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Mehrfeldprobleme in der Kontinuumsmechanik

Dorothee Knees

On the regularity of weak solutions of nonlinear elliptic transmission problems on polyhedral domains

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Contents

1 Introduction

This paper is concerned with the study of the global regularity of weak solutions of boundary transmission problems for nonlinear elliptic systems with p–structure, $1 < p < \infty$. The systems are defined in polygonal or polyhedral domains $\Omega = \cup_i \Omega_i \subset \mathbb{R}^d$, $d \geqslant 2$, and have the following form for $u : \Omega \to \mathbb{R}^m$, $u_i = u\big|_{\Omega_i}$:

$$\operatorname{div}_x (D_A W_i(\nabla u_i)) + f_i = 0 \quad \text{in } \Omega_i,\ 1 \leqslant i \leqslant M, \tag{1}$$

$$u_i - u_j = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{2}$$

$$D_A W_i(\nabla u_i)\vec{n}_{ij} + D_A W_j(\nabla u_j)\vec{n}_{ji} = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{3}$$

$$u = g \quad \text{on } \Gamma_D, \tag{4}$$

$$D_A W_i(\nabla u_i)\vec{n}_i = h \quad \text{on } \Gamma_N. \tag{5}$$

The functions $W_i : \mathbb{R}^{m \times d} \to \mathbb{R}$ can be interpreted as energy densities and satisfy growth conditions which will be specified in section 3. $D_A W_i(A)$ denotes the gradient of $W_i(A)$ for

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$A \in \mathbb{R}^{m \times d}$. It is admitted that the energy densities W_i have different growth properties on each subdomain. The transmission problems include for example the following equation:

$$\operatorname{div}\left(\mu(x)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + f = 0,$$

where $\mu(x)$ and $p(x)$ are piecewise constant with respect to the partition of Ω. The main result states, that the weak solution $u\big|_{\Omega_i}$ is in $W^{\frac{3}{2}-\epsilon, r_i}(\Omega_i)$ for a suitable $r_i \in [p_i, 2]$ if $p_i \in (1, 2]$ and from $W^{1+\frac{1}{p_i}-\epsilon, p_i}(\Omega_i)$ if $p_i > 2$, provided that the energy densities are ordered quasi-monotonely.

In the case of transmission problems for linear elliptic systems it is well known, that the structure of weak solutions in the neighborhood of cross points (points, where different subdomains come together) can be completely described by an asymptotic expansion, see [3, 13, 16, 17, 20, 24, 26, 27, 28]. The singular exponents in the expansion characterize the regularity of the solution. In the papers [2, 21, 30, 28, 14] estimates for the singular exponents were derived for transmission problems of the Laplace operator as well as for the equations of linear, isotropic elasticity with piecewise constant material parameters. It turned out, that a quasi-monotone distribution of the material parameters in combination with some geometrical conditions leads to piecewise $H^{\frac{3}{2}}$–regularity of weak solutions. There are also various examples which show, that the regularity can get very low (i.e. $H^{1+\epsilon}, \epsilon > 0$ small) if these conditions are violated.

For scalar nonlinear elliptic equations asymptotic expansions are known in some special cases, see [34, 5, 15, 22]. For systems or transmission problems it is an open question, whether the structure of weak solutions in the neighborhood of corners, edges or cross points can be described by such expansions completely. A very useful tool to deduce regularity results for these cases is the difference-quotient technique. This technique is widely used in order to derive interior regularity results, see for example [25, 35, 4, 31, 23], and was improved by C.Ebmeyer and J.Frehse in order to prove global regularity results on polyhedral domains, [7, 9, 10]. In this paper, the difference-quotient technique is applied to prove the main result. Test functions of the form $\xi(x) = \varphi^2(x)(u(x + he_l) - u(x))$, where u is a weak solution, φ is a cut-off function, $h > 0$ and e_l is a basis vector, are inserted into the weak formulation. The difficulty is, that the differences are taken across the transmission boundaries and due to the different growth properties of the differential operators on the subdomains, the functions ξ are not admissible test functions in general. Therefore, it is assumed, that the energy densities W_i of the transmission problem satisfy a quasi-monotonicity condition, which guarantees, that there exist vectors e_l for which ξ is admissible. The quasi-monotonicity condition, which will be introduced in this paper, is a considerable modification and generalization of the original definition by M.Dryja, M.V.Sarkis and O.B.Widlund. In [6] they defined quasi-monotonicity for the distribution of the parameters in Poisson's equation with piecewise constant coefficients. In this paper, we change the point of view and define quasi-monotonicity for the distribution of the energy densities which correspond to the transmission problem. The relation between the definition in [6] and our definition is discussed in chapter 4.

The presented regularity results generalize those from [11], where the homogeneous Dirichlet-problem for two subdomains with plane interface and $p_1 = p_2 = 2$ is considered. As a special case, our results can be applied to a class of linear elliptic transmission problems and to coupled linear elastic, not necessarily isotropic, materials and provide new estimates for the singular exponents in the asymptotic expansions.

The paper is organized as follows: In section 2, the domains and function spaces are defined following the approach in [19]. The weak formulation of the transmission problem and existence results are presented briefly in section 3. Here, the main theorem of monotone operators plays a crucial role. In section 4, the quasi-monotonicity is introduced and illustrated by various examples for two and three dimensional domains. The main theorem is stated and proved in section 4 using the difference-quotient technique. The paper closes with an appendix, where some essential inequalities are given, which follow from the growth properties and convexity of the energy densities W_i.

2 Domains and function spaces

Throughout the whole article it is assumed that $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, is a bounded polygonal or polyhedral domain with Lipschitz-boundary. It is further assumed that there exists a finite number of pairwise disjoint polyhedral domains $\Omega_i \subset \Omega$, $1 \leqslant i \leqslant M$, with Lipschitz-boundaries such that

$$\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega_i}, \quad \Gamma_{ij} := \partial\Omega_i \cap \partial\Omega_j.$$

On each of these subdomains a differential operator will be given and the growth properties of these operators may vary from subdomain to subdomain. Therefore, the following function spaces are introduced, which take into account the splitting of Ω (analogously to [19]):
For $1 \leqslant i \leqslant M$ let $p_i \in (1,\infty)$, $\vec{p} := (p_1, \dots, p_M)$ and $p_{\min} := \min\{p_i,\ 1 \leqslant i \leqslant M\}$. Then

$$L^{\vec{p}}(\Omega) := \left\{ u \in L^{p_{\min}}(\Omega):\ u\big|_{\Omega_i} \in L^{p_i}(\Omega_i) \right\},$$
$$W^{1,\vec{p}}(\Omega) := \left\{ u \in W^{1,p_{\min}}(\Omega):\ u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i) \right\},$$

where $u\big|_{\Omega_i}$ is the restriction of u to the subdomain Ω_i. These spaces are endowed with the following norms:

$$\|u\|_{L^{\vec{p}}(\Omega)} := \sum_{i=1}^{M} \left\| u\big|_{\Omega_i} \right\|_{L^{p_i}(\Omega_i)},$$
$$\|u\|_{W^{1,\vec{p}}(\Omega)} := \sum_{i=1}^{M} \left\| u\big|_{\Omega_i} \right\|_{W^{1,p_i}(\Omega_i)}.$$

Note, that we do not distinguish in the notation between scalar and vector valued functions or spaces. The next lemma states some essential properties of these spaces:

Lemma 2.1. *[19] Let $p_i \in (1,\infty)$ for $1 \leqslant i \leqslant M$. Then*

1. *$L^{\vec{p}}(\Omega)$ is a reflexive Banach space and the dual space is given by $\left(L^{\vec{p}}(\Omega)\right)' = L^{\vec{q}}(\Omega)$, where $\vec{q} = (q_1, \dots, q_M)$ and $q_i = p_i'$, i.e. $\frac{1}{p_i} + \frac{1}{q_i} = 1$.*
2. *$W^{1,\vec{p}}(\Omega)$ is a reflexive Banach space.*
3. *$\mathcal{C}^\infty(\overline{\Omega})$ is dense in $L^{\vec{p}}(\Omega)$ and also in $W^{1,\vec{p}}(\Omega)$.*

Since $W^{1,\vec{p}}(\Omega)$ is contained in $W^{1,p_{\min}}(\Omega)$, the trace operator

$$W^{1,\vec{p}}(\Omega) \to W^{1-\frac{1}{p_{\min}},p_{\min}}(\partial\Omega): \; u \to u\big|_{\partial\Omega}$$

is well defined, linear and continuous [13]. Analogously to [19], the space of traces of functions from $W^{1,\vec{p}}(\Omega)$ is defined as follows:

$$W^{\frac{\vec{p}-1}{\vec{p}},\vec{p}}(\partial\Omega) := \left\{ u\big|_{\partial\Omega} : \; u \in W^{1,\vec{p}}(\Omega) \right\},$$

where $\frac{\vec{p}-1}{\vec{p}} := (1-\frac{1}{p_1},\ldots,1-\frac{1}{p_M})$. The trace theorem [13] also shows, that the latter space is a subspace of $\{u \in L^1(\partial\Omega): \; u\big|_{(\partial\Omega\cap\partial\Omega_i)} \in W^{1-\frac{1}{p_i},p_i}(\partial\Omega \cap \partial\Omega_i)\}$. For the description of mixed boundary value problems, the following spaces are useful: Let $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, where Γ_D and Γ_N are open and disjoint.

$$\begin{aligned}
V^{\vec{p}}(\Omega) &= \left\{ u \in W^{1,\vec{p}}(\Omega): \; u\big|_{\Gamma_D} = 0 \right\}, \\
W^{(\vec{p}-1)/\vec{p}}(\Gamma_D) &= \left\{ u\big|_{\Gamma_D} : \; u \in W^{(\vec{p}-1)/\vec{p}}(\partial\Omega) \right\}, \\
\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N) &= \left\{ u\big|_{\Gamma_N} : \; u \in V^{\vec{p}}(\Omega) \right\} = \left\{ u\big|_{\Gamma_N} : \; u \in W^{(\vec{p}-1)/\vec{p}}(\partial\Omega) \text{ and } u\big|_{\Gamma_D} = 0 \right\}.
\end{aligned}$$

Finally, there is an equivalent characterization of the space $W^{1,\vec{p}}(\Omega)$.

Lemma 2.2. *Let $p_i \in (1,\infty)$ for $1 \leqslant i \leqslant M$. Then*

$$W^{1,\vec{p}}(\Omega) = \left\{ u \in L^{\vec{p}}(\Omega): \; u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i) \text{ and } \left(u\big|_{\Omega_i}\right)\big|_{\Gamma_{ij}} = \left(u\big|_{\Omega_j}\right)\big|_{\Gamma_{ij}} \right\}. \tag{6}$$

Moreover $W^{1,\vec{p}}(\Omega)$ is a closed subspace of $\left\{ u \in L^{\vec{p}}(\Omega): \; u\big|_{\Omega_i} \in W^{1,p_i}(\Omega_i) \right\}$.

In other words, the space $W^{1,\vec{p}}(\Omega)$ consists of all functions which are piecewise in $W^{1,p_i}(\Omega_i)$ and which do not jump at the interfaces Γ_{ij}.

Proof. Let $u \in W^{1,\vec{p}}(\Omega)$ be a scalar-valued function and $\varphi \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^d) = \{v : \Omega \to \mathbb{R}^d : \; v \in \mathcal{C}^\infty(\Omega), \operatorname{supp} v \subset \Omega\}$. Since $u \in W^{1,p_{\min}}(\Omega)$, there holds for the distributional derivative of u:

$$\begin{aligned}
0 = \langle \nabla u, \varphi\rangle - \int_\Omega \nabla u \cdot \varphi \, \mathrm{d}x &= -\int_\Omega u \operatorname{div}\varphi \, \mathrm{d}x - \int_\Omega \nabla u \cdot \varphi \, \mathrm{d}x \\
&= -\sum_{i=1}^M \int_{\Omega_i} \operatorname{div}(u_i\varphi)\, \mathrm{d}x \overset{\text{Gauss}}{=} -\sum_{i=1}^M \int_{\partial\Omega_i} u\,(\varphi\cdot\vec{n}_i)\, \mathrm{d}s \\
&= -\sum_{i=1}^M \sum_{j=1}^{i-1} \int_{\Gamma_{ij}} \left(\left(u\big|_{\Omega_i}\right)\big|_{\Gamma_{ij}} - \left(u\big|_{\Omega_j}\right)\big|_{\Gamma_{ij}} \right) (\varphi\cdot\vec{n}_{ij})\, \mathrm{d}s.
\end{aligned}$$

Since $\varphi \in \mathcal{C}_0^\infty(\Omega,\mathbb{R}^d)$ is arbitrary, it follows that $\left(u\big|_{\Omega_i}\right)\big|_{\Gamma_{ij}} - \left(u\big|_{\Omega_j}\right)\big|_{\Gamma_{ij}} = 0$ on Γ_{ij} and "$\subset$" is proved in (6). In order to prove the inverse relation one has to show, that functions from the space on the right hand side in (6) are elements of $W^{1,p_{\min}}(\Omega)$. To prove this, one has to calculate the distributional derivative of these functions. With the help of Gauss' Theorem the assertion follows. $\square$

The Sobolev embedding theorems can be carried over directly to the $W^{1,\vec{p}}(\Omega)$ spaces, see [19], and consequently there is also an inequality of Poincaré-Friedrichs' type:

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain with Lipschitz boundary which is decomposed into M pairwise disjoint polyhedral subdomains with Lipschitz boundaries; $1 < p_i < \infty$ for $1 \leqslant i \leqslant M$. If $V \subset W^{1,\vec{p}}(\Omega)$ is a closed subspace with the property*

$$u \in V,\ \nabla u = 0 \ \text{in}\ \Omega \implies u = 0 \ \text{in}\ \Omega,$$

then there exists a constant $c > 0$ such that for every $u \in V$: $\|u\|_{L^{\vec{p}}(\Omega)} \leqslant c\,\|\nabla u\|_{L^{\vec{p}}(\Omega)}$.

Proof. This lemma can be proved (as in the case $M = 1, p = 2$, [38]) by contradiction using that the embedding $W^{1,\vec{p}}(\Omega) \to L^{\vec{p}}(\Omega)$ is compact. □

Difference quotients of weak solutions will be estimated in the proof of the regularity results. Therefore we introduce the Nikolskii space, which takes difference quotients into account explicitly.

Definition 2.1 (Nikolskii space). *[1, 29] Let $\Omega \subset \mathbb{R}^d$ be an open domain, $s = m + \sigma$, where $m \geqslant 0$ is an integer and $0 < \sigma < 1$. For $1 < p < \infty$*

$$\mathcal{N}^{s,p}(\Omega) := \Big\{ u \in L^p(\Omega) :\ \|u\|_{\mathcal{N}^{s,p}(\Omega)} < \infty \Big\}, \tag{7}$$

where

$$\|u\|^p_{\mathcal{N}^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0 \\ h\in\mathbb{R}^d \\ 0<|h|<\eta}} \int_{\Omega_\eta} \frac{|D^\alpha u(x+h) - D^\alpha u(x)|^p}{|h|^{\sigma p}}\,\mathrm{d}x \tag{8}$$

and $\Omega_\eta = \{x \in \Omega :\ \operatorname{dist}(x, \partial\Omega) > \eta\}$.

