\in\mathcal{E}_S$. $c_{\beta,\mu}$ are constants, $c_e^{\alpha,\mu}\in H^{s-\mathrm{Re}\,\alpha}(\mathbb{R})$ and $\mathcal{R}_e$ is a smoothing operator, see [3, 22]. Finally ρ denotes the distance to the corner S, r_e the distance to edge e. $v_{\delta,e}$ are the singular functions given by (3.22), *$w_{\gamma,c}$ the singular functions in* (3.15).

Remark 3.2. The eigenfunctions and associated functions $\Psi_{\alpha,\mu,\kappa}$ of the corner pencil $\mathcal{A}_c$ are solutions of elliptic boundary transmission problems on non smooth two dimensional domains. These functions can also be splitted into singular functions and a smooth remainder.

Remark 3.3. If S is a point on an edge and no corner or crossing point then the sum $\sum_{\gamma\in\Lambda_S}$ vanishes in the expansion (3.23).

Corollary 3.1. *If for all corners and edges of Ω there holds that there are no eigenvalues of the corresponding operator pencils in the strip* $\mathrm{Re}\,\beta \in]-\frac{1}{2},\varepsilon]$ *for corners and* $\mathrm{Re}\,\alpha \in]0,\frac{1}{2}+\varepsilon]$ *for edges with a small $\varepsilon > 0$, then we have for a weak solution with right hand sides as in Theorem 3.2: $u\big|_{\Omega_i} \in H^{\frac{3}{2}+\varepsilon}(\Omega_i)$. The Sobolev embedding theorem then shows $u\big|_{\Omega_i} \in \mathcal{C}(\overline{\Omega_i})$.*

In chapter 5 we will deduce sufficient conditions on the geometry and parameters which guarantee that the strips in Corollary 3.1 are free of eigenvalues. For this we need a homotopy argument which will be presented in the next section.

4 A homotopy method

The homotopy method is used to carry over estimates for the eigenvalues of “easy” problems to more complicate problems. To do that we use a version of Rouché's Theorem for analytic operator valued functions. Further we need Lemma 3.1 and Lemma 3.2 which describe domains in the complex plane where no eigenvalues of the pencils $\mathcal{A}_c$ or $\mathcal{A}_e$ are situated.

Let $\mathcal{D} \subset \mathbb{C}$ be a domain, i.e. open and connected. We consider analytic Fredholm operator pencils $\mathcal{A} : \mathcal{D} \to \mathcal{L}(X,Y)$, X, Y Banach spaces. Let $\Gamma \subset \mathcal{D}$ be a closed simply connected curve, piecewise smooth, where the surrounded domain Q is contained in $\mathcal{D}$. The algebraic multiplicity $m(\Gamma,\mathcal{A})$ of $\mathcal{A}$ with respect to the contour Γ is

$$m(\Gamma,\mathcal{A}) := \sum_{\alpha_0} m(\alpha_0,\mathcal{A}).$$

The sum extends over all eigenvalues α_0 in the interior of the enclosed domain, $m(\alpha_0,\mathcal{A})$ denotes the algebraic multiplicity of the eigenvalue α_0. Since the pencil $\mathcal{A}$ is supposed to be a Fredholm pencil, $m(\Gamma,\mathcal{A})$ is finite, [5].

Theorem 4.1 (Rouché's Theorem). *[5, Thm 9.2] Let H be a separable Hilbert space, $\mathcal{D} \subset \mathbb{C}$ open and connected and $\gamma \subset \mathcal{D}$ a simply connected piecewise smooth curve where the corresponding included domain Q is contained in $\mathcal{D}$. Let further $S_1, S_2 : \mathcal{D} \to \mathcal{L}(H)$ be analytic operator valued functions such that S_1 is normal with respect to Γ, that means:*

1. *$S_1(\alpha)$ is invertible for all $\alpha \in \Gamma$,*
2. *$S_1(\alpha)$ is a Fredholm operator for all $\alpha \in Q$.*

If for every $\alpha \in \Gamma$

$$\left\| S_1(\alpha)^{-1} S_2(\alpha) \right\| < 1$$

in the operator norm, then $S_1 + S_2$ is also normal with respect to Γ and the algebraic multiplicities of S_1 and $S_1 + S_2$ coincide.
If in addition $S_1(\alpha)$ is invertible for all $\alpha \in Q$ then the same is true for $S_1 + S_2$.

As a corollary we get

Theorem 4.2. *Let H_1, H_2 be separable Hilbert spaces, $\Gamma \subset \mathbb{C}$ a simply connected piecewise smooth curve with corresponding interior domain Q.*
For $t \in [0,1]$ we consider a family of analytic Fredholm operator pencils of the form

$$\mathcal{A}_t : \mathbb{C} \to \mathcal{L}(H_1, H_2) : \alpha \to \mathcal{A}_t(\alpha) = (1-t)\mathcal{A}_0(\alpha) + t\mathcal{A}_1(\alpha)$$

where $\mathcal{A}_0, \mathcal{A}_1 : \mathbb{C} \to \mathcal{L}(H_1, H_2)$ are analytic Fredholm operator pencils. We assume for all $t \in [0,1]$ that the pencils $\mathcal{A}_t(\alpha)$ are invertible for all $\alpha \in \Gamma$ except a finite number of $\alpha_i \in \Gamma, 1 \leqslant i \leqslant l$. For these α_i we assume:
α_i is an eigenvalue of $\mathcal{A}_t(\cdot)$ for all $t \in [0,1]$ with algebraic multiplicity $m(\alpha_i)$ which is independent of t.
Then the operators $\mathcal{A}_t(\cdot)$, $t \in [0,1]$, have the same algebraic multiplicities with respect to Γ.
If there exists $t_0 \in [0,1]$ such that $\mathcal{A}_{t_0}(\alpha)$ is invertible for every $\alpha \in Q$ then this is true for all $\mathcal{A}_t$, $t \in [0,1]$.

Proof (sketch). [9] In a first step one shows the Theorem for the case $H_1 = H_2$ and when no eigenvalues are situated on the contour Γ. To do that one proves the following assertion:
For every $t_0 \in [0,1]$ exists a $\delta(t_0) > 0$ such that for all $t \in K(t_0) := \{t \in [0,1] : |t - t_0| < \delta(t_0)\}$ and for all $\alpha \in \Gamma$: $\left\|\mathcal{A}_{t_0}(\alpha)^{-1}(\mathcal{A}_t(\alpha) - \mathcal{A}_{t_0}(\alpha)\right\| < 1$. Here, $\|\cdot\|$ denotes the operator norm.
Applying Rouchés Theorem to $S_1 := \mathcal{A}_{t_0}$, $S_2 := \mathcal{A}_t - \mathcal{A}_{t_0}$, $t \in K(t_0)$, it follows that the algebraic multiplicity with respect to Γ of $\mathcal{A}_t$ is constant for $t \in K(t_0)$ and finally also for $t \in [0,1]$.
In the second step one shows that due to the assumptions on the eigenvalues on Γ there exists a small neighborhood U_i for every α_i which contains no further eigenvalues of $\mathcal{A}_t(\cdot), t \in [0,1]$. We define a new curve $\tilde{\Gamma}$ which coincides with Γ outside U_i and which is contained in $Q \cap U_i$ near the eigenvalues in such a way that $\tilde{\Gamma}$ does not contain eigenvalues of the $\mathcal{A}_t$. The proof for the case $H_1 = H_2$ is finished by applying the results of the first step to the contour $\tilde{\Gamma}$. The case $H_1 \neq H_2$ is a simple consequence. □

With the help of Lemma 3.1 and Lemma 3.2 one can extend Theorem 4.2 for operator pencils $\mathcal{A}_c$, $\mathcal{A}_e$ of section 3.2.2 to infinite strips of the form $\{\alpha \in \mathbb{C} : \text{ Re } \alpha \in [a,b]\}$, $-\infty < a < b < \infty$. To show that a given operator pencil $\mathcal{A}_e$ or A_c has no eigenvalues in the strip Re $\alpha \in]a,b[$ it suffices to find the eigenvalues and their algebraic multiplicities on the lines Re $\alpha \in \{a,b\}$ and to construct an operator family $\mathcal{A}_t$ as in Theorem 4.2 where $\mathcal{A}_0 = \mathcal{A}_c$ resp. $\mathcal{A}_0 = \mathcal{A}_e$ and where the distribution of the eigenvalues is known for $t = 1$. This method will be used in the next section to derive new estimates for the eigenvalues.