The relation between Nikolskii spaces and Sobolev-Slobodeckij spaces is described in the next lemma:

Lemma 2.4. *[1, 29, 36, 37] Let s, p be as in Definition 2.1. If $\Omega = \mathbb{R}^d$ or if $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, then the following embeddings are continuous:*

$$\text{for every } \varepsilon > 0 : \qquad \mathcal{N}^{s+\varepsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset \mathcal{N}^{s,p}(\Omega).$$

Proof. If $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, then there exist linear and continuous extension operators $E_1 : W^{s,p}(\Omega) \to W^{s,p}(\mathbb{R}^d)$ and $E_2 : \mathcal{N}^{s,p}(\Omega) \to \mathcal{N}^{s,p}(\mathbb{R}^d)$ for $s > 0$ and $1 < p < \infty$ (see [13, Theorem 1.4.1.3] for $W^{s,p}$ and [29, p. 381] for $\mathcal{N}^{s,p}$). Furthermore, the restriction operators from $\mathbb{R}^d$ to Ω are continuous as well. Therefore it suffices to prove Lemma 2.4 for the case $\Omega = \mathbb{R}^d$.
For s, p as in Definition 2.1 and $1 \leqslant r \leqslant \infty$ we denote by $B^s_{p,r}(\mathbb{R}^d)$ the Besov spaces on $\mathbb{R}^d$. For the definition see e.g. [33, 36]. There holds $B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d)$ and $B^s_{p,\infty}(\mathbb{R}^d) = \mathcal{N}^{s,p}(\mathbb{R}^d)$, [36, sections 1.3 and 2.2.9]. The following embeddings are continuous for $\epsilon > 0$, [37, sec. 2.3.2, Prop. 2] and [36, sec. 2.1.1]:

$$\mathcal{N}^{s+\epsilon,p}(\mathbb{R}^d) = B^{s+\epsilon}_{p,\infty}(\mathbb{R}^d) \subset B^s_{p,p}(\mathbb{R}^d) = W^{s,p}(\mathbb{R}^d) \subset B^s_{p,\infty}(\mathbb{R}^d) = \mathcal{N}^{s,p}(\mathbb{R}^d).$$

This completes the proof. Note, that in Lemma 2.4 the assumptions on Ω can be weakened: Lemma 2.4 is valid for domains for which continuous extension operators E_1 and E_2 exist. □

For inner products and norms of matrices $A, B \in \mathbb{R}^{m\times d}$, $m \geqslant 1, d \geqslant 2$, the following abbreviations are used:

$$A : B = \operatorname{tr}(B^T A) = \operatorname{tr}(AB^T) = \sum_{i=1}^{m}\sum_{j=1}^{d} A_{ij}B_{ij},$$

$$|A| = \sqrt{A:A} = \left(\sum_{i=1}^{m}\sum_{j=1}^{d} A_{ij}^2\right)^{1/2}.$$

For $R > 0$ and $x \in \mathbb{R}^d$, $B_R(x)$ denotes the open ball with center x and radius R: $B_R(x) = \{y \in \mathbb{R}^d : |x-y| < R\}$ and $\partial B_R(x) = \{y \in \mathbb{R}^d : |x-y| = R\}$.

3 Weak formulation of the transmission problem and existence of solutions

In this section we describe the assumptions on the structure of the boundary transmission problem (1)-(5) and give some short comments on the existence of weak solutions.
Let $\Omega \subset \mathbb{R}^d$ be a polygonal or polyhedral domain with Lipschitz boundary which is decomposed into M pairwise disjoint Lipschitz-polyhedrons Ω_i (compare section 2). $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, Γ_D and Γ_N open and disjoint; by $\vec{n}_{ij}$ we denote the exterior normal vector of Ω_i with respect to Γ_{ij}, $\vec{n}_{ij} = -\vec{n}_{ji}$ and $\vec{n}_i$ is the exterior normal vector of Ω_i with respect to $\partial\Omega_i \cap \partial\Omega$.
Let $m \geqslant 1$ and assume, that there are given M functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$. The boundary transmission problem reads:
Find $u : \Omega \to \mathbb{R}^m$, $u\big|_{\Omega_i} = u_i$, such that:

$$\begin{aligned} \operatorname{div}_x (D_A W_i(\nabla u_i)) + f_i &= 0 \quad \text{in } \Omega_i,\ 1 \leqslant i \leqslant M, && (9)\\ u_i - u_j &= 0 \quad \text{on } \Gamma_{ij}, && (10)\\ D_A W_i(\nabla u_i)\vec{n}_{ij} + D_A W_j(\nabla u_j)\vec{n}_{ji} &= 0 \quad \text{on } \Gamma_{ij}, && (11)\\ u &= g \quad \text{on } \Gamma_D, && (12)\\ D_A W_i(\nabla u_i)\vec{n}_i &= h \quad \text{on } \Gamma_N. && (13) \end{aligned}$$

Here and in the sequel, the following notation is used: Let $A, B, C \in \mathbb{R}^{m\times d}$

$$(D_A W_i(A))_{k,l} = \frac{\partial W_i(A)}{\partial A_{kl}}, \quad 1 \leqslant k \leqslant m, 1 \leqslant l \leqslant d, \quad D_A W_i(A) \in \mathbb{R}^{m\times d},$$

$$D_A W_i(A) : B = \sum_{k=1}^{m}\sum_{l=1}^{d} \frac{\partial W_i(A)}{\partial A_{kl}} B_{kl},$$

$$D_A^2 W_i(A)[B,C] = \sum_{k,j=1}^{m}\sum_{s,t=1}^{d} \frac{\partial^2 W_i(A)}{\partial A_{ks}\partial A_{jr}} B_{ks} C_{jr}, \ \left|D_A^2 W_i(A)\right| = \left(\sum_{k,j=1}^{m}\sum_{s,t=1}^{d} \left(\frac{\partial^2 W_i(A)}{\partial A_{ks}\partial A_{jr}}\right)^2\right)^{1/2},$$

$$\operatorname{div}_x (D_A W_i(\nabla u(x)) \in \mathbb{R}^m, \ (\operatorname{div}_x (D_A W_i(\nabla u(x)))_j = \sum_{l=1}^{d} \frac{\partial}{\partial x_l}\left((D_A W_i(\nabla u(x)))_{jl}\right).$$

In this paper it is assumed, that the functions W_i are of p–structure which means that the functions W_i and their derivatives satisfy the following growth properties (compare also [8, 9]): Let $p_i \in (1, \infty)$.

H0 $W_i \in \mathcal{C}^1(\mathbb{R}^{m\times d}) \cap \mathcal{C}^2(\mathbb{R}^{m\times d}\backslash\{0\})$.

H1 There exist $c_0^i \in \mathbb{R}, c_1^i, c_2^i > 0$, such that for every $A \in \mathbb{R}^{m\times d}$:

$$c_0^i + c_1^i \, |A|^{p_i} \leqslant W_i(A) \leqslant c_2^i \, (1 + |A|^{p_i}) \, .$$

H2 There exists $c^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}$:

$$|D_A W_i(A)| \leqslant c^i \left(1 + |A|^{p_i - 1}\right).$$

H3 There exists $c^i > 0$ such that for every $A \in \mathbb{R}^{m\times d}\backslash\{0\}$:

$$\left|D_A^2 W_i(A)\right| \leqslant c^i \left(1 + |A|^{p_i - 2}\right).$$

H4 Ellipticity condition, convexity of W_i: There exist $c_i > 0$ and $\kappa_i \in \{0, 1\}$ such that for every $A, B \in \mathbb{R}^{m\times d}$, $A \neq 0$:

$$D_A^2 W_i(A)[B, B] \geqslant c_i \, (\kappa_i + |A|)^{p_i - 2} \, |B|^2 \, .$$

We are now able to describe in which sense equations (9)-(13) shall be solved.

Definition 3.1. *Let $\Omega \subset \mathbb{R}^d, d \geqslant 2$, with $\overline{\Omega} = \bigcup_{i=1}^M \overline{\Omega_i}$ be a polygonal or polyhedral domain as introduced above, $m \in \mathbb{N}$. Assume, that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfy* **H0** - **H4** *with $p_i \in (1, \infty)$. Let $\vec{p} = (p_1, \ldots, p_M)$, $\vec{q} = (q_1, \ldots, q_M)$ with $q_i = p_i' = \frac{p_i}{p_i - 1}$ and $f \in L^{\vec{q}}(\Omega, \mathbb{R}^m)$, $g \in W^{(\vec{p}-1)/\vec{p}}(\Gamma_D, \mathbb{R}^m)$ and $h \in \left(\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N, \mathbb{R}^m)\right)'$.*
A function $u : \Omega \to \mathbb{R}^m$, $u \in W^{1,\vec{p}}(\Omega)$ is a weak solution of the boundary transmission problem (9)-(13) *if $u\big|_{\Gamma_D} = g$ and if for every $v \in V^{\vec{p}}(\Omega, \mathbb{R}^m)$:*

$$\sum_{i=1}^{M} \int_{\Omega_i} D_A W_i(\nabla u_i(x)) : \nabla v_i(x) \, \mathrm{d}x = \sum_{i=1}^{M} \int_{\Omega_i} f_i(x) v_i(x) \, \mathrm{d}x + \langle h, v \rangle, \tag{14}$$

$\langle \cdot, \cdot \rangle$ denotes the dual pairing between elements of $\left(\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N)\right)'$ and $\tilde{W}^{(\vec{p}-1)/\vec{p}}(\Gamma_N)$.

If a weak solution u and the right hand sides f, g, h in equation (14) are smooth enough, then u satisfies equations (9)-(13).

Remark 3.1. The functions W_i can be interpreted as energy density functions. Furthermore equation (14) is the weak Euler-Lagrange equation which is associated with the following minimizing problem: Find $u \in W^{1,\vec{p}}(\Omega)$ with $u\big|_{\Gamma_D} = g$ such that

$$\text{for every } v \ \in W^{1,\vec{p}}(\Omega) \text{ with } v\big|_{\Gamma_D} = g : \quad J(u) \leqslant J(v),$$

where $J(v) = \sum_{i=1}^M \int_{\Omega_i} W_i(\nabla v) \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x - \langle h, v \rangle$.

Remark 3.2. Note, that the coupling of linear homogeneously elliptic systems of second order with constant coefficients, where in addition the principal parts of the differential operators coincide with the differential operators themselves and which are Euler-Lagrange equations for minimizing problems, is also included here as a special case.

It shall be emphasized, that different exponents p_i for the functions W_i on each subdomain Ω_i are possible. The following existence result is a direct consequence of the theorem on monotone operators, see e.g. [39]:

Theorem 3.1 (Existence). *Let $\Omega \subset \mathbb{R}^d$ be a polyhedral domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ and assume that it is decomposed into M polyhedral subdomains Ω_i as introduced in section 2. For $1 \leqslant i \leqslant M$ let $p_i \in (1,\infty)$ and assume that $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies* **H0 - H4**. *Furthermore let $f \in L^{\vec{q}}(\Omega)$, where $q_i = p_i'$; $g \in W^{(\vec{p}-1)/\vec{p}}(\Gamma_D)$ and $h \in \left(W_0^{(\vec{p}-1)/\vec{p}}(\Gamma_N)\right)'$. If $\Gamma_D = \emptyset$, the following solvability condition shall be satisfied for every constant function v:*

$$\int_\Omega f v \,\mathrm{d}x + \langle h, v\rangle = 0. \tag{15}$$

Then there exists a weak solution $u \in W^{1,\vec{p}}(\Omega)$ of problem (14) with $u\big|_{\Gamma_D} = g$. If $\Gamma_D = \emptyset$, then u is unique, else u is unique up to constants.

Proof. The theorem can be proved with the main theorem of monotone operators, see for example [39]. Hypotheses **H0 - H4**, inequality (50) in the Appendix and Poincaré-Friedrichs' inequality guarantee that the nonlinear operator, which is related to the weak formulation, satisfies the assumptions of the main theorem of monotone operators. In particular, the operator $W^{1,\vec{p}}(\Omega) \to \left(W^{1,\vec{p}}(\Omega)\right)' : u \to \sum_{i=1}^M \int_{\Omega_i} D_A W_i(\nabla u_i(x)) : \nabla(\,\cdot\,)\,\mathrm{d}x$ is continuous and monotone on $W^{1,\vec{p}}(\Omega)$ and coercive on $V^{\vec{p}}(\Omega)$ if $\Gamma_D \neq \emptyset$. $\square$

Remark 3.3. (Physically nonlinear elasticity) Let $m = d \in \{2,3\}$ and assume that $D_A W_i(B)$ is symmetric if $B \in \mathbb{R}^{d\times d}$ is symmetric. It is reasonable to consider the following equation instead of equation (14):

$$\sum_{i=1}^M \int_{\Omega_i} D_A W_i(\varepsilon(u_i(x))) : \varepsilon(v_i(x))\,\mathrm{d}x = \sum_{i=1}^M \int_{\Omega_i} f_i(x) v_i(x)\,\mathrm{d}x + \langle h, v\rangle, \tag{16}$$

where $\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ is the linearized strain tensor corresponding to the displacement field u. For this equation, the statements of theorem 3.1 hold without any changes when $\Gamma_D \neq \emptyset$. In the case of $\Gamma_D = \emptyset$, one has to require that the solvability condition (15) is satisfied for every $v \in \ker\varepsilon$, which is the set of rigid body motions.

4 Regularity results for polyhedral domains

In this section, the main result on regularity of weak solutions of transmission problems on polyhedral domains is proved. The main theorem 4.1 states: if the energy densities W_i satisfy a quasi-monotonicity condition, then $u_i \in W^{\frac{3}{2}-\varepsilon,r_i}(\Omega_i)$ for a suitable $r_i \in [p_i, 2]$ for $p_i \in (1,2]$ and $u\big|_{\Omega_i} \in W^{1+\frac{1}{p}-\epsilon,p}(\Omega_i)$ if $p_i > 2$. As a special case, the theorem includes the earlier derived results for Poisson's equation and Lamé's equation with piecewise constant coefficients, [14]. The quasi-monotone distribution of the energy densities W_i is the essential assumption for our main theorem. The definition will be given in section 4.1 and is inspired by the definition of M.Dryja, M.V.Sarkis and O.B.Widlund in [6] for the distribution of the coefficients in Poisson's equation with piecewise constant coefficients. Let us remark, that our definition of

quasi-monotonicity is a considerable generalization of the definition in [6] and can be applied to a large class of linear and nonlinear boundary transmission problems.
The proof of the main result uses a difference quotient technique for polyhedrons, which was developed by C. Ebmeyer and J. Frehse in [8, 10], where they investigated the global regularity of weak solutions of nonlinear elliptic systems of $p-$structure on polyhedral domains.
Throughout the whole section various examples illustrate the condition of quasi-monotonicity. Furthermore, the obtained regularity results will be compared with known results for linear elliptic transmission problems.

4.1 Quasi-monotone distribution of energy densities

In the proof of the main theorem, $\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega_i}$ will be divided into a finite number of model domains, where it is assumed that each of these model domains coincides with the intersection of a ball with a collection of N suitable polyhedral cones (N depends on the model domain). This motivates the next definition:

Definition 4.1 (Polyhedral cone). *A set $\mathcal{K} \subset \mathbb{R}^d$, is a polyhedral cone with tip in S if*

1. *There exists $\mathcal{C} \subset \partial B_1(0)$, $\mathcal{C}$ open and not empty, such that*

$$\mathcal{K} = \left\{ x \in \mathbb{R}^d : \ \frac{x - S}{|x - S|} \in \mathcal{C} \right\}$$

2. *There is a finite number of hyperplanes E_i, $1 \leqslant i \leqslant n$, such that*

$$\partial\mathcal{K} = \bigcup_{i=1}^{n} \overline{E_i \cap \partial\mathcal{K}}.$$

Note, that $\mathcal{K}$ is open and $S \notin \mathcal{K}$.

Definition 4.2 (Quasi-monotonicity with respect to interior cross points).
Let $\mathcal{K}_1, \ldots, \mathcal{K}_N \subset \mathbb{R}^d$ be pairwise disjoint polyhedral cones with tip in 0 such that $\mathbb{R}^d = \cup_{i=1}^{N} \overline{\mathcal{K}_i}$.
For $s \in \mathbb{N}$ consider N functions $W_i : \mathbb{R}^s \to \mathbb{R} \cup \{\pm\infty\}$, $1 \leqslant i \leqslant N$.
The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if there exist numbers $k_1, \cdots, k_N \in \mathbb{R}$ and a basis $\{e_1, \ldots e_d\} \subset \mathbb{R}^d$ with $|e_l| = 1$, such that for every $h > 0$, $1 \leqslant l \leqslant d$ and $1 \leqslant i, j \leqslant N$ there holds:

$$\text{if } (\mathcal{K}_i + he_l) \cap \mathcal{K}_j \neq \emptyset, \text{ then } \quad W_j(A) + k_j \geqslant W_i(A) + k_i \quad \text{for every } A \in \mathbb{R}^s. \tag{17}$$

Here, $\mathcal{K}_i + he_l = \left\{ x \in \mathbb{R}^d : \ x = y + he_l, \ y \in \mathcal{K}_i \right\}$.

In the two dimensional case, this definition can be reformulated in a more illustrative way. Let $d = 2$ and assume that the polygonal cones $\mathcal{K}_i$ in definition 4.2 are given as follows: There are angles $\Phi_0 < \Phi_1 < \ldots < \Phi_N = \Phi_0 + 2\pi$ such that $\mathcal{K}_i = \{x \in \mathbb{R}^2 : \ 0 < r, \ \Phi_{i-1} < \varphi < \Phi_i\}$. Here, polar coordinates are used.

Lemma 4.1. *Let $d = 2$. The functions $W_i : \mathbb{R}^s \to \mathbb{R}$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if the following two conditions are satisfied:*

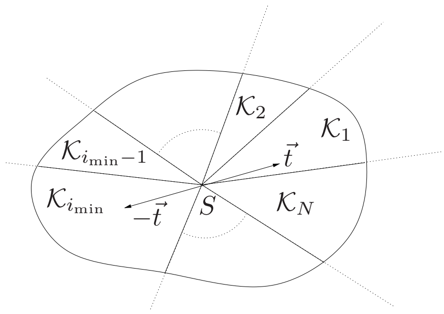

Figure 1: Example for the geometric condition at an interior cross point S

1. *There exist numbers $k_i \in \mathbb{R}$ and indices $i_{min}, i_{max} \in \{1, \dots, N\}$ such that for every $A \in \mathbb{R}^s$ (the indices are numbered modulo N):*

$$\begin{aligned} &W_{i_{max}}(A) + k_{i_{max}} \geqslant W_{i_{max}+1}(A) + k_{i_{max}+1} \geqslant \dots \\ &\qquad \geqslant W_{i_{min}-1}(A) + k_{i_{min}-1} \geqslant W_{i_{min}}(A) + k_{i_{min}} \leqslant W_{i_{min}+1}(A) + k_{i_{min}+1} \leqslant \dots \\ &\qquad\qquad \leqslant W_{i_{max}-1}(A) + k_{i_{max}-1} \leqslant W_{i_{max}}(A) + k_{i_{max}} \end{aligned}$$

.

2. *There exists a vector $\vec{t} \in \mathbb{R}^2, |\vec{t}| = 1$, such that $\vec{t} \in \mathcal{K}_{i_{max}}$ and $-\vec{t} \in \mathcal{K}_{i_{min}}$.*

The second condition in the previous lemma states that $\mathcal{K}_{i_{\min}}$ and $\mathcal{K}_{i_{\max}}$ are lying opposite, see also figure 1, where $i_{\max} = 1$.

Proof. If $\mathcal{K}_i$ and W_i satisfy conditions *1.* and *2.* in lemma 4.1, then it is easy to see that the functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ in the sense of definition 4.2: Choose $e_1 = \vec{t}$. From *2.* in lemma 4.1 and from the assumption that the cones $\mathcal{K}_i$ are open, it follows, that there exists a vector $\tilde{\vec{t}} \neq \vec{t}$ with $\tilde{\vec{t}} \in \mathcal{K}_{i_{\max}}$ and $-\tilde{\vec{t}} \in \mathcal{K}_{i_{\min}}$. Choose $e_2 = \tilde{\vec{t}}$. With this choice, relation (17) is satisfied.
It remains to prove, that conditions *1.* and *2.* of lemma 4.1 can be deduced from definition 4.2. Assume that $e_1 = \binom{1}{0}$ and that the cones $\mathcal{K}_i$, $1 \leqslant i \leqslant N$, are numbered counterclockwise in such a way, that the intersection of $\mathcal{K}_1$ with the upper half plane is not empty and that $e_1 \in \overline{\mathcal{K}_1}$. It follows from (17) that there holds for every $\mathcal{K}_i$, which is completely contained in the upper half plane:

$$\text{if } \mathcal{K}_i + e_1 \cap \mathcal{K}_j \neq \emptyset, \text{ then } j \leqslant i \text{ and } \; W_j(A) + k_j \geqslant W_i(A) + k_i \; \text{ for every } A \in \mathbb{R}^s.$$

On the other hand, there holds for every $\mathcal{K}_j$, which is completely contained in the lower half plane:

$$\text{if } \mathcal{K}_i + e_1 \cap \mathcal{K}_j \neq \emptyset, \text{ then } j \geqslant i \text{ and } \; W_j(A) + k_j \geqslant W_i(A) + k_i \; \text{ for every } A \in \mathbb{R}^s.$$

It follows that there exist $n \in \{1, \dots, N\}$ and $\tilde{n} \in \{n, n+1\}$ such that for every $A \in \mathbb{R}^s$:

$$W_1(A) + k_1 \geqslant W_2(A) + k_2 \geqslant \dots \geqslant W_n(A) + k_n \quad \text{and} \tag{18}$$
$$W_{\tilde{n}}(A) + k_{\tilde{n}} \leqslant W_{\tilde{n}+1}(A) + k_{\tilde{n}+1} \leqslant \dots \leqslant W_N(A) + k_N. \tag{19}$$

In order to find $i_{\min}$, $i_{\max}$ and $\vec{t}$, several cases have to be distinguished.
1. Case: $n = \tilde{n}$ and $e_1 \in \mathcal{K}_1$, i.e. the positive x_1-axis is contained in $\mathcal{K}_1$ and the negative x_1-axis is contained in $\mathcal{K}_n$. Then $i_{\min} = n$, $i_{\max} = 1$ and $\vec{t} = e_1$.
2. Case: $\tilde{n} = n + 1$ and $e_1 \in \mathcal{K}_1$, i.e. the negative x_1-axis is the interface between $\mathcal{K}_n$ and K_{n+1}. It follows from the assumptions (definition 4.2) that $W_N(A) + k_N \leqslant W_1(A) + k_1$ and therefore $i_{\max} = 1$. To find $i_{\min}$, assume without loss of generality that $e_2 \cdot \binom{0}{1} > 0$. Then it follows that $\mathcal{K}_{n+1} + e_2 \cap \mathcal{K}_n \neq \emptyset$ and therefore, by the assumptions of definition 4.2: $i_{\min} = n + 1$. Furthermore there exists $\theta \in (0, 1)$ such that $\vec{t} := \theta e_1 + (1 - \theta)e_2$ satisfies condition *2.* of lemma 4.1.
The remaining two cases, where either only the positive x_1-axis or the whole x_1-axis is part of the boundaries of $\mathcal{K}_1$ or $\mathcal{K}_n$, can be treated similarly. □

The following corollary is essential in the proof of the regularity results.

Corollary 4.1. *Let $\mathcal{K}_1, \dots, \mathcal{K}_N \subset \mathbb{R}^d$ be polyhedral cones as in definition 4.2. Assume that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ and that they satisfy* **H0-H1** *for some $p_i \in (1, \infty)$. Let $\{e_1, \dots, e_d\} \subset \mathbb{R}^d$ be the basis in definition 4.2. Then there holds for every $h > 0$, $1 \leqslant l \leqslant d$, $1 \leqslant i, j \leqslant N$:*
If $(\mathcal{K}_i + he_l) \cap \mathcal{K}_j \neq \emptyset$, then $p_j \geqslant p_i$. Furthermore, if $u \in W^{1,\vec{p}}(\mathbb{R}^d)$ and has compact support, then also $u(\cdot + he_l) \in W^{1,\vec{p}}(\mathbb{R}^d)$.

Proof. From $\mathcal{K}_i + he_l \cap \mathcal{K}_j \neq \emptyset$ it follows that $W_j(A) + k_j \geqslant W_i(A) + k_i$ for every $A \in \mathbb{R}^{m\times d}$ and therefore, by **H1**:

$$\forall\, A \in \mathbb{R}^{m\times d}: \qquad c_2^j(1 + |A|^{p_j}) + k_j \geqslant c_0^i + c_1^i\,|A|^{p_i} + k_i.$$

This is only possible if $p_j \geqslant p_i$.
We prove the second assertion: Let $u \in W^{1,\vec{p}}(\mathbb{R}^d)$ with compact support. Then, by the definition of the space $W^{1,\vec{p}}(\mathbb{R}^d)$: $u \in W^{1,p_{\min}}(\mathbb{R}^d)$ and $u\big|_{\mathcal{K}_i} \in W^{1,p_i}(\mathcal{K}_i)$. Obviously, $u(\cdot + he_l) \in W^{1,p_{\min}}(\mathbb{R}^d)$ for $h > 0$. It remains to show, that $u(\cdot + he_l)\big|_{\mathcal{K}_i} \in W^{1,p_i}(\mathcal{K}_i)$. Note, that $u(x + he_l)\big|_{\mathcal{K}_i} = u(y)\big|_{\mathcal{K}_i + he_l}$ with $y = x + he_l$. Furthermore, $\mathcal{K}_i + he_l = \bigcup_{j=1}^N \overline{\mathcal{K}_i + he_l \cap \mathcal{K}_j}$. Assume, that $\mathcal{K}_i + he_l \cap \mathcal{K}_j \neq \emptyset$. By the definition of $W^{1,\vec{p}}(\mathbb{R}^d)$, there holds $u\big|_{\mathcal{K}_i + he_l \cap \mathcal{K}_j} \in W^{1,p_j}(\mathcal{K}_i + he_l \cap \mathcal{K}_j)$ and, due to the first assertion of corollary 4.1, $p_j \geqslant p_i$. Since u has compact support, Hölder's inequality yields $u\big|_{\mathcal{K}_i + he_l \cap \mathcal{K}_j} \in W^{1,p_i}(\mathcal{K}_i + he_l \cap \mathcal{K}_j)$ for every j with $\mathcal{K}_i + he_l \cap \mathcal{K}_j \neq \emptyset$. Since $u \in W^{1,p_{\min}}(\mathbb{R}^d)$, the assertion follows by arguments which are similar to those in the proof of lemma 2.2. □

The next examples describe some possible choices for the functions W_i and cones $\mathcal{K}_i$ for $d = 2, 3$.

Example 4.1. For $\Phi_0 < \Phi_1 < \ldots < \Phi_N = \Phi_0 + 2\pi$ let $\mathcal{K}_i = \{x \in \mathbb{R}^2 :\ 0 < r,\ \Phi_{i-1} < \varphi < \Phi_i\}$. Consider the functions $W_i : \mathbb{R}^2 \to \mathbb{R} : A \to \frac{\mu_i}{2}\,|A|^2$ with $\mu_i > 0$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if there exists $i_{\min} \in \{2, \dots, N\}$ such that

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{i_{\min}} \leqslant \mu_{i_{\min}+1} \leqslant \cdots \leqslant \mu_N \leqslant \mu_1 \tag{20}$$

and $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, see figure 1. The constants k_i in definition 4.2 can be chosen as 0.
The transmission problem, which corresponds to the functions W_i, is Poisson's equation with

piecewise constant coefficients μ_i on $\mathcal{K}_i$. Historically, quasi-monotonicity was first defined by Dryja/Sarkis/Widlund in [6] for the distribution of these coefficients. In contrast to our definition they did not require the geometric assumption $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, which is hidden in definition 4.2.

Example 4.2. Let $\mathcal{K}_i \subset \mathbb{R}^2$, $1 \leqslant i \leqslant N$ be as in example 4.1 and assume that the functions $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfy **H0** and **H1** for some $p_i \in (1,\infty)$ with $p_i \neq p_j$ for $i \neq j$ and $p_1 = \max\{p_i,\, 1 \leqslant i \leqslant N\}$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if there exists $i_{\min} \in \{2,\ldots,N\}$ such that $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$ and

$$p_1 > p_2 > \ldots > p_{i_{\min}-1} > p_{i_{\min}} < p_{i_{\min}+1} < \ldots < p_N < p_1.$$

Example 4.3. Let $\mathcal{K}_i \subset \mathbb{R}^2$, $1 \leqslant i \leqslant N$ be as in example 4.1 and consider the functions $W_i : \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}$, $W_i(A) = \frac{1}{2}(\lambda_i + \mu_i)\left|\operatorname{tr} A\right|^2 + \mu_i \left|A^D\right|^2$, where $\mu_i > 0$, $\lambda_i + \mu_i > 0$ and $A^D = A - \frac{1}{2}(\operatorname{tr} A)I$. The functions W_i describe the elastic energy density for homogeneous, isotropic, linear elastic materials with Lamé constants λ_i, μ_i if A is replaced by $\varepsilon(u)$. If there exists an index $i_{\min} \in \{2,\ldots,N\}$ such that

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{i_{\min}} \leqslant \mu_{i_{\min}+1} \leqslant \cdots \leqslant \mu_N \leqslant \mu_1,$$
$$\lambda_1 + \mu_1 \geqslant \lambda_2 + \mu_2 \geqslant \ldots \geqslant \lambda_{i_{\min}} + \mu_{i_{\min}} \leqslant \lambda_{i_{\min}+1} + \mu_{i_{\min}+1} \leqslant \cdots \leqslant \lambda_N + \mu_N \leqslant \lambda_1 + \mu_1$$

and $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, then the functions W_i are distributed quasi-monotonely. This generalizes the definition of quasi-monotonicity for the coefficients of Lamé's equation in [14, definition 5.1].