5 Estimates of the eigenvalues

In this chapter we first give an overview over existing estimates for the width of the strips in Theorems 3.1 and 3.2. The main goal is to describe a class of boundary transmission problems for which the strip Re $\alpha \in]0, \frac{1}{2}]$ in the two-dimensional case and the strips Re $\beta_c \in]-\frac{1}{2}, 0]$ for corners and Re $\alpha_e \in]0, \frac{1}{2}]$ for edges in the three dimensional case are free of eigenvalues. Here a quasi-monotone distribution of the material parameters is an essential assumption which leads together with some geometric conditions to the estimates.

5.1 Two dimensional domains

For transmission problems for the Laplace operator on two dimensional domains there exist many estimates for the smallest positive eigenvalue of the operator pencil given by (3.1)-(3.5). General estimates without any restriction on the geometry or the parameters were developed by Costabel/Dauge/Nicaise in [2] and by Kühn, [14], who also developed recursion formulas (with respect to the number of subdomains) for equations whose roots are the eigenvalues. It turns out that there are no general bounds for the smallest positive eigenvalue in the case of more than two subdomains with pure Dirichlet (DD) or Neumann (NN) conditions or more than three subdomains in the case of interior crossing points. This is illustrated in the next example.

Example 5.1. Kellogg's example [8] shows that it is possible to get arbitrary small positive eigenvalues for the Laplace operator. Consider a domain as in fig. 8 with parameters $\mu_1 = \mu_3 = 1$, $\mu_2 = \mu_4 = h$. The eigenvalues which correspond to the interior crossing point are given by $\alpha = 2k$, $k \in \mathbb{Z}$, and the solutions of the equation $\cos(\alpha\pi) = 1 - \frac{8h}{(1+h)^2}$. For $h \to 0$ or $h \to \infty$ the smallest positive eigenvalue tends to 0, see also fig 8.

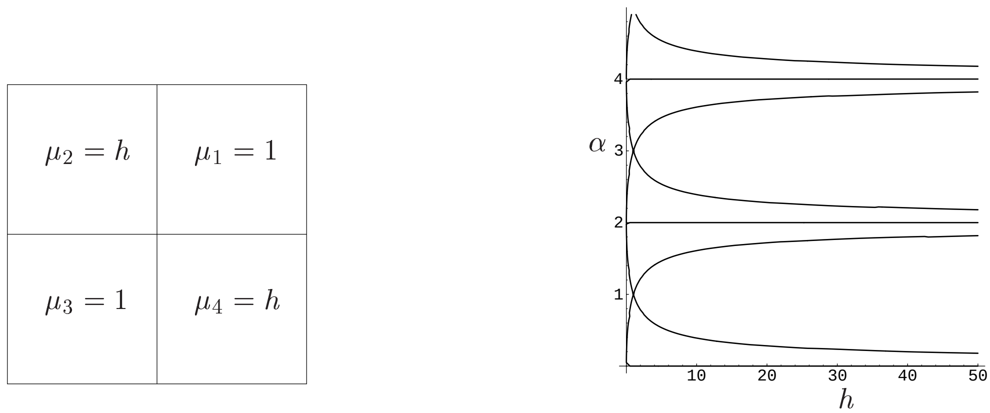


Figure 8: Kellogg's example: Domain and eigenvalues

Under the assumption of a quasi-monotone distribution of the material parameters (Def. 5.1 below) one can prove further estimates for the Laplace operator for an arbitrary number of subdomains. This was done in [25] without any additional condition on the geometry and in [9] under an additional geometrical assumption. Further estimates were derived in [19] where the author studies the influence of the opening angles of the subdomains in detail. The estimates for the 2D Laplace operator are collected in table 1.

For the Lamé operator, numerical examples indicate that analogous estimates are valid. While for one subdomain the position of the eigenvalues is well known, the situation is more difficult for several subdomains. In this paper we introduce a generalized quasi-monotonicity condition for pairs (λ_i, μ_i). Under this assumption and an additional geometric condition, which is exactly the same as in the case of the Laplace operator, we prove estimates for an arbitrary number of subdomains. These results are a generalization of those in [24], where estimates for two subdomains are given. The estimates for the Lamé operator are collected in table 2.
Estimates for arbitrary elliptic problems on one (n dimensional) subdomain with conical points and pure Dirichlet or Neumann conditions are developed in [11].

5.1.1 Quasi-monotonicity and geometric conditions

Let $\Omega \subset \mathbb{R}^2$ be a composite (see sect. 1). For a corner or crossing point S let $\Omega_1, \ldots, \Omega_m$ be those subdomains which contain S. $\mathcal{C}_i$ denotes the cone with vertex in S which coincides with Ω_i in a neighborhood of

S. Let the numbering of the subdomains be such that in polar coordinates $\mathcal{C}_i = \{x : 0 < r, \phi_{i-1} < \varphi < \phi_i\}$ where $\phi_0 < \ldots < \phi_m \leqslant \phi_0 + 2\pi$, $\mathcal{C}^S := \{x : \phi_0 < \varphi < \phi_m\}$. Finally we set $\Gamma_{ij} = \partial\mathcal{C}_i \cap \partial\mathcal{C}_j$, fig. 4. Quasi-monotonicity was first defined for transmission problems for the Laplace operator, [4]. We now generalize the definition to pairs (λ_i, μ_i) of Lamé constants. Here and in the sequel we denote by *DD* the case of pure Dirichlet conditions on $\mathcal{C}^S$, *NN* the case of pure Neumann conditions and by *DN* the case of mixed boundary conditions.

Definition 5.1 (Quasi-monotonicity). *[4]*

1. *If S is an exterior crossing point with pure Neumann conditions or mixed conditions, or if S is an interior crossing point, then the distribution of the parameters $\mu_1, \ldots, \mu_m$, $\lambda_1, \ldots, \lambda_m$ is quasi-monotone with respect to the crossing point S if there exists a unique $j \in \{1, \ldots, m\}$ such that*

$$\mu_1 \leqslant \ldots \leqslant \mu_j \geqslant \ldots \geqslant \mu_m \quad \textit{and} \quad \lambda_1 \leqslant \ldots \leqslant \lambda_j \geqslant \ldots \geqslant \lambda_m.$$

   In the case of mixed boundary conditions we require that Ω_j is the subdomain at the Dirichlet boundary.

2. *If S is an exterior crossing point with pure Dirichlet conditions, then the distribution of the parameters $\mu_1, \ldots, \mu_m$, $\lambda_1, \ldots, \lambda_m$ is quasi-monotone with respect to the crossing point S if there exists a unique $j \in \{1, \ldots, m\}$ such that*

$$\mu_1 \geqslant \ldots \geqslant \mu_j \leqslant \ldots \leqslant \mu_m \quad \textit{and} \quad \lambda_1 \geqslant \ldots \geqslant \lambda_j \leqslant \ldots \leqslant \lambda_m.$$

For the Laplace operator ignore the λ's and replace $\leqslant$ by $<$ resp. $\geqslant$ by $>$.

Example 5.2. In the case of the Laplace-operator and an interior crossing point where three subdomains meet, the parameters μ_i are always quasi-monotone. The parameters in Kellogg's example are not quasi-monotone.

The following geometric conditions are closely related to the definition of quasi-monotonicity:

GC 1 Let S be an exterior crossing point with pure Dirichlet conditions (DD) or pure Neumann conditions (NN). Let the parameters $\mu_1, \ldots, \mu_m, \lambda_1, \ldots, \lambda_m$ be quasi-monotone with j as that index for which we have the maximum in the case NN resp. the minimum in the case DD. The condition reads

$$\exists \vec{t} \in \mathbb{R}^2 \backslash \{0\} : \vec{t} \in \mathcal{C}_j \text{ and } \vec{n}_1^a \vec{t} < 0,\ \vec{n}_m^a \vec{t} < 0.$$

Here we denote by $\vec{n}_1^a$, $\vec{n}_m^a$ the exterior normal vectors on $\mathcal{C}_1$ resp. $\mathcal{C}_m$, see figure 9.

GC 2 Let S be an interior crossing point; let the parameters $\mu_1, \ldots, \mu_m, \lambda_1, \ldots, \lambda_m$ be quasi-monotone, j the index where we have the maximum and k the index where we have the minimum. Then the condition is

$$\exists \vec{t} \in \mathbb{R}^2 \backslash \{0\} : \vec{t} \in \mathcal{C}_j \text{ and } -\vec{t} \in \mathcal{C}_k.$$

Remark 5.1. If condition **GC 1** holds for an exterior crossing point S then the opening angle of the domain is $< 2\pi$ in a neighborhood of S, so S is not a crack tip.