Example 4.4. Let $\mathcal{K}_i \subset \mathbb{R}^2$, $1 \leqslant i \leqslant N$ be as in example 4.1. Consider the functions $W_i : \mathbb{R}^s \to \mathbb{R}$ with $W_i(A) = C_i A \cdot A$, where $C_i \in \mathbb{R}^{s\times s}$ is symmetric and positive definite. Let λ_i be the smallest and Λ_i the largest eigenvalue of C_i. If there exists $i_{\min} \in \{2,\ldots,N\}$ such that

$$\lambda_1 \geqslant \Lambda_2 \geqslant \lambda_2 \geqslant \Lambda_3 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_{i_{\min}-1} \geqslant \Lambda_{i_{\min}} \leqslant \lambda_{i_{\min}+1} \leqslant \Lambda_{i_{\min}+1} \leqslant \ldots \lambda_N \leqslant \Lambda_N \leqslant \lambda_1 \tag{21}$$

and $-\mathcal{K}_1 \cap \mathcal{K}_{i_{\min}} \neq \emptyset$, then the functions W_i are distributed quasi-monotonely. Condition (21) can be weakened if more details are known on the eigenvectors of the matrices C_i. Note, that example 4.3 is a special case of this example.
If $s = 2$, then the corresponding boundary transmission problem reads as follows for $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$: $\operatorname{div}(C_i \nabla u) + f = 0$ in Ω_i together with boundary and transmission conditions. These equations describe transmission problems for anisotropic Laplace operators.

Example 4.5. Consider a cube which is decomposed into two subdomains as in figure 2 (left). Any two functions $W_i : \mathbb{R}^s \to \mathbb{R}$ which satisfy either a) or b)

a) $\exists k_1, k_2 \in \mathbb{R}$: $\forall A \in \mathbb{R}^s$: $W_1(A) + k_1 \geqslant W_2(A) + k_2$
b) $\exists k_1, k_2 \in \mathbb{R}$: $\forall A \in \mathbb{R}^s$: $W_1(A) + k_1 \leqslant W_2(A) + k_2$

are quasi-monotonely distributed. In the filled Fichera-corner, see figure 2 (right), the quasi-monotonicity condition is satisfied, if e.g. $W_1(A) + k_1 \leqslant W_2(A) + k_2 \leqslant W_3(A) + k_3$ for every $A \in \mathbb{R}^s$. For this case, a possible choice of the vectors e_i is indicated in figure 2.

The next definition describes quasi-monotonicity for the case, when the cones $\mathcal{K}_i$ do not fill $\mathbb{R}^d$ completely. The definition depends on the kind of the prescribed boundary conditions.

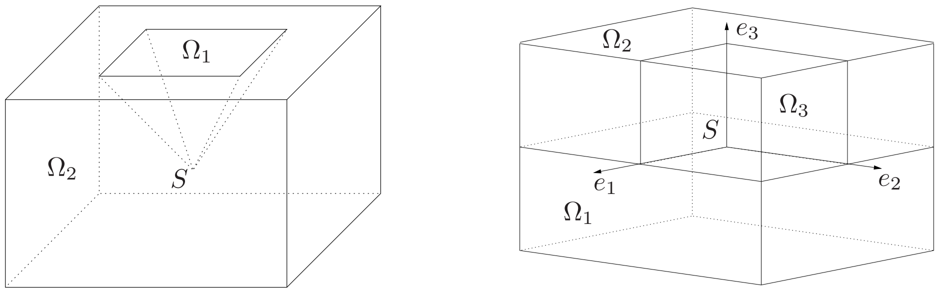

Figure 2: Examples for interior cross points

Definition 4.3 (Quasi-monotonicity for cross points on the boundary). *Let $\mathcal{K}_i \subset \mathbb{R}^d$, $1 \leqslant i \leqslant N$, be pairwise disjoint polyhedral cones with tip in 0, $\mathcal{C}_i = \mathcal{K}_i \cap \partial B_1(0)$. Set $\mathcal{C} := \operatorname{int}\left(\bigcup_{i=1}^N \overline{\mathcal{C}_i}\right)$ and assume that $\mathcal{C}_0 := \partial B_1(0) \backslash \overline{\mathcal{C}}$ is not the empty set. Further let $\mathcal{K} := \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \mathcal{C}\}$ and $\mathcal{K}_0 := \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \mathcal{C}_0\}$. Suppose that $\mathcal{K}$ has a Lipschitz boundary and consider N functions $W_i : \mathbb{R}^s \to \mathbb{R}$ for $1 \leqslant i \leqslant N$ and a fixed $s \geqslant 2$.*

<u>Dirichlet conditions on $\partial\mathcal{K}$:</u> Choose $W_0(A) := \infty$ for $A \in \mathbb{R}^s$. The functions $W_i : \mathbb{R}^s \to \mathbb{R}$, $1 \leqslant i \leqslant N$, are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$, $1 \leqslant i \leqslant N$, if the functions $W_0, W_1, \ldots, W_N$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_0, \ldots, \mathcal{K}_N$ in the sense of definition 4.2.

<u>Neumann conditions on $\partial\mathcal{K}$:</u> Choose $W_0(A) := -\infty$ for $A \in \mathbb{R}^s$. The functions $W_i : \mathbb{R}^s \to \mathbb{R}$, $1 \leqslant i \leqslant N$, are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$, $1 \leqslant i \leqslant N$, if the functions $W_0, W_1, \ldots, W_N$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_0, \ldots, \mathcal{K}_N$ in the sense of definition 4.2.

<u>Mixed conditions on $\partial\mathcal{K}$:</u> Assume that $\partial\mathcal{C} = \overline{\gamma_D} \cup \overline{\gamma_N}$, where γ_D and γ_N are nonempty, open and disjoint sets; $\Gamma_D = \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \gamma_D\}$, $\Gamma_N = \{x \in \mathbb{R}^d : \frac{x}{|x|} \in \gamma_N\}$. The functions $W_1, \ldots, W_N : \mathbb{R}^s \to \mathbb{R}$ are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ and the splitting of the boundary into Γ_D and Γ_N if there holds:
There exist two disjoint polyhedral cones $\mathcal{K}_{-\infty}, \mathcal{K}_\infty$ with $\overline{\mathcal{K}_0} = \overline{\mathcal{K}_{-\infty}} \cup \overline{\mathcal{K}_\infty}$ and $\Gamma_D \subset \partial\mathcal{K}_\infty, \Gamma_N \subset \partial\mathcal{K}_{-\infty}$, such that the functions $W_{-\infty}, W_\infty, W_1, \ldots, W_N$ with $W_{-\infty}(A) = -\infty$, $W_\infty(A) = \infty$, are distributed quasi-monotonely with respect to the cones $\mathcal{K}_\infty, \mathcal{K}_{-\infty}, \mathcal{K}_1, \ldots, \mathcal{K}_N$ in the sense of definition 4.2.

Remark 4.1. It follows from definition 4.3 that for every $h > 0, 1 \leqslant l \leqslant d$:

$$\begin{aligned} x + he_l \notin \mathcal{K} \quad & \text{for every } x \in \Gamma_D, \\ x + he_l \in \overline{\mathcal{K}} \quad & \text{for every } x \in \Gamma_N. \end{aligned}$$

The next lemma reformulates definition 4.3 for the two dimensional case. Assume, that $\mathcal{K} \subset \mathbb{R}^2$ is given in the following way (polar coordinates): There exist angles $\Phi_0 < \Phi_1 < \ldots < \Phi_N < \Phi_0 + 2\pi$ such that $\mathcal{K}_i = \{x \in \mathbb{R}^2 : r > 0, \Phi_{i-1} < \varphi < \Phi_i\}$, $\mathcal{K} = \{x \in \mathbb{R}^2 : r > 0, \Phi_0 < \varphi < \Phi_N\}$ and $\mathcal{K}_0 = \{x \in \mathbb{R}^2 : r > 0, \Phi_N < \varphi < \Phi_0 + 2\pi\}$.

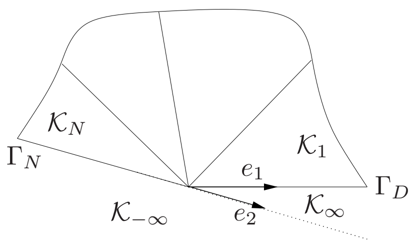

Figure 3: Two dimensional domain with mixed boundary conditions

Lemma 4.2. *Consider N functions $W_i : \mathbb{R}^s \to \mathbb{R}$, $1 \leqslant i \leqslant N$.*
Dirichlet conditions on $\partial\mathcal{K}$: Let $\partial\mathcal{K} \subset \Gamma_D$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if

1. *There exist constants $k_1, \dots, k_N \in \mathbb{R}$ and $i_{min} \in \{1, \dots, N\}$ such that for every $A \in \mathbb{R}^s$:*

$$W_1(A) + k_1 \geqslant \ldots \geqslant W_{i_{min}}(A) + k_{i_{min}} \leqslant \ldots \leqslant W_N(A) + k_N.$$

2. *There exists $\vec{t} \in \mathbb{R}^2$ such that $\vec{t} \in \mathcal{K}_{i_{min}}$ and $-\vec{t} \in \mathcal{K}_0$.*

Neumann conditions on $\partial\mathcal{K}$: Let $\partial\mathcal{K} \subset \Gamma_N$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if

1. *There exist constants $k_1, \dots, k_N \in \mathbb{R}$ and $i_{max} \in \{1, \dots, N\}$ such that for every $A \in \mathbb{R}^s$:*

$$W_1(A) + k_1 \leqslant \ldots \leqslant W_{i_{max}}(A) + k_{i_{max}} \geqslant \ldots \geqslant W_N(A) + k_N.$$

2. *There exists $\vec{t} \in \mathbb{R}^2$ such that $\vec{t} \in \mathcal{K}_{i_{max}}$ and $-\vec{t} \in \mathcal{K}_0$.*

Mixed conditions on $\partial\mathcal{K}$: Assume that $\partial\mathcal{K} \cap \partial\mathcal{K}_1 \subset \Gamma_D$ and $\partial\mathcal{K}_N \cap \partial\mathcal{K} \subset \Gamma_N$. The functions W_i are distributed quasi-monotonely with respect to the cones $\mathcal{K}_i$ if and only if

1. *There exist constants $k_i \in \mathbb{R}$ such that $W_1(A) + k_1 \geqslant W_2(A) + k_2 \geqslant \ldots \geqslant W_N(A) + k_N$.*

2. *$\measuredangle(\Gamma_D, \Gamma_N) = \Phi_N - \Phi_0 < \pi$, $\measuredangle$ denotes the interior opening angle.*

Proof. The assertions for the case of pure Dirichlet or Neumann conditions on $\partial\mathcal{K}$ follow directly from definition 4.3 in combination with lemma 4.1.
In the case of mixed boundary conditions assume, that *1.* and *2.* in lemma 4.2 hold. Then a possible choice for $e_1, e_2, \mathcal{K}_\infty, \mathcal{K}_{-\infty}$ is the following, see also figure 3: $e_1 = \binom{\cos\Phi_0}{\sin\Phi_0}$, $e_2 = \binom{\cos(\Phi_N+\pi)}{\sin(\Phi_N+\pi)}$, $\mathcal{K}_{-\infty} = \{x : r > 0, \Phi_N < \varphi < \Phi_N + \pi\}$ and $\mathcal{K}_\infty = \{x : r > 0, \Phi_N + \pi < \varphi < \Phi_0 + 2\pi\}$.
On the other hand, if the functions W_i satisfy definition 4.3, part *3.*, for some cones $\mathcal{K}_\infty, \mathcal{K}_{-\infty}$ and a basis e_1, e_2, then $\operatorname{int} \overline{\mathcal{K}_\infty \cup \mathcal{K}_{-\infty}} = \{x : r > 0, \Phi_N < \varphi < \Phi_0 + 2\pi\}$ and lemma 4.1 states, that there exists $\vec{t} \in \mathbb{R}^2$ with $\vec{t} \in \mathcal{K}_\infty$ and $-\vec{t} \in \mathcal{K}_{-\infty}$. This shows, that $\Phi_0 + 2\pi - \Phi_N > \pi$. The remaining part of lemma 4.2 again follows by lemma 4.1 with $\mathcal{K}_{i_{\max}} = \mathcal{K}_\infty$, $\mathcal{K}_{i_{\min}} = \mathcal{K}_{-\infty}$. $\square$

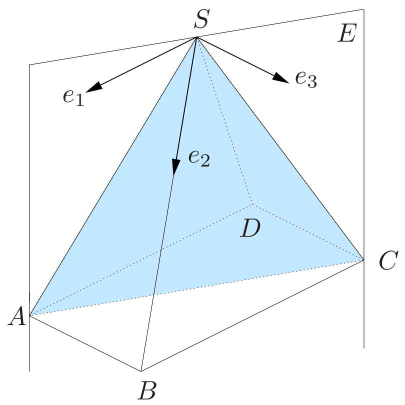

Figure 4: Example for mixed boundary conditions

Example 4.6. Assume that $\mathcal{K}, \mathcal{K}_i \subset \mathbb{R}^2$, $1 \leqslant i \leqslant N$, are given as in lemma 4.2 and that the numbering is counterclockwise. Consider the functions $W_i(A) = \frac{\mu_i}{2} |A|^2$, $\mu_i > 0$, $A \in \mathbb{R}^2$. These functions are distributed quasi-monotonely if there exists $i_0 \in \{1, \ldots, N\}$ such that

$$\begin{aligned} &\mu_1 \geqslant \ldots \geqslant \mu_{i_0} \leqslant \ldots \leqslant \mu_N && \text{in the Dirichlet case,} \\ &\mu_1 \leqslant \ldots \leqslant \mu_{i_0} \geqslant \ldots \geqslant \mu_N && \text{in the Neumann case,} \end{aligned}$$

and $-\mathcal{K}_{i_0} \cap \mathcal{K}_0 \neq \emptyset$. In the case of mixed boundary conditions with $\Gamma_D \subset \partial\mathcal{K}_1$ and $\Gamma_N \subset \partial\mathcal{K}_N$ the parameters μ_i are distributed quasi-monotonely if

$$\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_N$$

and $\measuredangle(\Gamma_D, \Gamma_N) < \pi$, where $\measuredangle$ denotes the interior opening angle.

In the same way, examples 4.2–4.4 can be carried over to the case of a cross point on the boundary.

Example 4.7. Mixed boundary conditions on one subdomain, $d = 3$: Consider the pyramid $\mathcal{K}$, given by A, B, C, D, S, in figure 4 with $AB \parallel CD$, $BC \parallel AD$ and let $N = 1$ (only one subdomain). Assume, that the faces ABS and BCS are parts of the Dirichlet boundary and CDS and DAS are parts of the Neumann boundary. Let $W : \mathbb{R}^{m\times 3} \to \mathbb{R}$ satisfy **H1**. Then one can find a basis $e_1, \ldots, e_3$ and cones $\mathcal{K}_{-\infty}, \mathcal{K}_\infty$ such that the assumptions in definition 4.3, part *3.* are satisfied with $N = 1$. A possible choice is plotted in figure 4, where $e_1 \parallel BC$, $e_3 \parallel AB$ and $e_2 \parallel SB$. $\mathcal{K}_{-\infty}$ can be chosen as the complementary of $\mathcal{K}$ in the rear half space with respect to the plane E. Furthermore $\mathcal{K}_\infty = \mathbb{R}^3 \backslash \overline{\mathcal{K} \cup \mathcal{K}_{-\infty}}$. This example shows, that for $N = 1$ and mixed boundary conditions the assumptions in definition 4.3 for this case are slightly weaker than the assumptions in [8, 9]. There, for $d = 3$ at most three faces may intersect at points S with changing boundary conditions.