Figure 9: Examples for **GC1**, **GC2**

5.1.2 Estimates for the Laplace-operator, 2D

It is easy to verify that the eigenvalues of the operator bundles which describe the corner singularities for the Laplace-operator on two-dimensional domains are real and that there are no associated eigenfunctions for non vanishing eigenvalues, [9]. Thus the asymptotic expansion of a solution u in $\mathcal{H}^1(\Omega)$ has the following simple form:

$$\eta^S u = u_{\text{reg}} + \eta^S \sum_{0<\alpha<1} \sum_{\mu=1}^{I(\alpha)} c_{\alpha,\mu} r^\alpha v_{\alpha,\mu}$$

with the same notation as in Theorem 3.1. $v_{\alpha,\mu}$ are the eigenfunctions to the eigenvalues α, there are no logarithmic terms.
In table 1 estimates of the smallest positive eigenvalue are listed. We use the following notations: m is the number of subdomains which meet at crossing point S; Φ is the opening angle of the whole domain at S, $0 < \Phi \leqslant 2\pi$; Φ_i' is the opening angle of subdomain $\mathcal{C}_i$. We assume $\Phi_i' > 0$ and have $\sum_i \Phi_i' = \Phi$. Further $\tilde{\Phi} = \max_{1\leqslant i\leqslant m}\{\Phi_i'\}$ is the maximal opening angle of the subdomains. By α_0 we denote the smallest positive eigenvalue.

The estimates in the last row can be proved by a homotopy method. Since the arguments are the same as in the 3D case we omit the details here and refer to the proof of Theorem 5.1.
In contrast to the case of one subdomain we get the estimate $\alpha_0 > \frac{1}{2}$ in the case DN for two or more subdomains also for $\Phi_m = \pi$. The reason is that one can prove the estimate for two subdomains directly. For more than two subdomains, the proof is based on a homotopy argument (see the proof of Theorem 5.1) which carries over the known estimate for two subdomains to the case of $m > 2$ subdomains.
The estimates are sharp in the sense that for every estimate in table 1 there exists an example for which the smallest eigenvalue is arbitrarily close to the bound given there. If the conditions in table 1 are violated, there are examples for which the smallest positive eigenvalue is lower than the bounds given in the table.
Further estimates which take into account the parameters and opening angles more precisely were derived in [14] for two subdomains and in [19] for an arbitrary number of subdomains. For the special case of an interior crossing point, which is the intersection of two straight lines, Petzoldt [25] proved $\alpha > \frac{1}{2}$ under the only assumption that the parameters are distributed quasimonotonely.

Table 1: Estimates for the Laplace operator, 2D

	DD or NN	interior points	DN
$m = 1$	$\alpha_0 = \frac{\pi}{\Phi} \geqslant \frac{1}{2}$		$\alpha_0 = \frac{\pi}{2\Phi} \geqslant \frac{1}{4}$
$m = 2$			quasi-mon., $\Phi \leqslant \pi$:
	$\alpha_0 \geqslant \frac{\pi}{2\tilde{\Phi}} > \frac{\pi}{2\Phi} \geqslant \frac{1}{4}$	$\alpha_0 > \frac{1}{2}$	$\alpha_0 > \frac{1}{2}$
	(Costabel/Dauge/Nicaise [2])		[9]
$m = 3$	—	$\alpha_0 > \frac{\pi}{2\tilde{\Phi}} \geqslant \frac{1}{4}$	—
$m \geqslant 3$	quasi-monotonicity: (Petzoldt [25])		
	$\alpha_0 > \frac{1}{4}$	$\alpha_0 > \frac{1}{4}$	$\alpha_0 > \frac{1}{4}$
$m \geqslant 3$	quasi-monotonicity + geom. cond. **GC 1,2**, [9]:		quasi-mon., $\Phi \leqslant \pi$, [9]:
	$\alpha_0 > \frac{1}{2}$	$\alpha_0 > \frac{1}{2}$	$\alpha_0 > \frac{1}{2}$

5.1.3 Estimates for the Lamé-operator, 2D

In contrast to the Laplace-operator the eigenvalues of the operator bundles related to the Lamé-operator can also be non real and associated eigenfunctions can exist. Thus the asymptotic expansion in Theorem 3.1 cannot be simplified in general. The estimates are collected in table 2, where we use the following notations: m is the number of subdomains which meet at crossing point S. Φ is the opening angle of the whole domain at S, $0 < \Phi \leqslant 2\pi$ and α_0 is an eigenvalue with smallest positive real part.

The estimates for $m \geqslant 2$ subdomains can be shown by a homotopy method which uses the same arguments as in the proof of Theorem 5.1 for 3D corner singularities, thus we omit the details here.

5.2 Three dimensional domains

The singularities for 3D polyhedral domains can be divided into edge and corner singularities, see Theorem 3.2. The eigenvalues of the operator bundles which are related to the edges are in the case of the Laplace-operator completely given by the eigenvalues of the corner bundles of the Laplace-operator in the corresponding two-dimensional domain. In the case of the Lamé-operator the eigenvalues are given by those for the two dimensional Lamé-operator and those of the two dimensional Laplace-operator on the corresponding 2D domain, see Remark 3.1. Therefore, we consider only corner singularities in this section.

5.2.1 Laplace-operator, 3D

The following properties of the eigenvalues of the corner bundles for the 3D Laplace are well known:

Table 2: Estimates for the Lamé operator, 2D

	DD	NN	interior points	DN
$m = 1$				[22]
$0 < \Phi < \pi$	$\mathrm{Re}\ \alpha_0 > 1$	$\mathrm{Re}\ \alpha_0 > 1$		$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$
$\Phi = \pi$	$\alpha_0 = 1$	$\alpha_0 = 1$		$\mathrm{Re}\ \alpha_0 = \frac{1}{2}$
$\pi < \Phi < 2\pi$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$		$\mathrm{Re}\ \alpha_0 > \frac{1}{4}$
$\Phi = 2\pi$	$\alpha_0 = \frac{1}{2}$	$\alpha_0 = \frac{1}{2}$		$\mathrm{Re}\ \alpha_0 = \frac{1}{4}$
$m = 2$, [24]		$(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) \geqslant 0$,		quasi-monotonicity
	GC 1:	**GC 1:**		$\Phi < \pi$:
	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$
$m \geqslant 3$, [9]	quasi-monotonicity			
	GC 1:	**GC 1:**	**GC 2:**	$\Phi < \pi$:
	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$	$\mathrm{Re}\ \alpha_0 > \frac{1}{2}$

Lemma 5.1. *The eigenvalues of the bundle given by* (3.8) *are real, there are no associated eigenfunctions. In the cases DD and DN there are no eigenvalues in* $[-1, 0]$*. In the case NN and in the case of interior crossing points there are no eigenvalues in the interval* $(-1, 0)$*.* -1 *and* 0 *are eigenvalues with geometric multiplicity=algebraic multiplicity=1. The eigenfunctions to the eigenvalue* $\alpha = 0$ *are the constant functions.*

Proof. The Lemma is a direct consequence of the properties of the eigenvalues of the Laplace-Beltrami operator. □

Corollary 5.1. *Let* $u \in \mathcal{H}^1(\Omega)$ *be a solution of* (2.4)-(2.8) *with right hand sides as in Theorem 3.2. Then there exists* $\varepsilon > 0$ *such that*

$$\eta^S u = u_{reg} + \eta^S \sum_{e \in \mathcal{E}_S} \sum_{\substack{\alpha_e \in (0, \frac{1}{2}+\varepsilon) \\ 1 \leqslant \mu \leqslant I(\alpha_e)}} \mathcal{R}_e(c_e^{\alpha_e,\mu}) r^{\alpha_e} \Phi_{e;\alpha,\mu}, \tag{5.1}$$

where $u_{reg}\big|_{\Omega_i} \in H^{\frac{3}{2}+\varepsilon}(\Omega_i)$*,* $\Phi_{e;\alpha,\mu}$ *is eigenfunction to the eigenvalue* α_e *of the operator bundle corresponding to edge e, there are no logarithmic terms.*

5.3 Lamé-operator, 3D

For the Lamé-operator there is no result like Lemma 5.1. Estimates of the real parts of the eigenvalues are only possible for problems under assumptions which are a generalization of the quasi-monotonicity and the geometric conditions in section 5.1. It is still an open question whether there exist examples where the corner bundles have eigenvalues in the strip Re $\alpha \in (-\frac{1}{2}, 0)$, which would result in unbounded deformation fields.