4.2 Regularity of weak solutions of the transmission problem

Consider the transmission problem (14). The assumptions for the main theorem are as follows:

A1 $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, is a polygonal or polyhedral domain with Lipschitz boundary, $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$, Γ_D and Γ_N open and disjoint. Furthermore, $\overline{\Omega} = \cup_{i=1}^M \overline{\Omega_i}$, where Ω_i is a polyhedral domain with Lipschitz boundary, $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$.

A2 For $1 \leqslant i \leqslant M$, $W_i : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies **H0-H4** for some $p_i \in (1,\infty)$ and $\kappa_i \in \{0,1\}$.

A3 There exists a finite number of balls $B_l(x_l)$ with center $x_l \in \overline{\Omega}$ such that $\Omega \subset \bigcup_l B_l(x_l)$ and $\Omega \cap B_l(x_l)$ coincides with an appropriate polyhedral cone $\mathcal{K}_l$ with tip in x_l, i.e. $\overline{\Omega} \cap B_l(x_l) = \overline{\mathcal{K}_l} \cap B_l(x_l)$. Let $\Omega_{l,1}, \dots, \Omega_{l,N(l)}$ be those subdomains of Ω with $x_l \in \overline{\Omega_{l,j}}$, $1 \leqslant j \leqslant N(l)$, and $W_{l,1}, \dots, W_{l,N(l)}$ the corresponding energy densities. We assume, that there exist $N(l)$ pairwise disjoint polyhedral cones $\mathcal{K}_{l,j}$ with tip in x_l, such that

$$\overline{\mathcal{K}_l} = \bigcup_{j=1}^{N(l)} \overline{\mathcal{K}_{l,j}} \text{ and } \overline{\mathcal{K}_{l,j}} \cap B_l(x_l) = \overline{\Omega_{l,j}} \cap B_l(x_l) \text{ for } 1 \leqslant j \leqslant N(l).$$

On each of the composed cones $\mathcal{K}_l$, the corresponding energy densities $W_{l,j}$, $1 \leqslant j \leqslant N(l)$, are distributed quasi-monotonely.

A4 $f \in L^{\vec{q}}(\Omega)$ where $q_i = p_i' = \frac{p_i}{p_i - 1}$.

A5 Dirichlet-datum: $u\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ where g is an element of $W^{2,(\vec{p},p_{\max})}(\hat{\Omega})$ with $\nabla g \in L^\infty(\hat{\Omega})$ for some domain $\hat{\Omega} \supset\supset \Omega$. The space $W^{2,(\vec{p},p_{\max})}(\hat{\Omega})$ is defined as follows: $W^{2,(\vec{p},p_{\max})}(\hat{\Omega}) = \left\{ g \in W^{2,p_{\min}}(\hat{\Omega}) :\ g\big|_{\Omega_i} \in W^{2,p_i}(\Omega_i),\ g\big|_{\hat{\Omega}\setminus\Omega} \in W^{2,p_{\max}}(\hat{\Omega}\setminus\Omega) \right\}$ and $p_{\max} = \max_i\{p_i\}$.

A6 Neumann-datum: $H \in W^{1,\vec{q}}(\Omega, \mathbb{R}^{m\times d}) \cap L^\infty(\Omega, \mathbb{R}^{m\times d})$ and $D_A W_i(\nabla u)\vec{n} = H\vec{n}$ on Γ_N.

The assumption, that the Dirichlet-datum g is defined on a larger region $\hat{\Omega} \supset\supset \Omega$ is for technical reasons. Note, that for the Neumann-datum no extension to $\hat{\Omega}$ is needed.

Theorem 4.1 (Main Theorem). *Assume that assumptions* **A1-A6** *are satisfied and that $u \in W^{1,\vec{p}}(\Omega)$ is a weak solution of problem (14). Then for every $\epsilon, \delta > 0$ and $1 \leqslant i \leqslant M$, there holds:*

$$\text{if } p_i \in (1,2]: \qquad u\big|_{\Omega_i} \in \mathcal{N}^{\frac{3}{2}, r_i - \epsilon}(\Omega_i) \cap W^{\frac{3}{2}-\delta, r_i}(\Omega_i), \tag{22}$$

$$\text{if } p_i \in [2,\infty): \qquad u\big|_{\Omega_i} \in \mathcal{N}^{1+\frac{1}{p_i}, p_i}(\Omega_i) \subset W^{1+\frac{1}{p_i}-\epsilon, p_i}(\Omega_i), \tag{23}$$

with $r_i = \frac{2dp_i}{2d-2+p_i}$. Note, that $p_i \leqslant r_i \leqslant 2$ for $p_i \in (1,2]$. Furthermore, if $p_i \in [2,\infty)$ and $\kappa_i = 1$ in **H4**, *then*

$$u\big|_{\Omega_i} \in \mathcal{N}^{\frac{3}{2},2}(\Omega_i) \cap \mathcal{N}^{1+\frac{1}{p_i}, p_i}(\Omega_i). \tag{24}$$

If $p_i \in (1,2]$ for every $i \in \{1, \dots, M\}$, then

$$u \in \mathcal{N}^{\frac{3}{2}, r_{min} - \epsilon}(\Omega) \tag{25}$$

globally, where $r_{min} = \frac{2dp_{min}}{2d-2+p_{min}}$.

Before we prove the main theorem in section 4.4, we first give some corollaries and remarks and compare the results in theorem 4.1 with known results for linear elliptic boundary-transmission problems.

Remark 4.2. Theorem 4.1 has local character, that means: If there is a subset $\tilde{\Omega} \subset \Omega$, for which the assumptions of theorem 4.1 are satisfied, then $u\big|_{\tilde{\Omega}}$ has the regularity which is given in theorem 4.1.

Corollary 4.2. *Let the assumptions be the same as in theorem 4.1 with $p_i \in (1,2]$ for every $i \in \{1,\ldots,M\}$ and assume that $d = 2$. Then by lemma 2.4 and the standard embedding theorems for Sobolev-Slobodeckii spaces:*

$$u \in W^{\frac{3}{2}-\epsilon,\frac{4p_{min}}{2+p_{min}}}(\Omega) \subset \mathcal{C}(\overline{\Omega}) \qquad \textit{for every } \epsilon > 0, \ \textit{small.}$$

Remark 4.3. In the case $M = 1$, i.e. the problem reduces to a boundary value problem on a single domain, the result of theorem 4.1 is well known for $p_i \in (1,2]$ (if $g = 0$ and $\kappa = 0$ in **H4**) and is derived by C. Ebmeyer and J. Frehse in [10, 9]. For $p > 2$, theorem 4.1 sharpens the results in [9]. In the proof, Ebmeyer and Frehse developed and applied a difference quotient technique, which will be adapted for the proof of theorem 4.1. In the case of two coupled nonlinear elliptic systems with a plane interface, $p_1 = p_2 = 2$ and pure Dirichlet conditions, theorem 4.1 is a special case of the results in [11]. There, the authors require a geometric condition, but they do not need a quasi-monotone distribution of the energy densities W_i.

Remark 4.4. Assume, that $m = d$ and that $D_A W_i(B)$ is symmetric for symmetric $B \in \mathbb{R}^{d\times d}$. Then theorem 4.1 also holds if in equation (14) ∇u is replaced by $\varepsilon(u)$. The necessary changes in the proof will be indicated. Therefore, transmission problems for linear and special classes of physically nonlinear elastic materials are covered as well by theorem 4.1.

Remark 4.5. There exist higher local regularity results and results for smooth interfaces, see for example [31, 23], where for the case $\kappa_i = 0$ in assumption **H4** and $1 < p_i < 2$ the regularity $u\big|_{\tilde{\Omega}_i} \in W^{2,\frac{dp_i}{d-2+p_i}}(\tilde{\Omega}_i)$ is derived for $\tilde{\Omega}_i \subset\subset \Omega_i$. The same result is obtained at plane parts of the boundary of Ω_i, if assumption **H3** is replaced by **H3'**: $\left|D_A^2 W_i(A)\right| \leqslant c^i \left|A\right|^{p_i-2}$, see [32].

Example 4.8. (Coupling of a linear with a nonlinear equation) Consider an L-shaped domain $\Omega \subset \mathbb{R}^2$ which is decomposed into two subdomains Ω_1, Ω_2; $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ (see figure 5). The functions $W_i : \mathbb{R}^{m\times 2} \to \mathbb{R}$ are chosen as follows ($A \in \mathbb{R}^{m\times 2}$):

$$W_1(A) = \frac{1}{2}(C_1 A) : A \quad \text{for a fixed } C_1 \in \mathbb{R}^{(m\times 2)\times(m\times 2)}, \text{ symmetric and positive definite,}$$
$$W_2 : \mathbb{R}^{m\times 2} \to \mathbb{R} \quad \text{satisfies } \mathbf{H0\text{-}H4} \text{ for some } p_2 \in (1,\infty),\ p_2 \neq 2.$$

The corresponding boundary-transmission problem for $u : \Omega \to \mathbb{R}^m$ reads:

$$\begin{aligned} \operatorname{div}\,(C_1 \nabla u) + f_1 &= 0 \quad \text{in } \Omega_1, \\ \operatorname{div}\,(D_A W_2(\nabla u)) + f_2 &= 0 \quad \text{in } \Omega_2, \end{aligned}$$

together with boundary and transmission conditions. Assume, that the given data f, g, h satisfy the assumptions of theorem 4.1. Choose $S_0 \in \Gamma_{12}\backslash\{S_1, S_2\}$. Since $p_1 = 2 \neq p_2$, it

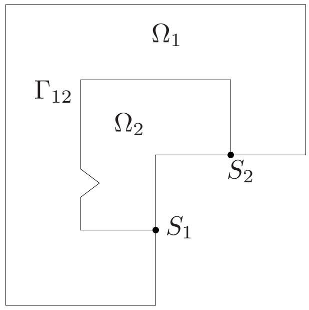

Figure 5: L-shaped domain

follows, that the energy densities W_1 and W_2 are distributed quasi-monotonely with respect to S_0, see example 4.2. Let $U(S_0) \subset \Omega$ be a neighborhood of S_0 with $\overline{U(S_0)} \cap \partial\Omega = \emptyset$. Then theorem 4.1 can be applied to $u\big|_{U(S_0)}$ and one obtains for every $\delta > 0$:

$$u\big|_{U(S_0)\cap\Omega_1} \in W^{\frac{3}{2}-\delta,2}(U(S_0) \cap \Omega_1), \tag{26}$$

$$u\big|_{U(S_0)\cap\Omega_2} \in \begin{cases} W^{\frac{3}{2}-\delta,\frac{4p_2}{2+p_2}}(U(S_0) \cap \Omega_2) & \text{if } p_2 < 2, \\ W^{1+\frac{1}{p_2}-\delta,p_2}(U(S_0) \cap \Omega_2) & \text{if } p_2 > 2. \end{cases} \tag{27}$$

This example illustrades, that in the general case of two polygonal or polyhedral subdomains with Lipschitz boundaries, where a linear PDE ($p_1 = 2$) is coupled with a nonlinear PDE ($p_2 \neq 2$), the quasimonotonicity condition **A3** is satisfied at every point $S_0 \in \Gamma_{12} \backslash \partial\Omega$. Therefore, theorem 4.1 can be applied locally in a neighborhood of these points S_0.

4.3 Comparison to results for linear elliptic boundary-transmission problems

For simplicity assume $d = 2$ and $m \in \{1, 2\}$. Let $\Omega \subset \mathbb{R}^2$, $\Omega_i = \cup_{i=1}^M \Omega_i$, be a polygonal domain and choose $B_i \in \mathrm{Lin}(\mathbb{R}^{m\times 2}, \mathbb{R}^{m\times 2})$ symmetric and positive definite. For $u_i : \Omega_i \to \mathbb{R}^m$ set

$$W_i(u_i) := \begin{cases} \frac{1}{2} B_i(\nabla u_i) \cdot \nabla u_i & \text{if } m = 1, \\ \frac{1}{2} B_i(\varepsilon(u_i)) : \varepsilon(u_i) & \text{if } m = 2, \end{cases} \qquad F_i(Du_i) := \begin{cases} B_i \nabla u_i & \text{if } m = 1, \\ B_i(\varepsilon(u_i)) & \text{if } m = 2. \end{cases}$$

Due to the assumptions on B_i, the operator $\operatorname{div} F_i(Du_i)$ is linear and elliptic. Consider the following boundary transmission problem for f, g, h as in theorem 4.1 ($p_i = 2$):

$$\begin{aligned} \operatorname{div} F_i(Du_i) + f &= 0 && \text{in } \Omega_i, \\ u_i - u_j &= 0 && \text{on } \Gamma_{ij}, \\ F_i(Du_i)\vec{n}_{ij} + F_j(Du_j)\vec{n}_{ji} &= 0 && \text{on } \Gamma_{ij}, \\ u_i &= g && \text{on } \partial\Omega_i \cap \Gamma_D, \\ F_i(Du_i)\vec{n}_i &= h && \text{on } \partial\Omega_i \cap \Gamma_N. \end{aligned}$$

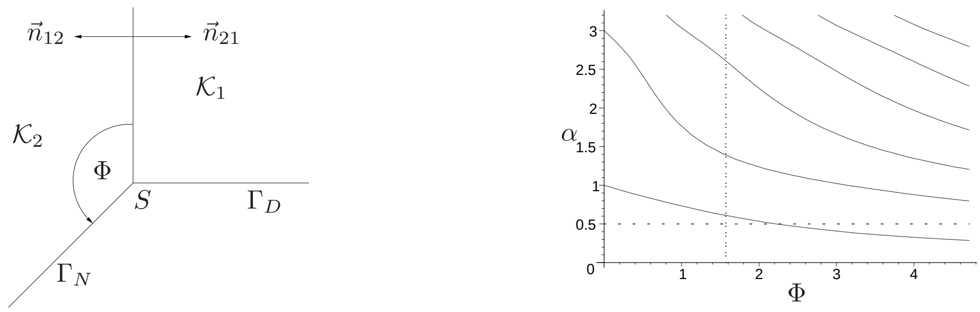

Figure 6: Domain and singular exponents for example 4.9

For $m = 2$ these equations can be interpreted as the field equations of coupled linear elastic bodies with elasticity matrices B_i. The regularity theory for linear elliptic boundary transmission problems states, that every weak solution $u \in W^{1,2}(\Omega)$ with $u_i = u\big|_{\Omega_i}$ has an asymptotic expansion of the following form in the neighborhood of interior cross points S or cross points on the boundary (polar coordinates r, φ with respect to S are used) [3, 13, 16, 17, 20, 24, 26, 27, 28]:

$$\eta^S u = \eta^S u_{\text{reg}} + \eta^S \sum_{\operatorname{Re}\alpha \in (0,1)} r^\alpha \, v_\alpha^S(\ln r, \varphi), \tag{28}$$

where η^S is a cut-off function, $\eta^S u_{\text{reg}}\big|_{\Omega_i} \in W^{2,2}(\Omega_i)$ and α is an eigenvalue of a corresponding eigenvalue problem, for details see e.g. [3, 26, 27, 28]. The functions $v_\alpha^S(\ln r, \varphi)$ contain in general powers of $\ln r$ and generalized eigenfunctions. It holds, that $r^\alpha \, v_\alpha^S\big|_{\Omega_i} \in W^{1+\operatorname{Re}\alpha-\epsilon,2}(\Omega_i)$ for arbitrary $\epsilon > 0$, see [13, Thm. 1.4.5.3].

Assume now, that the matrices B_i are distributed quasi-monotonely with respect to the cross point S. A sufficient condition for this is described in example 4.4. Then by theorem 4.1: $\eta^S u\big|_{\Omega_i} \in W^{\frac{3}{2}-\epsilon,2}(\Omega_i)$ and $\eta^S u \in W^{\frac{3}{2}-\epsilon,2}(\Omega)$ for every $\epsilon > 0$. It follows, that $\operatorname{Re}\alpha \geqslant \frac{1}{2}$ in the asymptotic expansion (28). In an earlier work, estimates for the eigenvalues were derived for Poisson's and Lamé's equations with piecewise constant coefficients. There, the same assumptions as in theorem 4.1 were used and by a homotopy argument it was proved, that $\operatorname{Re}\alpha > \frac{1}{2}$, [14]. This indicates, that the results in theorem 4.1 are nearly optimal (up to ϵ).

The following linear example shows, that if the assumptions of theorem 4.1 are violated, then one cannot expect the regularity $\eta^S u_i \in W^{\frac{3}{2}-\epsilon,2}(\Omega_i)$.

Example 4.9. Consider a domain $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \subset \mathbb{R}^2$, where Ω_1 and Ω_2 coincide in the neighborhood of $S = (0,0)$ with the cones (polar coordinates, figure 6):

$$\begin{aligned}
\mathcal{K}_1 &= \{x \in \mathbb{R}^2 : |x| > 0,\ 0 < \varphi < \frac{\pi}{2}\}, \\
\mathcal{K}_2 &= \{x \in \mathbb{R}^2 : |x| > 0, \frac{\pi}{2} < \varphi < \frac{\pi}{2} + \Phi\}, \quad \Phi > 0.
\end{aligned}$$

Dirichlet-conditions are prescribed on $\partial\Omega \cap \partial\mathcal{K}_1$, Neumann-conditions on $\partial\Omega \cap \partial\mathcal{K}_2$. The problem under consideration is: Find a solution of the following linear boundary transmission problem for the Poisson equation with piecewise constant coefficients $\mu_1, \mu_2 > 0$:

$$\begin{aligned} \mu_i \triangle u_i + f_i &= 0 \quad \text{in } \Omega_i,\ i = 1, 2, \\ u &= g \quad \text{on } \Gamma_D, \\ \frac{\partial u}{\partial \vec{n}} &= h \quad \text{on } \Gamma_N, \\ u_1 - u_2 &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2, \\ \mu_1 \frac{\partial u_1}{\partial \vec{n}_{12}} + \mu_2 \frac{\partial u_2}{\partial \vec{n}_{21}} &= 0 \quad \text{on } \partial\Omega_1 \cap \partial\Omega_2. \end{aligned}$$

Let the data f_i, g, h satisfy the assumptions of theorem 4.1 with $p_1 = p_2 = 2$. Weak solutions of this boundary transmission problem admit an asymptotic expansion of the following type near the cross point S, [28]:

$$\eta^S(x) u(x) = u_{\text{reg}}(x) + \eta^S(x) \sum_{0<\alpha<1} c_\alpha \, |x|^\alpha \, v_\alpha(\varphi),$$

where η^S is a cut-off function with respect to S, $u_{\text{reg}}\big|_{\Omega_i} \in W^{2,2}(\Omega_i)$, c_α are constants which are determined by the data f_i, g, h; α is the singular exponent and v_α the corresponding eigenfunction. Note, that the singular exponents are real numbers in our special case and that there are no logarithmic terms in the singular expansion. The singular exponents α solve the following equation, [28]:

$$-\mu_2 \sin(\alpha\Phi) \sin(\alpha\frac{\pi}{2}) + \mu_1 \cos(\alpha\Phi) \cos(\alpha\frac{\pi}{2}) = 0.$$

Choose $\mu_1 = 1, \mu_2 = \frac{1}{2}$. For $\Phi < \frac{\pi}{2}$, the quasi-monotonicity condition in theorem 4.1 is satisfied and therefore the smallest positive singular exponent $\alpha_{\min}$ is larger than or equal to $\frac{1}{2}$. For $\Phi \geqslant \frac{\pi}{2}$, the quasi-monotonicity condition is violated and if Φ is large enough, one obtains $\alpha_{\min} < \frac{1}{2}$. In this case, one can guarantee $u\big|_{\Omega_i} \in W^{1+\alpha_{\min}-\epsilon,2}(\Omega_i)$, only. The behavior of the singular exponents is illustrated in figure 6, where the exponents α are plotted versus the opening angle Φ of subdomain Ω_2.

4.4 Proof of main theorem 4.1

In the proof of the main theorem, a difference quotient technique is used. This technique is frequently applied to derive interior regularity results, [25, 35, 4, 31, 23], and is modified by C.Ebmeyer and J. Frehse, [10, 8], in order to prove global regularity results on polygonal or polyhedral domains. The main idea is to insert test functions of the form $\xi_j(x) = \varphi^2(u(x + he_j) - u(x))$ into the weak formulation and to apply the convexity inequality (50) from the Appendix. This leads to estimates in Nikolskii-spaces and by the embedding-lemma 2.4 to regularity results in Sobolev-Slobodeckij spaces. The main problem is, that the differences $u(x + he_j) - u(x)$ are taken across the interfaces and one has to check whether ξ_j is an admissible test function in $V^{\vec{p}}(\Omega)$. Due to the quasi-monotonicity condition, there exists a basis $\{e_j,\ 1 \leqslant l \leqslant d\} \subset \mathbb{R}^d$, such that the functions ξ_j are indeed admissible test functions. Furthermore, in the proof occur differences of the form $W_i(\nabla u(x)) - W_j(\nabla u(x))$, which have to be estimated in an appropriate way. Here, the quasi-monotonicity condition is also very useful. The proof is organized as follows: The case of pure Dirichlet-conditions will be proved in detail. For the remaining cases (Neumann, mixed and pure interface problems) the necessary changes in the proof will be indicated.

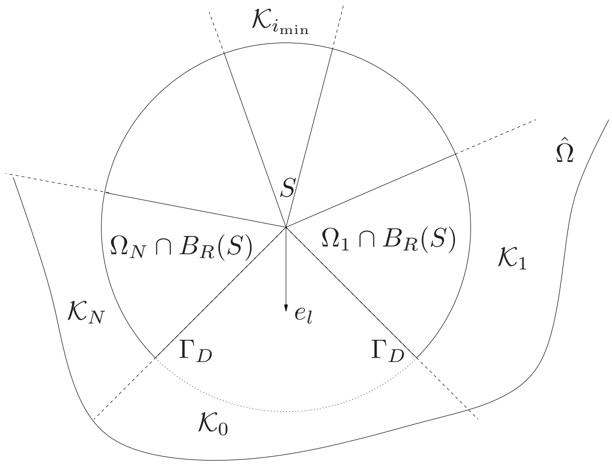

Figure 7: Example for the notation with Dirichlet conditions

Cross point on the boundary of Ω with pure Dirichlet conditions

Let $S \subset \partial\Omega$ and assume, that there exists $R > 0$ such that $B_R(S) \subset \hat{\Omega}$ and $\Omega \cap B_R(S) = \mathcal{K} \cap B_R(S)$, where $\mathcal{K}$ is an appropriate polyhedral cone with tip in S and $\partial\mathcal{K} \cap B_R(S) \subset \Gamma_D$. Assume further, that for every $j \in \{1, \ldots, M\}$ with $\Omega_j \cap B_R(S) \neq \emptyset$ there exists a polyhedral cone $\mathcal{K}_j$ with tip in S, such that $\Omega_j \cap B_R(S) = \mathcal{K}_j \cap B_R(S)$. Note, that after a suitable renumbering, $\overline{\mathcal{K}} = \bigcup_{i=1}^N \overline{\mathcal{K}_i}$, see also figure 7. Due to the assumptions in theorem 4.1, the cones $\mathcal{K}_i$ and functions W_i, $1 \leqslant i \leqslant N$, satisfy the quasi-monotonicity conditions in definition 4.3, part *1.*; $\mathcal{K}_0 := \mathbb{R}^d \backslash \overline{\mathcal{K}}$.

Let $u \in W^{1,\vec{p}}(\Omega)$ be a weak solution of problem (14) with right hand sides g, f, h as in theorem 4.1; $R''' = R/2, h_0 = R'' = R/4, R' = R/8$. Choose $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ with $\operatorname{supp}\varphi \subset B_{R''}(S)$, $\varphi\big|_{B_{R'}(S)} = 1$ and $0 \leqslant \varphi \leqslant 1$. Let further be e_l one of the basis vectors given by definition 4.3. For the definition of an appropriate test function, an extension of u across the Dirichlet boundary is needed:

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ g(x) & \text{if } x \in \hat{\Omega}\backslash\Omega. \end{cases} \tag{29}$$

For the extended function $\tilde{u}$ it holds:

$$\varphi^2 \tilde{u} \in W^{1,(\vec{p},p_{\max})}(B_R(S)) = \Big\{ v \in W^{1,p_{\min}}(B_R(S)) : v\big|_{\Omega_i \cap B_R(S)} \in W^{1,p_i}(\Omega_i \cap B_R(S)), \\ v\big|_{\mathcal{K}_0 \cap B_R(S)} \in W^{1,p_{\max}}(\mathcal{K}_0 \cap B_R(S)) \Big\}.$$

This follows since $\varphi^2\tilde{u}\big|_{\mathcal{K}_i} \in W^{1,p_i}(\mathcal{K}_i)$ for $1 \leqslant i \leqslant N$, $\varphi^2\tilde{u}\big|_{\mathcal{K}_0} \in W^{1,p_{\max}}(\mathcal{K}_0)$ and since, by the definition of $\tilde{u}$, $\varphi^2\tilde{u}$ does not jump across interfaces: $\left(\varphi^2\tilde{u}\big|_{\mathcal{K}_i}\right)\big|_{\Gamma_{ij}} = \left(\varphi^2\tilde{u}\big|_{\mathcal{K}_j}\right)\big|_{\Gamma_{ij}}$ for $0 \leqslant i, j, \leqslant N$.

The regularity results (22) and (23) will be derived in two steps. In a first step we prove inequality (30) here after. This is the essential inequality from which we deduce in a second step estimates for Nikolskii-norms of $\tilde{u}$ and u.

First step: We prove the following inequality:
There is a constant $c > 0$ such that for $1 \leqslant l \leqslant d$ and $0 < h < h_0$:

$$\sum_{i=1}^{N} \int_{\Omega_i} \varphi^2(x) \left(\kappa_i + |\nabla \tilde{u}(x + he_l)| + |\nabla \tilde{u}(x)|\right)^{p_i - 2} |\tilde{u}(x + he_l) - \tilde{u}(x)|^2 \, \mathrm{d}x \leqslant ch, \tag{30}$$

with κ_i from **H4**.

Proof of inequality (30): Define as test function for $0 < h < h_0$:

$$\xi(x) = \varphi^2(x) \left(\tilde{u}(x + he_l) - g(x + he_l) - (\tilde{u}(x) - g(x))\right) \equiv \varphi^2(x) \triangle_h (\tilde{u}(x) - g(x)), \ \ x \in \Omega.$$

From the quasi-monotonicity assumptions and by corollary 4.1 it follows, that $\xi \in W^{1,\vec{p}}(\Omega)$. Furthermore, $\xi\big|_{\Gamma_D} = 0$ and therefore $\xi \in V^{\vec{p}}(\Omega)$ is an admissible test function. Inserting ξ into the variational formulation (14) and rearranging the terms yields:

$$\begin{aligned} \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u)) : \nabla(\triangle_h \tilde{u}) \, \mathrm{d}x &= \int_{\Omega} f \xi \, \mathrm{d}x + \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u) : \triangle_h \nabla g \, \mathrm{d}x \\ &\quad - \sum_{i=1}^{N} \int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h (\tilde{u} - g) \otimes \nabla \varphi^2) \, \mathrm{d}x. \end{aligned} \tag{31}$$

For $a \in \mathbb{R}^m, b \in \mathbb{R}^d$, $a \otimes b = (a_i b_j)_{ij} \in \mathbb{R}^{m \times d}$ denotes the tensor product. Inequality (50) with $A = \nabla \tilde{u}(x + he_l)$, $B = \nabla \tilde{u}(x) = \nabla u(x)$ for $x \in \Omega$, applied to the left hand side of equation (31) results in ($c > 0$ is independent of h):

$$\begin{aligned} & c \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 (\kappa_i + |\nabla \tilde{u}(x + he_l)| + |\nabla \tilde{u}(x)|)^{p_i - 2} |\triangle_h \nabla \tilde{u}(x)|^2 \, \mathrm{d}x \\ & \quad \overset{(50)}{\leqslant} \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \triangle_h W_i(\nabla \tilde{u}) \, \mathrm{d}x - \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u) : \triangle_h \nabla \tilde{u} \, \mathrm{d}x \\ & \quad \overset{(31)}{=} \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \triangle_h W_i(\nabla \tilde{u})) \, \mathrm{d}x - \int_{\Omega} f \xi \, \mathrm{d}x \\ & \qquad - \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u) : \triangle_h \nabla g \, \mathrm{d}x \\ & \qquad + \sum_{i=1}^{N} \int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h (\tilde{u} - g) \otimes \nabla \varphi^2) \, \mathrm{d}x \\ & \quad = I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{32}$$

In the next steps, the integrals $I_1, \ldots, I_4$ will be estimated. By Hölder's inequality one gets:

$$|I_2| \leqslant \sum_{i=1}^{N} \|\varphi f\|_{L^{q_i}(\Omega_i)} \|\varphi \triangle_h (\tilde{u} - g)\|_{L^{p_i}(\Omega_i)} .$$

Put $\tilde{\Omega}_i := \{x \in \mathbb{R}^d : x = y + he_l, 0 \leqslant h < h_0, y \in \Omega_i\} \supset \Omega_i$. Due to the quasi-monotonicity and the special choice of the extension of u to $\tilde{u}$, it is $(\tilde{u} - g)\big|_{\tilde{\Omega}_i} \in W^{1,p_i}(\tilde{\Omega}_i)$. This follows by arguments which are similar to those in the proof of lemma 2.2. By [12, Lemma 7.23] one obtains

$$\|\varphi \triangle_h (\tilde{u} - g)\|_{L^{p_i}(\Omega_i)} \leqslant \|\triangle_h (\tilde{u} - g)\|_{L^{p_i}(\Omega_i \cap \operatorname{supp} \varphi)} \leqslant ch \|\nabla(\tilde{u} - g)\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp} \varphi)},$$

where the constant c depends on the vector e_l but is independent of h. Therefore

$$|I_2| \leqslant ch \sum_{i=1}^{N} \|\varphi f\|_{L^{q_i}(\Omega_i)} \|\nabla(\tilde{u} - g)\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp} \varphi)}. \tag{33}$$