5.3.1 Estimates for one subdomain

Dirichlet-problem, [18]: Let $S^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ and $G \subset S^2$ be a domain. For the Dirichlet-problem on cones of the form $\mathcal{C} = \{x \in \mathbb{R}^3 : \frac{x}{|x|} \in G\}$ there holds: The strip Re $\alpha \in [-1, 0]$ does not contain any eigenvalues of the corresponding corner operator bundle.
Note that the domains described by $\mathcal{C}$ can also be non-Lipschitz domains.

Neumann-problem, [11]: Let φ be a positively homogeneous function of degree one, piecewise smooth in $\mathbb{R}^2 \backslash \{0\}$. Consider cones of the form $\mathcal{C} = \{x \in \mathbb{R}^3 : x_3 = \varphi(x_1, x_2)\}$. The corresponding corner pencil has no eigenvalues in the strip Re $\alpha \in (-1, 0)$. $\alpha = 0$ and $\alpha = -1$ are the only eigenvalues on the lines Re $\alpha = -1$ resp. Re $\alpha = 0$. The geometric and algebraic multiplicities coincide and equal to 3. The eigenfunctions for $\alpha = 0$ are given by the constant functions.

Mixed problem, [22]: For the mixed problem we need further restrictions on the geometry. If **G 1** holds (see below), then the corner bundle has no eigenvalues in the strip Re $\alpha \in [-1, 0]$.
If for example $\mathcal{C}$ is convex and $|\mathcal{D}| = 1$ or $|\mathcal{N}| = 1$ then the geometric condition is satisfied.
To describe **G 1** we introduce the following notation: Let $\mathcal{C}$ be a polyhedral cone $\subset \mathbb{R}^3$ with vertex in 0. We assume that the boundary of $\mathcal{C}$ can be divided in the following way into plane oriented faces $(\gamma_i, \vec{n}_i)$:

$$\partial\mathcal{C} = \bigcup_{i=1}^{n} \overline{\gamma_k},\ \gamma_i \cap \gamma_j = \emptyset \text{ for } i \neq j.$$

$\vec{n}_i$ is the exterior normal vector on $\mathcal{C}$ with respect to γ_i. We set $\mathcal{F} = \{(\gamma_i, \vec{n}_i),\ 1 \leqslant i \leqslant n\} = \mathcal{D} \cup \mathcal{N}$, $\mathcal{D}$ and $\mathcal{N}$ disjoint and not empty. We further define the following index sets: $\tilde{\mathcal{D}} := \{k \in \{1, \ldots, n\} : (\gamma_k, \vec{n}_k) \in \mathcal{D}\}$, $\tilde{\mathcal{N}} := \{k \in \{1, \ldots, n\} : (\gamma_k, \vec{n}_k) \in \mathcal{N}\}$. $\cup_{k \in \tilde{\mathcal{D}}} \gamma_k$ is the Dirichlet boundary, $\cup_{k \in \tilde{\mathcal{N}}} \gamma_k$ the Neumann boundary. Finally

$$\mathcal{C}_{\mathcal{D}} := \left\{ x \in \mathbb{R}^3 : x = \sum_{k \in \tilde{\mathcal{D}}} \lambda_k \vec{n}_k,\ \lambda_k \geqslant 0,\ \sum_{k \in \tilde{\mathcal{D}}} \lambda_k > 0 \right\},$$

$$\mathcal{C}_{\mathcal{N}} := \left\{ x \in \mathbb{R}^3 : x = \sum_{k \in \tilde{\mathcal{N}}} \lambda_k \vec{n}_k,\ \lambda_k \geqslant 0,\ \sum_{k \in \tilde{\mathcal{N}}} \lambda_k > 0 \right\}.$$

G 1: $\mathcal{C}$ has no cracks and $\mathcal{C}_{\mathcal{D}} \cap \mathcal{C}_{\mathcal{N}} = \emptyset$.

It is an open question whether the estimates in the cases NN and DN still hold when the geometric conditions are violated.

5.3.2 Estimates for $m \geqslant 2$ subdomains

The estimates in this section we prove only for domains which have no cracks, thus we introduce a slightly simplified notation in comparison to section 3:
Let $\mathcal{C} \subset \mathbb{R}^3$ be a polyhedral cone with vertex in 0 and $\overline{\mathcal{C}} = \cup_{i=1}^{m} \overline{\mathcal{C}_i}$, where $\mathcal{C}_i$ are polyhedral cones

with vertex in 0, pairwise disjoint. We assume that neither $\mathcal{C}$ nor $\mathcal{C}_i, 1 \leqslant i \leqslant m$, have cracks; $\mathcal{C}$ and $\mathcal{C}_i$ need not have Lipschitz-boundaries. We further assume that if $\mathrm{mes}_{N-1}(\partial\mathcal{C} \cap \partial\mathcal{C}_i) \neq 0$, then there exist plane faces $\gamma_{i,l}$, pairwise disjoint, such that $\mathrm{mes}_{N-1}(\bigcup_{l=1}^{n(i)} \overline{\gamma_{i,l}}) = \mathrm{mes}_{N-1}(\partial\mathcal{C}_i \cap \partial\mathcal{C})$. We set $\mathcal{F} = \{\gamma_{i,l},\, 1 \leqslant i \leqslant m\} = \mathcal{D} \cup \mathcal{N}$. The interface between $\mathcal{C}_i, \mathcal{C}_j$ is divided into plane pieces $\gamma_{ij,l}$ such that $\gamma_{ij,l} \subset \partial\mathcal{C}_i \cup \partial\mathcal{C}_j$, and we set $\mathcal{G} := \{\gamma_{ij,l},\, 1 \leqslant i, j \leqslant m\}$.

If conversely $\gamma \in \mathcal{G}$ then there exist $\mathcal{C}_i, \mathcal{C}_j$ with γ as a part of the common interface. We set $\mathcal{C}_1(\gamma) := \mathcal{C}_i$, $\mathcal{C}_2(\gamma) := \mathcal{C}_j$ and $\mu_1^\gamma := \mu_i$, $\mu_2^\gamma := \mathcal{C}_j$, (the same for λ). Finally we denote by $\vec{n}_1^\gamma$ the exterior normal vector to γ with respect to $\mathcal{C}_1(\gamma)$; $\vec{u}_1^\gamma := \vec{u}_i\big|_\gamma$ and $\vec{u}_2^\gamma := \vec{u}_j\big|_\gamma$. If $\gamma \in \mathcal{F}$ we denote by $\vec{n}^\gamma$ the exterior normal vector to the cone whose boundary contains γ.
If it is clear to which boundary γ we refer, we omit the index γ. We are now ready to give the main theorem of this work.

Theorem 5.1. *($\geqslant 2$ subdomains, Lamé-operator) We assume that the Lamé constants satisfy $\mu_i > 0$, $\lambda_i + \mu_i > 0$.*

1. *Pure Dirichlet conditions, $\mathcal{F} = \mathcal{D}$: Let $\mathcal{C}$ be an arbitrary polyhedral cone which is divided into subcones $\mathcal{C}_i$. Let further $\vec{t}_1, \vec{t}_2, \vec{t}_3 \in \mathbb{R}^3$ with $\vec{t}_i\vec{t}_j = \delta_{ij}$ such that the following three conditions are satisfied:*

$$\begin{array}{lll} (a) & \forall\gamma \in \mathcal{D}: & \vec{n}^\gamma\vec{t}_1 \leqslant 0, \\ & \forall\gamma \in \mathcal{G}: & \vec{t}_1\vec{n}_1^\gamma(\lambda_1^\gamma - \lambda_2^\gamma) \geqslant 0,\ \vec{t}_1\vec{n}_1^\gamma(\mu_1^\gamma - \mu_2^\gamma) \geqslant 0, \\ (b) & \forall\gamma \in \mathcal{D} \text{ with } \vec{n}^\gamma\vec{t}_1 = 0: & \vec{t}_2\vec{n}^\gamma \leqslant 0, \\ & \forall\gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma\vec{t}_1 = 0: & \vec{t}_2\vec{n}_1^\gamma(\lambda_1^\gamma - \lambda_2^\gamma) \geqslant 0,\ \vec{t}_2\vec{n}_1^\gamma(\mu_1^\gamma - \mu_2^\gamma) \geqslant 0, \\ (c) & \forall\gamma \in \mathcal{D} \text{ with } \vec{n}^\gamma \parallel \vec{t}_3: & \vec{t}_3\vec{n}^\gamma < 0, \\ & \forall\gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma \parallel \vec{t}_3: & \vec{t}_3\vec{n}_1^\gamma(\lambda_1^\gamma - \lambda_2^\gamma) \geqslant 0,\ \vec{t}_3\vec{n}_1^\gamma(\mu_1^\gamma - \mu_2^\gamma) \geqslant 0. \end{array}$$

   Then there are no eigenvalues of the corresponding corner bundle $\mathcal{A}_c$ in the strip $\mathrm{Re}\,\alpha \in [-1, 0]$.

2. *Pure Neumann conditions, $\mathcal{F} = \mathcal{N}$: Let $\mathcal{C}$ be a cone which is given by a function φ as in section 5.3.1 and divided into subcones $\mathcal{C}_i$. Let further $\vec{t}_1, \vec{t}_2, \vec{t}_3 \in \mathbb{R}^3$ with $\vec{t}_i\vec{t}_j = \delta_{ij}$ such that the following three conditions are satisfied:*

$$\begin{array}{lll} (a) & \forall\gamma \in \mathcal{N}: & \vec{n}^\gamma\vec{t}_1 \leqslant 0, \\ & \forall\gamma \in \mathcal{G}: & \vec{t}_1\vec{n}_1^\gamma(\lambda_1^\gamma - \lambda_2^\gamma) \leqslant 0,\ \vec{t}_1\vec{n}_1^\gamma(\mu_1^\gamma - \mu_2^\gamma) \leqslant 0, \\ (b) & \forall\gamma \in \mathcal{N} \text{ with } \vec{n}^\gamma\vec{t}_1 = 0: & \vec{t}_2\vec{n}^\gamma \leqslant 0, \\ & \forall\gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma\vec{t}_1 = 0: & \vec{t}_2\vec{n}_1^\gamma(\lambda_1^\gamma - \lambda_2^\gamma) \leqslant 0,\ \vec{t}_2\vec{n}_1^\gamma(\mu_1^\gamma - \mu_2^\gamma) \leqslant 0, \\ (c) & \forall\gamma \in \mathcal{N} \text{ with } \vec{n}^\gamma \parallel \vec{t}_3: & \vec{t}_3\vec{n}^\gamma < 0, \\ & \forall\gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma \parallel \vec{t}_3: & \vec{t}_3\vec{n}_1^\gamma(\lambda_1^\gamma - \lambda_2^\gamma) \leqslant 0,\ \vec{t}_3\vec{n}_1^\gamma(\mu_1^\gamma - \mu_2^\gamma) \leqslant 0. \end{array}$$

   Then the only eigenvalues of the corner bundle $\mathcal{A}_c$ in the strip $\mathrm{Re}\,\alpha \in [-1, 0]$ *are* -1 *and* 0*. The algebraic and geometric multiplicities coincide and equal to* 3*. The eigenfunctions of $\alpha = 0$ are the constant functions.*

3. *Mixed conditions: Let $\mathcal{C}$ be a cone which satisfies condition* **G1** *in section 5.3.1 and which is divided into subcones $\mathcal{C}_i$. Let further $\vec{t}_1, \vec{t}_2, \vec{t}_3 \in \mathbb{R}^3$ with $\vec{t}_i\vec{t}_j = \delta_{ij}$ such that the following three conditions are satisfied:*

$$\begin{array}{lll}
(a) & \forall \gamma \in \mathcal{D}: & \vec{n}^\gamma \vec{t}_1 \leqslant 0, \\
& \forall \gamma \in \mathcal{N}: & \vec{n}^\gamma \vec{t}_1 \geqslant 0, \\
& \forall \gamma \in \mathcal{G}: & \vec{t}_1 \vec{n}_1^\gamma (\lambda_1^\gamma - \lambda_2^\gamma) \geqslant 0,\ \vec{t}_1 \vec{n}_1^\gamma (\mu_1^\gamma - \mu_2^\gamma) \geqslant 0, \\
(b) & \forall \gamma \in \mathcal{D} \text{ with } \vec{n}^\gamma \vec{t}_1 = 0: & \vec{t}_2 \vec{n}^\gamma \leqslant 0, \\
& \forall \gamma \in \mathcal{N} \text{ with } \vec{n}^\gamma \vec{t}_1 = 0: & \vec{t}_2 \vec{n}^\gamma \geqslant 0, \\
& \forall \gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma \vec{t}_1 = 0: & \vec{t}_2 \vec{n}_1^\gamma (\lambda_1^\gamma - \lambda_2^\gamma) \geqslant 0,\ \vec{t}_2 \vec{n}_1^\gamma (\mu_1^\gamma - \mu_2^\gamma) \geqslant 0, \\
(c) & \forall \gamma \in \mathcal{D} \text{ with } \vec{n}^\gamma \parallel \vec{t}_3: & \vec{t}_3 \vec{n}^\gamma < 0, \\
& \forall \gamma \in \mathcal{N} \text{ with } \vec{n}^\gamma \parallel \vec{t}_3: & \vec{t}_3 \vec{n}^\gamma > 0, \\
& \forall \gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma \parallel \vec{t}_3: & \vec{t}_3 \vec{n}_1^\gamma (\lambda_1^\gamma - \lambda_2^\gamma) \geqslant 0,\ \vec{t}_3 \vec{n}_1^\gamma (\mu_1^\gamma - \mu_2^\gamma) \geqslant 0.
\end{array}$$

Then there are no eigenvalues of the corner bundle $\mathcal{A}_c$ in the strip Re $\alpha \in [-1, 0]$.

4. *Interior crossing points: Let $\mathcal{C} = \mathbb{R}^3$ be divided into polyhedral subcones $\mathcal{C}_i$. If there exist $\vec{t}_1, \vec{t}_2, \vec{t}_3 \in \mathbb{R}^3$ with $\vec{t}_i \vec{t}_j = \delta_{ij}$ such that*

$$\begin{array}{lll}
(a) & \forall \gamma \in \mathcal{G}: & \vec{t}_1 \vec{n}_1^\gamma (\lambda_1^\gamma - \lambda_2^\gamma) \leqslant 0,\ \vec{t}_1 \vec{n}_1^\gamma (\mu_1^\gamma - \mu_2^\gamma) \leqslant 0, \\
(b) & \forall \gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma \vec{t}_1 = 0: & \vec{t}_2 \vec{n}_1^\gamma (\lambda_1^\gamma - \lambda_2^\gamma) \leqslant 0,\ \vec{t}_2 \vec{n}_1^\gamma (\mu_1^\gamma - \mu_2^\gamma) \leqslant 0, \\
(c) & \forall \gamma \in \mathcal{G} \text{ with } \vec{n}_1^\gamma \parallel \vec{t}_3: & \vec{t}_3 \vec{n}_1^\gamma (\lambda_1^\gamma - \lambda_2^\gamma) \leqslant 0,\ \vec{t}_3 \vec{n}_1^\gamma (\mu_1^\gamma - \mu_2^\gamma) \leqslant 0,
\end{array}$$

then the only eigenvalues of the corner bundle $\mathcal{A}_c$ in the strip Re $\alpha \in [-1, 0]$ *are* -1 *and* 0. *The algebraic and geometric multiplicities coincide and equal to* 3. *The eigenfunctions of $\alpha = 0$ are the constant functions.*

The conditions in the cases DD and NN for two subdomains with a plane interface are exactly the conditions in [24]. Therefore the proof of Theorem 5.1 is a generalization of the proof in [24]. Rewriting Theorem 5.1 for two dimensional domains shows that the conditions are satisfied if and only if the parameters of the two dimensional problem are quasi-monotone and if the conditions **GC 1** resp. **GC 2** are satisfied. The conditions in Theorem 5.1 can be seen as a generalized quasi-monotonicity with additional geometric conditions.

Corollary 5.2. *If for every corner the assumptions of Theorem 5.1 are satisfied, then for all edges e, the corresponding edge bundles $\mathcal{A}_e(\beta)$ have no eigenvalues in the strip* Re $\beta \in (0, \frac{1}{2}]$. *Thus $u\big|_{\Omega_i} \in \mathcal{C}(\overline{\Omega}_i)$ (if the data are as in Theorem 3.2).*

Proof. From the assumptions about the corners one can easily derive the assumptions of section 5.1.1 for two dimensional problems. The assertion follows with Remark 3.1. □

The proof of Theorem 5.1 is based on a homotopy argument for an operator family $\mathcal{A}_{c,t}$ which describes for $t = 1$ the given operator pencil A_c on m subdomains and for $t = 0$ an operator pencil on one subdomain where we know the distribution of the eigenvalues. The conditions **G1** in the case DN and the assumption that $\mathcal{C}$ can be described by a function φ in the case NN are required to guarantee that the problems on one subdomain do not have eigenvalues in the strip Re $\alpha \in (-1, 0)$.