The same considerations can be made for I_3 and I_4 using assumption **H2** which yields $D_A W_i(\nabla u) \in L^{q_i}(\Omega_i)$. One finally gets

$$|I_3| \leqslant ch \sum_{i=1}^{N} \|\varphi D_A W_i(\nabla u)\|_{L^{q_i}(\Omega_i)} \left\|D^2 g\right\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp} \varphi)}, \tag{34}$$

$$|I_4| \leqslant ch \sum_{i=1}^{N} \|\varphi D_A W_i(\nabla u)\|_{L^{q_i}(\Omega_i)} \|\nabla(\tilde{u} - g)\|_{L^{p_i}(\tilde{\Omega}_i \cap \operatorname{supp} \varphi)}. \tag{35}$$

Again, c is a constant which is independent of h. It remains to estimate I_1. Here, it is essential, that the functions W_i are distributed quasi-monotonely. Let $k_1, \ldots, k_N$ be the numbers from definition 4.3. (Do not confuse k_i from definition 4.3 with κ_i from **H4**). It is

$$I_1 \equiv \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \triangle_h \left(W_i(\nabla \tilde{u}) + k_i\right) \, \mathrm{d}x = \ldots$$

and by the product rule for differences, $\triangle_h(fg)(x) = (\triangle_h f)(x) g(x) + f(x + he_l) \triangle_h g(x)$, it follows:

$$\begin{aligned} \ldots &= \sum_{i=1}^{N} \int_{\Omega_i} \triangle_h \left(\varphi^2 (W_i(\nabla \tilde{u}) + k_i)\right) \, \mathrm{d}x - \sum_{i=1}^{N} \int_{\Omega_i} (\triangle_h \varphi^2)(W_i(\nabla \tilde{u}(x + he_l)) + k_i) \, \mathrm{d}x \\ &= I_{11} + I_{12}. \end{aligned}$$

By assumption **H1** and the fact, that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, it holds with a constant c which is independent of h:

$$|I_{12}| \leqslant ch \sum_{i=1}^{N} \left(\|\nabla \tilde{u}(\cdot + he_l)\|_{L^{p_i}(\Omega_i)}^{p_i} + k_i |\Omega_i| \right) \leqslant ch \sum_{i=1}^{N} \left(\|\nabla \tilde{u}\|_{L^{p_i}(\tilde{\Omega}_i)}^{p_i} + k_i |\Omega_i| \right). \tag{36}$$

In the next estimates, the following notation is used: $\Omega_0 = \mathcal{K}_0 \cap B_R(S)$. Note, that for $1 \leqslant i \leqslant N$, $0 < h < h_0$:

$$\Omega_i \cap \operatorname{supp} \varphi \cap \left(\bigcup_{j=0}^{N} \overline{\Omega_j + he_l} \right) = \Omega_i \cap \operatorname{supp} \varphi, \quad (\Omega_i + he_l) \cap \operatorname{supp} \varphi \cap \left(\bigcup_{j=0}^{N} \overline{\Omega_j} \right) = \Omega_i + he_l \cap \operatorname{supp} \varphi. \tag{37}$$

It follows that

$$
\begin{aligned}
I_{11} &= \sum_{i=1}^{N} \int_{\Omega_i + he_l} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x - \int_{\Omega_i} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{\Omega_i + he_l \setminus \Omega_i} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x - \int_{\Omega_i \setminus \Omega_i + he_l} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x \\
&\overset{(37)}{=} \sum_{i=1}^{N} \sum_{\substack{j=0,\\ j\neq i}}^{N} \int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x - \int_{\Omega_i \cap \Omega_j + he_l} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x \\
&= \sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x - \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left(W_i(\nabla\tilde{u})) + k_i\right) \,\mathrm{d}x \\
&\quad + \sum_{\substack{i,j=1,\\ j\neq i}}^{N} \int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 \left(W_i(\nabla\tilde{u}) + k_i - W_j(\nabla\tilde{u}) - k_j\right) \,\mathrm{d}x \qquad (38) \\
&= I_{111} + I_{112}.
\end{aligned}
$$

Since the functions W_i are distributed quasi-monotonely it follows, that $\Omega_i \cap \Omega_0 + he_l = \emptyset$ for $h > 0$ and $1 \leqslant i \leqslant N$, compare definition 4.3. It remains, taking into account the definition of $\tilde{u}$ and **H1**, **A5**:

$$
\begin{aligned}
I_{111} &= \sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 (W_i(\nabla\tilde{u}) + k_i) \,\mathrm{d}x \\
&\overset{(29)}{=} \sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 (W_i(\nabla g) + k_i) \,\mathrm{d}x \overset{\mathbf{A5}}{\leqslant} c \sum_{i=1}^{N} |\Omega_i + he_l \cap \Omega_0 \cap \operatorname{supp}\varphi| \leqslant ch.
\end{aligned}
$$

Again due to the quasi-monotonicity of the functions W_i it holds: if $\Omega_i + he_l \cap \Omega_j \neq \emptyset$, then $W_j(A) + k_j \geqslant W_i(A) + k_i$ for every $A \in \mathbb{R}^{m\times d}$. Therefore

$$I_{112} \leqslant 0.$$

Collecting these estimates finally yields

$$I_1 \leqslant ch,$$

where $c > 0$ is a constant which is independent of h. This finishes the proof of inequality (30).

Second step: In this step, we derive estimates for the Nikolskii-norms of u on the basis of inequality (30).
Since the addends on the left hand side of inequality (30) are nonnegative, it holds for $1 \leqslant i \leqslant N$:

$$\int_{\Omega_i} \varphi^2 \left(\kappa_i + |\nabla\tilde{u}(x + he_l)| + |\nabla\tilde{u}(x)|\right)^{p_i - 2} \left|\triangle_h \nabla\tilde{u}(x)\right|^2 \,\mathrm{d}x \leqslant ch. \qquad (39)$$

Applying inequality (51) with $\alpha_i = p_i/2$ to each subdomain separately yields for $1 \leqslant i \leqslant N$:

$$\int_{\Omega_i} \varphi^2 \left|\triangle_h (\kappa_i + |\nabla\tilde{u}_i|)^{\frac{p_i}{2}}\right|^2 \,\mathrm{d}x \leqslant ch$$

Since $\varphi\big|_{B_{R'}(S)} = 1$ it follows for $\Omega'_{i,\eta} := \{x \in B_{R'}(S) \cap \Omega_i : \operatorname{dist}(x, \partial(B_{R'}(S) \cap \Omega_i)) > \eta\}$

$$\sup_{\substack{\eta>0\\0<h<\eta}} \int_{\Omega'_{i,\eta}} h^{-1} \left| \triangle_h (\kappa_i + |\nabla u_i|)^{\frac{p_i}{2}} \right|^2 \mathrm{d}x \leqslant c$$

and therefore

$$(\kappa_i + |\nabla u_i|)^{\frac{p_i}{2}} \in \mathcal{N}^{\frac{1}{2},2}(\Omega_i \cap B_{R'}(S)).$$

Assume first, that $p_i \in (1,2]$. The remaining part of the proof for this case follows exactly the considerations in [9] and is given here for completeness. Lemma 2.4 and the embedding theorems for Sobolev Slobodeckii spaces state, that

$$(\kappa_i + |\nabla u_i|)^{\frac{p_i}{2}} \in W^{\frac{1}{2}-\delta,2}(\Omega'_i) \subset L^{\frac{2d}{d-1}-\epsilon}(\Omega'_i) \tag{40}$$

for every δ and $\epsilon = \epsilon(\delta) > 0$, where $\Omega'_i = \Omega_i \cap B_{R'}(S)$. Thus, $\nabla u_i \in L^{\frac{dp_i}{d-1}-\varepsilon}(\Omega'_i)$. By standard embedding theorems, the space $W^{1,p_i}(\Omega'_i)$ is continuously embedded in $L^{\frac{dp_i}{d-1}-\epsilon}(\Omega'_i)$. This together with the previous estimate for ∇u_i shows, that $u_i \in W^{1,\frac{dp_i}{d-1}-\epsilon}(\Omega'_i)$ for every $\epsilon > 0$. Choose $\sigma_i = r_i - \delta$ for arbitrary $\delta > 0$, where $r_i = \frac{2dp_i}{2d-2+p_i}$ as in theorem 4.1. For $1 < p_i \leqslant 2$ it is $1 < \sigma_i \leqslant \frac{dp_i}{d-1}$ and therefore $u_i \in W^{1,\sigma_i}(\Omega'_i)$. Thus for $0 < h < \eta < h_0$, $1 \leqslant l \leqslant d$ and $M_h := \{x \in \Omega'_i : \nabla\tilde{u}(x + he_l) = \nabla\tilde{u}(x) = 0\}$ it holds (apply Hölder's inequality)

$$\begin{aligned}
\int_{\Omega'_{i,\eta}} \left| h^{-\frac{1}{2}} \triangle_h \nabla u \right|^{\sigma_i} \mathrm{d}x &= \int_{\Omega'_{i,\eta}\setminus M_h} \left| h^{-\frac{1}{2}} \triangle_h \nabla u_i \right|^{\sigma_i} (\kappa_i + |\nabla u_i(x)| + |\nabla u_i(x+he_l)|)^{\frac{\sigma_i}{2}(p_i-2)} \\
&\qquad\qquad (\kappa_i + |\nabla u_i(x)| + |\nabla u_i(x+he_l)|)^{-\frac{\sigma_i}{2}(p_i-2)} \,\mathrm{d}x \\
&\leqslant \left(\int_{\Omega'_{i,\eta}} \left| h^{-\frac{1}{2}} \triangle_h \nabla u \right|^2 (\kappa_i + |\nabla u_i(x+he_l)| + |\nabla u(x)|)^{p_i-2} \,\mathrm{d}x \right)^{\frac{\sigma_i}{2}} \\
&\qquad \times \left(\int_{\Omega'_{i,\eta}} (\kappa_i + |\nabla u_i(x)| + |\nabla u_i(x+he_l)|)^{\frac{\sigma_i(2-p_i)}{2-\sigma_i}} \,\mathrm{d}x \right)^{\frac{2-\sigma_i}{2}}.
\end{aligned}$$

By inequality (39) the first factor is bounded independently of h and η. Furthermore, $1 < \frac{\sigma_i(2-p_i)}{2-\sigma_i} < \frac{dp_i}{d-1}$ and thus the second term is bounded independently of h and η as well. It follows:

$$\sup_{\substack{\eta>0,\\0<h<\eta}} \int_{\Omega'_{i,\eta}} \left| h^{-\frac{1}{2}} \triangle_h \nabla u_i \right|^{\sigma_i} \mathrm{d}x \leqslant c$$

and relation (22) of theorem 4.1 is proved for $p_i \in (1,2]$. For the proof of the global result (25) note, that for arbitrary $A, B \in \mathbb{R}^{m\times d}$: $(|A| + |B|)^{p_i-2} \geqslant (1 + |A| + |B|)^{p_i-2} \geqslant (1+|A|+|B|)^{p_{\min}-2}$ and proceed as subsequent to equation (30) with Ω_i replaced by $\operatorname{supp}\varphi \cap \Omega$.

Assume now, that $p_i > 2$. The following two inequalities can be deduced from (39):

$$\int_{\Omega_i} \varphi^2 \left|\triangle_h \nabla\tilde{u}(x)\right|^{p_i} \mathrm{d}x \leqslant ch, \tag{41}$$

$$\int_{\Omega_i} \varphi^2 \left|\triangle_h \nabla\tilde{u}(x)\right|^{2} \mathrm{d}x \leqslant ch \qquad \text{if } \kappa_i = 1. \tag{42}$$

This yields the assertions (23) and (24) and completes the proof of the Dirichlet case.

Cross point on the boundary of Ω with pure Neumann conditions

Note first, that it follows by the special structure of the Neumann data, compare **A6**:

$$\langle H\vec{n}, v\rangle = \int_\Omega \left(H^T v\right) \vec{n}\,\mathrm{d}s = \int_\Omega H^T : \nabla v\,\mathrm{d}x + \int_\Omega (\operatorname{div} H) v\,\mathrm{d}x \quad \text{for every } v \in V^{\vec{p}}(\Omega).$$

Therefore, the weak formulation (14) is equivalent to: for every $v \in V^{\vec{p}}(\Omega)$

$$\sum_{i=1}^M \int_{\Omega_i} D_A W_i(\nabla u_i) : \nabla v_i\,\mathrm{d}x = \sum_{i=1}^M \int_{\Omega_i} (f_i + \operatorname{div} H_i) v_i\,\mathrm{d}x + \sum_{i=1}^M \int_{\Omega_i} H_i^T : \nabla v_i\,\mathrm{d}x. \tag{43}$$

Let $S \subset \partial\Omega$ and assume, that there exists $R > 0$ such that $B_R(S) \subset \hat{\Omega}$ and $\Omega \cap B_R(S) = \mathcal{K} \cap B_R(S)$, where $\mathcal{K}$ is an appropriate polyhedral cone with tip in S and $\partial\mathcal{K} \cap B_R(S) \subset \Gamma_N$. Assume further, that for every $j \in \{1, \ldots, M\}$ with $\Omega_j \cap B_R(S) \neq \emptyset$ there exists a polyhedral cone $\mathcal{K}_j$ with tip in S, such that $\Omega_j \cap B_R(S) = \mathcal{K}_j \cap B_R(S)$. Note, that $\overline{K} = \bigcup_{i=1}^N \overline{\mathcal{K}_i}$. Due to the assumptions in theorem 4.1, the cones $\mathcal{K}_i$ and functions W_i, $1 \leqslant i \leqslant N$, satisfy the conditions in definition 4.3, part *2.*; $\mathcal{K}_0 = \mathbb{R}^d \backslash \overline{\mathcal{K}}$.
Let $u \in W^{1,\vec{p}}(\Omega)$ be a weak solution of problem (14), $R''' = R/2, h_0 = R'' = R/4, R' = R/8$ and choose $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ with $\operatorname{supp}\varphi \subset B_{R''}(S)$, $\varphi\big|_{B_{R'}(S)} = 1$ and $0 \leqslant \varphi \leqslant 1$. Let further be e_l one of the basis vectors given by definition 4.3. For $0 < h < h_0$ the following function

$$\xi(x) := \varphi^2(x)(u(x + he_l) - u(x)) = \varphi^2(x) \triangle_h u(x), \qquad x \in \Omega$$

is an admissible test function in $V^{\vec{p}}(\Omega)$. This is due to the quasi-monotonicity condition, compare also corollary 4.1 and remark 4.1. Note, that no extension of u across the Neumann boundary is needed. The next goal is to prove, that inequality (30) also holds in the case of pure Neumann conditions (with u instead of $\tilde{u}$). Inserting ξ into equation (43) and rearranging the terms yields:

$$\begin{aligned}\sum_{i=1}^N \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u) : \triangle_h \nabla u\,\mathrm{d}x &= \sum_{i=1}^N \int_{\Omega_i} (f + \operatorname{div} H)\xi\,\mathrm{d}x + \sum_{i=1}^N \int_{\Omega_i} H^T : \nabla\xi\,\mathrm{d}x \\ &\quad - \sum_{i=1}^N \int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h u \otimes \nabla\varphi^2)\,\mathrm{d}x.\end{aligned} \tag{44}$$