Example 5.3. Consider Fichera's corner in fig. 10. If all the boundaries which contain crossing point S are Dirichlet boundaries and if $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ and $\mu_1 \leqslant \mu_2 \leqslant \mu_3$ then the conditions of Theorem 5.1 1. are satisfied. If all boundaries are Neumann boundaries, we have to replace $\leqslant$ by $\geqslant$ to satisfy Theorem 5.1 2. If we consider mixed boundary conditions at S with Dirichlet conditions at the parts of $\partial\Omega_1$ and $\partial\Omega_3$ which contain S, then condition 3. of Theorem 5.1 is satisfied if $\lambda_3 \leqslant \lambda_1 \leqslant \lambda_2$ and $\mu_3 \leqslant \mu_1 \leqslant \mu_2$ (this is the only possible choice in that case).

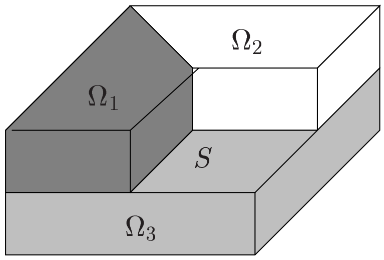


Figure 10: Fichera's corner with 3 subdomains

5.3.3 Proof of Theorem 5.1

The proof is based on a homotopy argument. We begin with the Dirichlet problem. In a first step we prove that there are no eigenvalues of the corresponding operator bundle $\mathcal{A}_c$ on the lines Re $\alpha = -1$ and Re $\alpha = 0$. Due to the symmetry of the eigenvalues (Lemma 3.2) we can restrict ourself to the line Re $\alpha = 0$. In a second step we construct an operator family $\mathcal{A}_{c,t}(\alpha)$ for which we can apply Theorem 4.2.

First step: We prove by contradiction that there are no eigenvalues of the operator-bundle $\mathcal{A}_c(\alpha)$ on the line Re $\alpha = 0$.
Let $\mathcal{C}$ be a polyhedral cone with tip in 0 and properties as in Theorem 5.1.1 (Dirichlet problem). Assume that there is an eigenvalue α with Re $\alpha = 0$ and a corresponding eigenfunction $\vec{v} \neq 0$. Then the function $\vec{u} := \rho^\alpha \vec{v}$ (ρ distance to 0) satisfies for $k \in \{1,2,3\}$:

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \int_{\mathcal{C}_i(\delta)} \partial_k(\lambda_i \, |\mathrm{tr}\, \varepsilon(\vec{u}_i)|^2 + 2\mu_i \, |\varepsilon(\vec{u}_i)|^2) \, dx && (5.2) \\ &- \sum_{i=1}^{m} \int_{\partial\mathcal{C}_i(\delta)} 2\,\mathrm{Re}\, \langle \sigma_i(\vec{u}_i)\vec{n}_i ; \partial_k \vec{u}_i \rangle \, ds. && (5.3) \end{aligned}$$

Here we have set $\mathcal{C}_{i,\delta} := \{\vec{x} \in \mathcal{C}_i \, : \, \delta < |\vec{x}| < 1\}$ for a given $\delta \in (0,1)$. With $\langle \cdot, \cdot \rangle$ we denote the inner product in $\mathbb{C}^3$; for $A, B \in \mathbb{C}^{3\times 3}$ we define $A : B = \mathrm{tr}(\overline{A}^t B)$. Equation (5.2) can be derived by Green's formula and the product rule, taking $\partial_k \vec{u}$ as a test function. Using Gauss Theorem we get:

$$\begin{aligned} 0 &= \sum_{i=1}^{m} \int_{\partial\mathcal{C}_i(\delta)} \big(\lambda_i \, |\mathrm{tr}\, \varepsilon(\vec{u}_i)|^2 + 2\mu_i \, |\varepsilon(\vec{u}_i)|^2 \big) n_k^i \, ds \\ &- \sum_{i=1}^{m} \int_{\partial\mathcal{C}_i(\delta)} 2\,\mathrm{Re}\, \langle \sigma_i(\vec{u}_i)\vec{n}_i ; \partial_k \vec{u}_i \rangle \, ds, && (5.4) \end{aligned}$$

where $\vec{n}_i$ denotes the exterior normal vector on $\mathcal{C}_i(\delta)$. The integrals over the sets $\{x \in \mathcal{C}_i : |x| = 1\}$ and $\{x \in \mathcal{C}_i : |x| = \delta\}$ vanish because of the special form of $\vec{u}$. It remains ($\gamma_\delta = \{x \in \gamma : \delta < |x| < 1\}$):

$$\begin{aligned} 0 &= \sum_{\gamma\in\mathcal{G}} (\vec{n}_1^\gamma)_k \int_{\gamma_\delta} \sigma_1^\gamma(\vec{u}_1^\gamma) : \varepsilon(\vec{u}_1^\gamma) - \sigma_2^\gamma(\vec{u}_2^\gamma) : \varepsilon(\vec{u}_2^\gamma)\, ds \\ &- \sum_{\gamma\in\mathcal{G}} 2\,\mathrm{Re} \int_{\gamma_\delta} \langle \sigma_1^\gamma(\vec{u}_1^\gamma)\vec{n}_1^\gamma; \partial_k u_1^\gamma - \partial_k u_2^\gamma\rangle\, ds \\ &+ \sum_{\gamma\in\mathcal{D}} \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)(\vec{n}_\gamma)_k - 2Re\langle \sigma_\gamma(\vec{u}_\gamma)\vec{n}_\gamma, \partial_k \vec{u}_\gamma\rangle\, ds. \end{aligned} \tag{5.5}$$

Let $\gamma \in \mathcal{D}$. Using the Dirichlet conditions we get $\frac{\partial \vec{u}}{\partial \vec{t}} = 0$ on γ for all $\vec{t}$ with $\vec{t}\vec{n}_\gamma = 0$. Thus we have $\partial_k \vec{u} = (\vec{n}_\gamma)_k \frac{\partial u}{\partial \vec{n}_\gamma}$ and therefore:

$$\langle \sigma_\gamma(\vec{u}_\gamma)\vec{n}_\gamma; \partial_k \vec{u}_\gamma\rangle = (\vec{n}_\gamma)_k \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma). \tag{5.6}$$

<u>Local coordinates on the interfaces:</u> Let $\gamma \in \mathcal{G}$. We introduce a local Cartesian coordinate system which is spanned by $\vec{a}_1, \vec{a}_2, \vec{n}_1^\gamma$ with positive orientation. We set $Q_\gamma := (\vec{a}_1, \vec{a}_2, \vec{n}_1^\gamma)$ and $\nabla_\gamma \vec{u} := \nabla \vec{u} Q$. For $\vec{w} := Q^T \vec{u}$ we further set $E(\vec{w}) := \frac{1}{2}(\nabla_\gamma \vec{w} + (\nabla_\gamma \vec{w})^T)$, $S_i(\vec{w}) := \lambda_i \operatorname{tr} E(\vec{w}) + 2\mu_i E(\vec{w})$. One finally gets the following relations between the original coordinates and the transformed system:

$$\begin{gathered} \varepsilon(\vec{u}) = QE(\vec{w})Q^T,\ \operatorname{tr}\varepsilon(\vec{u}) = \operatorname{tr} E(\vec{w}),\ |\operatorname{tr}\varepsilon(\vec{u})|^2 = |trE(\vec{w})|^2, \\ |\varepsilon(\vec{u})|^2 = |E(\vec{w})|^2,\ \sigma_i(\vec{u}) = QS_i(\vec{w})Q^T,\ \sigma_i(\vec{u}) : \varepsilon(\vec{u}) = S_i(\vec{w}) : E(\vec{w}). \end{gathered}$$

The transmission conditions for $\vec{x} \in \gamma$ are transformed as follows:

$$\vec{u}_i(\vec{x}) - \vec{u}_j(\vec{x}) = 0 \ \Leftrightarrow\ \vec{w}_i(\vec{x}) - \vec{w}_j(\vec{x}) = 0, \tag{5.7}$$

$$\sigma_i(\vec{u}_i)\vec{n}_1^\gamma - \sigma_j(\vec{u}_j)\vec{n}_1^\gamma = 0 \ \Leftrightarrow\ \big(S_i(\vec{w}_i) - S_j(\vec{w}_j)\big)\begin{pmatrix}0\\0\\1\end{pmatrix} = 0. \tag{5.8}$$

Inserting the Dirichlet transmission condition we get

$$\langle \sigma_1^\gamma(\vec{u}_1^\gamma)\vec{n}_1^\gamma; \partial_k u_1^\gamma - \partial_k u_2^\gamma\rangle = (\vec{n}_1^\gamma)_k \langle S_1^\gamma(\vec{w}_1^\gamma)\begin{pmatrix}0\\0\\1\end{pmatrix}; \frac{\partial \vec{w}_1^\gamma}{\partial \vec{n}_1^\gamma} - \frac{\partial \vec{w}_2^\gamma}{\partial \vec{n}_1^\gamma}\rangle, \quad \vec{x} \in \gamma.$$

Inserting the last equation and (5.6) into (5.5) gives:

$$\begin{aligned} 0 &= \sum_{\gamma\in\mathcal{G}} \vec{n}_1^\gamma \int_{\gamma_\delta} \big(S_1^\gamma(\vec{w}_1^\gamma) : E(\vec{w}_1^\gamma) - S_2^\gamma(\vec{w}_2^\gamma) : E(\vec{w}_2^\gamma)\big)\, ds \\ &- \sum_{\gamma\in\mathcal{G}} \vec{n}_1^\gamma 2\,\mathrm{Re} \int_{\gamma_\delta} \langle S_1^\gamma(\vec{w}_1^\gamma)\begin{pmatrix}0\\0\\1\end{pmatrix}; \frac{\partial \vec{w}_1^\gamma}{\partial \vec{n}_1^\gamma} - \frac{\partial \vec{w}_2^\gamma}{\partial \vec{n}_1^\gamma}\rangle\, ds \\ &- \sum_{\gamma\in\mathcal{D}} \vec{n}_\gamma \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\, ds. \end{aligned} \tag{5.9}$$

Using again the transmission conditions on γ we get the relations, see also [23]:

$$
\begin{aligned}
E_{ii}(\vec{w}_1) &= E_{ii}(\vec{w}_2),\ i = 1, 2,\\
\mu_1 E_{i3}(\vec{w}_1) &= \mu_2 E_{i3}(\vec{w}_2),\ i = 1, 2,\\
2\mu_1 E_{33}(\vec{w}_1) + \lambda_1 \operatorname{tr} E(\vec{w}_1) &= 2\mu_2 E_{33}(\vec{w}_2) + \lambda_2 \operatorname{tr} E(\vec{w}_2)\\
(\lambda_2 + 2\mu_2)E_{33}(\vec{w}_2) &= (2\mu_1 + \lambda_1)E_{33}(\vec{w}_1) + (\lambda_1 - \lambda_2)\sum_{i=1}^{2} E_{ii}(\vec{w}_1),
\end{aligned}
$$

After short calculations analogous to those in [23] we get from (5.9) and the above relations:

$$
\begin{aligned}
0 \quad = \quad & \sum_{\gamma\in\mathcal{G}} \vec{n}_1^\gamma \int_{\gamma_\delta} \Big(2(\mu_1-\mu_2)(|E_{11}(\vec{w}_1)|^2 + |E_{22}(\vec{w}_1)|^2 + 2\,|E_{12}(\vec{w}_1)|^2)\\
&\qquad + 4\frac{\mu_1}{\mu_2}(\mu_1-\mu_2)(|E_{13}(\vec{w}_1)|^2 + |E_{23}(\vec{w}_1)|^2)\\
&\qquad + \frac{(\lambda_1-\lambda_2)(\lambda_1+2\mu_2)}{\lambda_2+2\mu_2}\ |\operatorname{tr} E(\vec{w}_1)|^2\\
&\qquad + 4\frac{(\lambda_1-\lambda_2)(\mu_1-\mu_2)}{\lambda_2+2\mu_2}\ \operatorname{Re}\,\big(E_{33}(\vec{w}_1)\overline{\operatorname{tr} E(\vec{w}_1)}\big)\\
&\qquad + 2\frac{(\lambda_2+2\mu_1)(\mu_1-\mu_2)}{\lambda_2+2\mu_2}\ |E_{33}(\vec{w}_1)|^2\ \Big)\,ds\\
& - \sum_{\gamma\in\mathcal{D}} \vec{n}_\gamma \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\,ds\\
=: \quad & \sum_{\gamma\in\mathcal{G}} \vec{n}_1^\gamma \int_{\gamma_\delta} B_1^\gamma(\vec{w}_1^\gamma)\,ds - \sum_{\gamma\in\mathcal{D}} \vec{n}_\gamma \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\,ds. \qquad (5.10)
\end{aligned}
$$

The last equation is the essential equation of this proof. Scalar multiplication of (5.10) with $\vec{t}_1$ of Theorem 5.1 gives:

$$
0 = \sum_{\substack{\gamma\in\mathcal{G}\\ \vec{n}_1^\gamma\vec{t}_1\neq 0}} \vec{n}_1^\gamma\,\vec{t}_1 \int_{\gamma_\delta} B_1^\gamma(\vec{w}_1^\gamma)\,ds - \sum_{\substack{\gamma\in\mathcal{D}\\ \vec{n}^\gamma\vec{t}_1\neq 0}} \vec{n}_\gamma\,\vec{t}_1 \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\,ds. \qquad (5.11)
$$

Assumption 1.(a) of Theorem 5.1 together with Lemma 5.3 (subsequent to this proof) shows

$$
\begin{aligned}
\forall\,\gamma\in\mathcal{G} \text{ with } \vec{n}_1^\gamma\vec{t}_1 \neq 0: &\quad \vec{n}_1^\gamma\,\vec{t}_1 \int_{\gamma_\delta} B_1^\gamma(\vec{w}_1^\gamma)\,ds \geqslant 0,\\
\forall\,\gamma\in\mathcal{D} \text{ with } \vec{n}^\gamma\vec{t}_1 \neq 0: &\quad \vec{n}_\gamma\,\vec{t}_1 \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\,ds \leqslant 0.
\end{aligned}
$$

Thus equation (5.11) is satisfied iff

$$
\forall\,\gamma\in\mathcal{G} \text{ with } \vec{n}_1^\gamma\vec{t}_1 \neq 0: \quad B_1^\gamma(\vec{w}_1^\gamma) = 0, \qquad (5.12)
$$

$$
\forall\,\gamma\in\mathcal{D} \text{ with } \vec{n}^\gamma\vec{t}_1 \neq 0: \quad \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma) = 0. \qquad (5.13)
$$

In the same way we conclude for the remaining boundaries and interfaces. Thus we finally get (5.12), (5.13) for every $\gamma\in\mathcal{D}$ resp. $\gamma\in\mathcal{G}$. Using Lemma 5.3, we get from these equations for all $\gamma\in\mathcal{G}$:
<u>1. Case, $\mu_1^\gamma - \mu_2^\gamma \neq 0$:</u> $E(\vec{w}_1) = 0 = E(\vec{w}_2)$ on γ and therefore $\varepsilon(\vec{u}_1^\gamma) = 0 = \varepsilon(\vec{u}_2^\gamma)$ on γ.

2. Case, $\mu_1^\gamma = \mu_2^\gamma, \lambda_1^\gamma \neq \lambda_2^\gamma$: $\operatorname{tr}\varepsilon(\vec{u}_1^\gamma) = 0 = \operatorname{tr}\varepsilon(\vec{u}_2^\gamma)$ on γ.
For $\gamma \in \mathcal{D}$ we have $\varepsilon(\vec{u}^\gamma) = 0$ on γ.