Applying inequality (50) to (44) results in

$$\begin{aligned}&c \sum_{i=1}^N \int_{\Omega_i} \varphi^2 (\kappa_i + |\nabla u(x + he_l)| + |\nabla u(x)|)^{p_i - 2} \left|\triangle_h \nabla u\right|^2\,\mathrm{d}x \\ &\overset{(50)}{\leqslant} \sum_{i=1}^N \int_{\Omega_i} \varphi^2 \triangle_h W_i(\nabla u)\,\mathrm{d}x - \sum_{i=1}^N \int_{\Omega_i} \varphi^2 D_A W_i(\nabla u) : \triangle_h \nabla u\,\mathrm{d}x \\ &\overset{(44)}{=} \sum_{i=1}^N \int_{\Omega_i} \varphi^2 \triangle_h W_i(\nabla u)\,\mathrm{d}x - \sum_{i=1}^N \int_{\Omega_i} (f + \operatorname{div} H)\xi\,\mathrm{d}x \\ &\quad - \sum_{i=1}^N \int_{\Omega_i} H^T : \nabla\xi\,\mathrm{d}x + \sum_{i=1}^N \int_{\Omega_i} D_A W_i(\nabla u) : (\triangle_h u \otimes \nabla\varphi^2)\,\mathrm{d}x \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{45}$$

The constant c is independent of h. The integrals I_2 and I_4 can be estimated as in the case of pure Dirichlet conditions, compare (33)-(35), and one gets

$$|I_2| + |I_4| \leqslant ch$$

for some $c > 0$ which is independent of h. Let $k_1, \dots k_N$ be the numbers from definition 4.3. Then by the product rule for differences:

$$\begin{aligned} I_1 &= \sum_{i=1}^{N} \int_{\Omega_i} \triangle_h \left((\varphi^2 (W_i(\nabla u) + k_i) \right) \mathrm{d}x - \int_{\Omega_i} (\triangle_h \varphi^2)(W_i(\nabla u)(x + he_l) + k_i) \, \mathrm{d}x \\ &= I_{11} + I_{12}. \end{aligned}$$

As in (36) it follows that $|I_{12}| \leqslant ch$. Furthermore, with $\Omega_0 = \mathcal{K}_0 \cap B_R(S)$, I_{11} can be transformed analogously to (38):

$$\begin{aligned} I_{11} = &\sum_{i=1}^{N} \int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 (W_i(\nabla u) + k_i) \, \mathrm{d}x - \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 (W_i(\nabla u) + k_i) \, \mathrm{d}x \\ &+ \sum_{\substack{i,j=1 \\ j \neq i}}^{N} \int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 (W_i(\nabla u) + k_i - (W_j(\nabla u) + k_j)) \, \mathrm{d}x. \end{aligned} \tag{46}$$

Due to the quasi-monotonicity condition, it is $W_i(\nabla u) + k_i - (W_j(\nabla u) + k_j) \leqslant 0$ if $\Omega_i + he_l \cap \Omega_j \cap \operatorname{supp} \varphi \neq \emptyset$ and in addition $\Omega_i + he_l \cap \Omega_0 = \emptyset$. Therefore it remains

$$I_{11} \leqslant - \sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 (W_i(\nabla u) + k_i) \, \mathrm{d}x \overset{\mathbf{H1}}{\leqslant} - \sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left(c_1^i \, |\nabla u|^{p_i} + c_0^i + k_i \right) \, \mathrm{d}x. \tag{47}$$

Estimation of I_3: By the product rule for differences

$$\begin{aligned} I_3 = &- \sum_{i=1}^{N} \int_{\Omega_i} \triangle_h (\varphi^2 H^T : \nabla u) \, \mathrm{d}x + \sum_{i=1}^{N} \int_{\Omega_i} (\triangle_h \varphi^2) H^T (x + he_l) : \nabla u(x + he_l) \, \mathrm{d}x \\ &+ \sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \triangle_h H^T : \nabla u(x + he_l) \, \mathrm{d}x - \sum_{i=1}^{N} \int_{\Omega_i} H^T : (\triangle_h u \otimes \nabla \varphi^2) \, \mathrm{d}x \\ = &\, I_{31} + I_{32} + I_{33} + I_{34}. \end{aligned}$$

By the usual arguments, compare (33)-(35),

$$|I_{32}| + |I_{33}| + |I_{34}| \leqslant ch,$$

where c is independent of h. Analogously to the considerations in (38), keeping in mind that $\Omega_i + he_l \cap \Omega_0 \cap \operatorname{supp} \varphi = \emptyset$, one obtains

$$\begin{aligned} I_{31} = &- \sum_{i=1}^{N} \left(\int_{\Omega_i + he_l \cap \Omega_0} \varphi^2 H^T : \nabla u \, \mathrm{d}x - \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 H^T : \nabla u \, \mathrm{d}x \right) \\ &- \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \left(\int_{\Omega_i + he_l \cap \Omega_j} \varphi^2 H^T : \nabla u \, \mathrm{d}x - \int_{\Omega_i \cap \Omega_j + he_l} \varphi^2 H^T : \nabla u \, \mathrm{d}x \right) \\ = &\sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 H^T : \nabla u \, \mathrm{d}x - 0, \end{aligned}$$

since $\Omega_i + he_l \cap \Omega_0 = \emptyset$, see also definition 4.3 and remark 4.1. By Hölder's and Young's inequality and since $H \in L^\infty(\Omega)$ it follows for arbitrary $\delta_i > 0$ ($c, \tilde{c}$ independent of h):

$$\begin{aligned}
|I_{31}| &\leqslant \sum_{i=1}^{N} \delta_i^{-1} \left\| \varphi^{\frac{2}{q_i}} H^T \right\|_{L^{q_i}(\Omega_i \cap \Omega_0 + he_l)} \delta_i \left\| \varphi^{\frac{2}{p_i}} |\nabla u| \right\|_{L^{p_i}(\Omega_i \cap \Omega_0 + he_l)} \\
&\leqslant c \sum_{i=1}^{N} \left(\delta_i^{-q_i} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left| H^T \right|^{q_i} \, \mathrm{d}x + \delta_i^{p_i} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 |\nabla u|^{p_i} \, \mathrm{d}x \right) \\
&\overset{\mathbf{A6}}{\leqslant} \tilde{c} h \sum_{i=1}^{N} \delta_i^{-q_i} + \sum_{i=1}^{N} c \delta_i^{p_i} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 |\nabla u|^{p_i} \, \mathrm{d}x. \qquad (48)
\end{aligned}$$

For $1 \leqslant i \leqslant N$ choose $\delta_i = \left(\frac{c_1^i}{c} \right)^{\frac{1}{p_i}}$ where c_1^i is the constant from assumption **H1**. Then with (47) and (48):

$$\begin{aligned}
I_{11} + |I_{31}| &\leqslant \tilde{c} h \sum_{i=1}^{N} \delta_i^{-q_i} + \sum_{i=1}^{N} \int_{\Omega_i \cap \Omega_0 + he_l} \varphi^2 \left(c \delta_i^{p_i} |\nabla u|^{p_i} - c_1^i |\nabla u|^{p_i} - k_i - c_0^i \right) \mathrm{d}x \\
&\leqslant \tilde{c} h \sum_{i=1}^{N} \delta_i^{-q_i} + \sum_{i=1}^{N} (k_i + \left| c_0^i \right|) \left| \Omega_i \cap \Omega_0 + he_l \right| \\
&\leqslant c^* h,
\end{aligned}$$

where c^* is independent of h. Collecting the estimates, one obtains for (45):

$$\sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 (\kappa_i + |\nabla u(x + he_l)| + |\nabla u(x)|)^{p_i - 2} \left| \triangle_h \nabla u \right|^2 \, \mathrm{d}x \leqslant ch.$$

The remaining part of the proof is completely analogous to the considerations in the second step for the Dirichlet problem.

Cross point on the boundary with mixed boundary conditions

Consider a cross point $S \in \partial\Omega$ with mixed boundary conditions in its neighborhood. Let $e_1, \ldots, e_d$ be a basis as in definition 4.3, part *3*. Assume, that $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ is a suitable cut-off function. For the choice of the test function ξ one has to distinguish two cases, see also remark 4.1. If $\operatorname{supp} \varphi \cap (\Gamma_D + he_l) \subset \Omega$ for $0 < h < h_0$, then choose ξ as in the case of pure Neumann boundary conditions. Else choose ξ as in the case of pure Dirichlet conditions. Proceeding analogously to these two cases yields the assertion.

Interior cross point

Choose $\xi(x) = \varphi^2(x)(u(x+he_l) - u(x))$ as test function, where φ is a suitable cut-off function with $\operatorname{supp} \varphi \subset \Omega$, and proceed analogous to the case of pure Neumann conditions. This completes the proof of theorem 4.1.

Remark 4.6. If in the weak formulation (14) ∇u is replaced by $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, then the proof of the regularity result for equation (16) is completely analogous to the one of equation

(14), one has to replace ∇u by $\varepsilon(u)$, only, and (30) changes to the following inequality:

$$\sum_{i=1}^{N} \int_{\Omega_i} \varphi^2 \left(\kappa_i + |\varepsilon(\tilde{u}(x+he_l))| + |\varepsilon(\tilde{u}(x))|\right)^{p_i-2} |\triangle_h \varepsilon(\tilde{u}(x))|^{p_i-2} \,\mathrm{d}x \leqslant ch.$$

This leads to $\varepsilon(u_i) \in L^{\frac{dp_i}{d-1}-\epsilon}(\Omega_i')$. By Korn's inequality, the estimates can be carried over to ∇u and considerations analogous to those in the second step of the proof for the Dirichlet problem can be carried out in the case $p_i \in (1, 2]$. In the case $p_i > 2$, the argumentation is similar to (41)-(42) and again the estimates can be carried over to ∇u by Korn's inequality.

A Some essential inequalities

Lemma A.1. *1. For $A, B \in \mathbb{R}^s$, $|B| \geqslant |A|$ and $t \in [0, \frac{1}{4}]$ it holds,[35, formula (2.20)] :*

$$4\,|B + t(A-B)| \geqslant |A| + |B|\,. \tag{49}$$

*2. Assume, that $W : \mathbb{R}^{m\times d} \to \mathbb{R}$, $d \geqslant 2$, satisfies **H0** and **H4** for some $p \in (1,\infty)$ and $\kappa \in \{0,1\}$. Then there exists $c > 0$ such that for every $A, B \in \mathbb{R}^{m\times d}$:*

$$W(A) - W(B) \geqslant D_A W(B) : (A-B) + c\,(\kappa + |B| + |A|)^{p-2}\,|A-B|^2 \tag{50}$$

3. Let $\kappa \in \{0,1\}$, $\alpha > 0$. There exists a constant $c > 0$, such that for every $x, y \in \mathbb{R}^s$:

$$|(\kappa + |x|)^\alpha - (\kappa + |y|)^\alpha| \leqslant c\,(\kappa + |x| + |y|)^{\alpha-1}\,|x-y|\,. \tag{51}$$

Remark A.1. For the case $1 < p < 2$ and $W(A) = |A|^p$ inequality (50) is proved in [18, Lemma 4.2].

Proof. Proof of inequality (49): For $0 \leqslant t \leqslant \frac{1}{4}$ and $A, B \in \mathbb{R}^s$ with $|B| \geqslant |A|$ it holds:

$$|B + t(A-B)| \geqslant |(1-t)\,|B| - t\,|A|| \geqslant \left|\frac{3}{4}\,|B| - \frac{1}{4}\,|A|\right| \geqslant \frac{1}{2}\,|B| \geqslant \frac{1}{4}\,|B| + \frac{1}{4}\,|A|\,.$$

Proof of inequality (50): For $t \in [0,1]$ set $f(t) = W(B + t(A-B))$. Assume first, that $B + t(A-B) \neq 0$ for every $t \in [0,1]$. In this case,

$$\begin{aligned}
W(A) - W(B) &= f(1) - f(0) = f'(0) + \int_0^1 (1-t) f''(t)\,\mathrm{d}t \\
&= D_A W(B) : (A-B) + \int_0^1 (1-t) D_A^2 W(B + t(A-B))[A-B, A-B]\,\mathrm{d}t \\
&\overset{\mathbf{H4}}{\geqslant} D_A W(B) : (A-B) + c \int_0^1 (1-t)\,(\kappa + |B + t(A-B)|)^{p-2}\,\mathrm{d}t\,|A-B|^2
\end{aligned} \tag{52}$$

If $1 < p \leqslant 2$, then

$$\begin{aligned}
\ldots &\overset{1<p\leqslant 2}{\geqslant} D_A W(B) : (A-B) + c \int_0^1 (1-t)\,(\kappa + t\,|A| + (1-t)\,|B|)^{p-2}\,\mathrm{d}t\,|A-B|^2 \\
&\geqslant D_A W(B) : (A-B) + c\,(\kappa + |B| + |A|)^{p-2}\,|A-B|^2\,.
\end{aligned}$$

In the case $p > 2$, it follows from (52) by inequality (49) for $|B| \geqslant |A|$ and $p > 2$:

$$\begin{aligned}
W(A) - W(B) &\geqslant D_A W(B) : (A-B) + c \int_0^{\frac{1}{4}} (1-t)\,(\kappa + |B + t(A-B)|)^{p-2}\,\mathrm{d}t\,|A-B|^2 \\
&\overset{(49)}{\geqslant} D_A W(B) : (A-B) + \frac{c}{4^{p-2}} \int_0^{\frac{1}{4}} (1-t)\,\mathrm{d}t\,(\kappa + |B| + |A|)^{p-2}\,|A-B|^2\,.
\end{aligned}$$

On the other hand, if $|A| \geqslant |B|$, then a change of variables, $t = 1 - s$, and reasoning similarly to the case $|B| \geqslant |A|$ yields the assertion.

If there exists $t_0 \in (0,1]$ with $B + t_0(A - B) = 0$, then consider $A_\delta := A + \delta C$ for $\delta > 0, C \in \mathbb{R}^{m\times d}\backslash\{0\}$. Note, that $B + t(A_\delta - B) \neq 0$ for every $t \in [0,1]$ and by the first step, inequality (50) holds for A_δ and B for every $\delta > 0$. Taking the limit $\delta \to 0$ yields the assertion.

Proof of inequality (51): Assume first, that $\alpha > 1$. For $|x| > |y| \geqslant 0$, Taylor's expansion yields

$$\begin{aligned} 0 \leqslant (\kappa + |x|)^\alpha - (\kappa + |y|)^\alpha &\leqslant \int_0^1 \alpha\,(\kappa + t\,|x| + (1-t)\,|y|)^{\alpha-1}\,|x-y|\;\mathrm{d}t \\ &\leqslant \alpha \int_0^1 (\kappa + |x| + |y|)^{\alpha-1}\;\mathrm{d}t\,|x-y| \end{aligned}$$

and (51) is proved for $\alpha > 1$. Assume now, that $0 < \alpha < 1$ and $|x| > |y| > 0$. Then

$$\begin{aligned} &0 \leqslant ((\kappa + |x|)^\alpha - (\kappa + |y|)^\alpha)\,(2\kappa + |x| + |y|) \\ &= (\kappa + |x|)^{\alpha+1} - (\kappa + |y|)^{\alpha+1} + (\kappa + |x|)(\kappa + |y|)\,\underbrace{\left((\kappa + |x|)^{\alpha-1} - (\kappa + |y|)^{\alpha-1}\right)}_{\leqslant 0} \\ &\overset{1.\text{ step}}{\leqslant} c(\kappa + |x| + |y|)^\alpha\,|x-y| \leqslant c(2\kappa + |x| + |y|)^\alpha\,|x-y|\,. \end{aligned}$$

The lemma is proved. □

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