Since $\vec{u}$ is a solution of the homogeneous boundary transmission problem we conclude: For every $i = 1, \dots, m$ the function $\operatorname{tr}\varepsilon(\vec{u}_i)$ is a solution of the following problem in cone $\mathcal{C}_i$:

$$\triangle(\operatorname{tr}\varepsilon(\vec{u}_i)) = 0 \quad \text{in } \mathcal{C}_i, \tag{5.14}$$
$$\operatorname{tr}\varepsilon(\vec{u}_i) = 0 \quad \text{on } \partial\mathcal{C}_i. \tag{5.15}$$

Further $\operatorname{tr}\varepsilon(\vec{u}) = \rho^{-1+i\beta}\tilde{v}$ (follows from the ansatz for $\vec{u}$) which finally leads to $\operatorname{tr}\varepsilon(\vec{u}_i) = 0$ on $\mathcal{C}_i$. This follows with the help of Lemma 5.1.
1. Case: If for a sub cone $\mathcal{C}_i$ there holds $\varepsilon(\vec{u}_i) = 0$ on $\partial\mathcal{C}_i$, then each component $\varepsilon_{kl}(\vec{u}_i)$ of $\varepsilon(\vec{u}_i)$ is a solution of:

$$\triangle\varepsilon_{kl}(\vec{u}_i) = 0 \quad \text{in } \mathcal{C}_i, \tag{5.16}$$
$$\varepsilon_{kl}(\vec{u}_i) = 0 \quad \text{on } \partial\mathcal{C}_i. \tag{5.17}$$

Again with Lemma 5.1 there follows $\varepsilon(\vec{u}_i) = 0$ in $\mathcal{C}_i$ and finally $\vec{u}_i = const$ on $\mathcal{C}_i$.
2. Case: If $\mu_i = \mu_j$ for two neighboring cones $\mathcal{C}_i, \mathcal{C}_j$, then $\sigma_i(\vec{u}_i) = 2\mu_i\varepsilon(\vec{u}_i) = 2\mu_i\varepsilon(\vec{u}_j) = \sigma_i(\vec{u}_j)$ on $\gamma = \partial\mathcal{C}_i \cap \partial\mathcal{C}_j$. Here we used $\operatorname{tr}\varepsilon(\vec{u}_i) = 0$ on $\mathcal{C}_i$. Therefore $\vec{u}$ is a solution of

$$-(\mu_i\triangle\vec{u} + (\lambda_i + \mu_i)\operatorname{grad}\operatorname{div}\vec{u}) = 0 \quad \text{in } \mathcal{C}_i \cup \mathcal{C}_j,$$

the cone $\mathcal{C}_i \cup \mathcal{C}_j$ can be considered as one cone with the parameters μ_i, λ_i. Rejoining all neighboring cones with $\mu_i = \mu_j$ results in a cone $\tilde{\mathcal{C}}$ for which $\tilde{\mu} \neq \mu_k$ for all neighboring cones $\mathcal{C}_k$. The same considerations as in the first case lead to $\vec{u}|_{\tilde{\mathcal{C}}} = const$.
Finally we have $\vec{u} = const$ on $\mathcal{C}$ and together with the Dirichlet conditions: $\vec{u} = 0$ on $\mathcal{C}$. This is a contradiction to the assumption $\vec{u} \neq 0$, thus the line $\operatorname{Re}\alpha = 0$ does not contain eigenvalues of the operator pencil which corresponds to the Dirichlet problem. Using the symmetry of the eigenvalues the same is true for the line $\operatorname{Re}\alpha = -1$.

Second step: Applying the homotopy argument of Theorem 4.2 to the operator family $\mathcal{A}_t$ given by $\mu_i(t) := (1-t)\mu_1 + t\mu_i$, $\lambda_i(t) := (1-t)\lambda_1 + t\lambda_i$ finishes the proof since there are no eigenvalues in the strip $\operatorname{Re}\alpha \in [-1, 0]$ for $t = 0$ and for all $t \in [0,1]$ there are no eigenvalues of $\mathcal{A}_t$ on the lines $\operatorname{Re}\alpha = -1, \operatorname{Re}\alpha = 0$.

Mixed problems: We use the same arguments as for the Dirichlet problem. The essential equation here is

$$0 = \sum_{\gamma\in\mathcal{G}} \vec{n}_1^\gamma \int_{\gamma_\delta} B_1^\gamma(\vec{w}_1^\gamma)\, ds - \sum_{\gamma\in\mathcal{D}} \vec{n}_\gamma \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\, ds + \sum_{\gamma\in\mathcal{N}} \vec{n}_\gamma \int_{\gamma_\delta} \sigma(\vec{u}) : \varepsilon(\vec{u})\, ds. \tag{5.18}$$

which replaces equation (5.10). By analogous arguments we conclude that there are no eigenvalues on the lines $\operatorname{Re}\alpha \in \{-1, 0\}$. Condition **G1** guarantees that the problem with $\mu_i = \mu_j, \lambda_i = \lambda_j$ on all subdomains has no eigenvalues in the strip $\operatorname{Re}\alpha \in [-1, 0]$.

Neumann problem: Here, equation (5.10) is replaced by

$$0 = \sum_{\gamma\in\mathcal{G}} \vec{n}_1^\gamma \int_{\gamma_\delta} B_1^\gamma(\vec{w}_1^\gamma)\, ds + \sum_{\gamma\in\mathcal{N}} \vec{n}_\gamma \int_{\gamma_\delta} \sigma_\gamma(\vec{u}_\gamma) : \varepsilon(\vec{u}_\gamma)\, ds. \tag{5.19}$$

With similar arguments to the Dirichlet case one proves that $\alpha = 0$ is the only eigenvalue on the line $\operatorname{Re} \alpha = 0$ with the constant functions as eigenfunctions. By symmetry, -1 is the only eigenvalue on the line $\operatorname{Re} \alpha = -1$. By calculations similar to those in [13, pp. 127] one can prove that there are no associated eigenfunctions for the eigenvalue $\alpha = 0$. Thus the geometric multiplicity = algebraic multiplicity = 3. The proof finishes with a homotopy argument.

Interior crossing points: The case of one subdomain (i.e. $\mathcal{C} = \mathbb{R}^3$) is treated in Lemma 5.2 subsequent to the proof. For the case of $m \geqslant 2$ subdomains we proceed as in the Neumann problem, where equation (5.19) is replaced by

$$0 = \sum_{\gamma \in \mathcal{G}} \vec{n}_1^{\gamma} \int_{\gamma_\delta} B_1^{\gamma}(\vec{w}_1^{\gamma})\, ds. \tag{5.20}$$

This finishes the proof of Theorem 5.1. □

5.3.4 Two auxiliary Lemmata

Lemma 5.2. *The eigenvalue problem corresponding to the equation*

$$\mu \triangle \vec{u} + (\lambda + \mu)\, \mathrm{grad}\ \mathrm{div}\ \vec{u} = 0, \quad x \in \mathbb{R}^3 \backslash \{0\} \tag{5.21}$$

(here, 0 is the "vertex" of the cone) has exactly the eigenvalues $\alpha_k = 0$, $k \in \mathbb{Z}$. The eigenvalue $\alpha_0 = 0$ has geometric multiplicity = algebraic multiplicity = 3 and has the constant functions as eigenfunctions.

Proof. Let $\vec{u} = r^{\alpha} \vec{v}$ be a solution of (5.21). Then $\operatorname{tr} \varepsilon(\vec{u}) = r^{\alpha-1} \tilde{\vec{v}}$ is a solution of

$$\triangle \operatorname{tr}(\varepsilon(\vec{u})) = 0 \qquad \text{in } \mathbb{R}^3.$$

Thus $\alpha - 1 = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \lambda}$, where λ is an eigenvalue of the Laplace-Beltrami operator on the whole sphere S^2 and $\tilde{\vec{v}}$ is a corresponding eigenvector. The eigenvalues of the Laplace-Beltrami have the form $\lambda_n = n(n+1)$, $n \in \mathbb{N}_0$, the geometric multiplicity is given by $K_n = 2n+1$ [20], there are no associated eigenfunctions. □

Lemma 5.3. *[24] Let $\mu_1, \mu_2 > 0$, $\lambda_i + \mu_i > 0$, $i = 1, 2$. For $x, y \in \mathbb{C}$ we set*

$$b(x,y) := (\lambda_1 - \lambda_2)(\lambda_1 + 2\mu_2)\,|x|^2 + 4(\lambda_1 - \lambda_2)(\mu_1 - \mu_2) \operatorname{Re}(y\overline{x}) + 2(\lambda_2 + 2\mu_1)(\mu_1 - \mu_2)\,|y|^2 .$$

There holds:

i.) $\mu_1 \geqslant \mu_2$ *and* $\lambda_1 \geqslant \lambda_2 \Rightarrow b(x,y) \geqslant 0, \forall\, x, y \in \mathbb{C}$.

ii.) $\mu_1 > \mu_2$ *and* $\lambda_1 \geqslant \lambda_2 \Rightarrow b(x,y) > 0, \forall\, x, y \in \mathbb{C}, y \neq 0$.

iii.) $\mu_1 \leqslant \mu_2$ *and* $\lambda_1 \leqslant \lambda_2 \Rightarrow b(x,y) \leqslant 0, \forall\, x, y \in \mathbb{C}$.

iv.) $\mu_1 < \mu_2$ *and* $\lambda_1 \leqslant \lambda_2 \Rightarrow b(x,y) < 0, \forall\, x, y \in \mathbb{C}, y \neq 0$.

v.) $\mu_1 = \mu_2$ *and* $\lambda_1 \neq \lambda_2$*, then* $b(x,y) = 0 \Leftrightarrow x = 0$.

The Lemma is proven with similar arguments to those in [24].

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