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Global stress regularity for convex and some nonconvex variational problems on Lipschitz-domains

Dorothee Knees*

Abstract

A global regularity theorem for stress fields which correspond to minimisers of convex and some special nonconvex variational problems is derived for Lipschitz-domains. In the first part it is assumed that the energy densities defining the variational problem are convex but not necessarily strictly convex and satisfy a convexity inequality. The regularity result for this case is derived with a difference quotient technique. In the second part the regularity results are carried over from the convex case to special nonconvex variational problems taking advantage of the relation between nonconvex variational problems and the corresponding (quasi-) convexified problems. The results are applied amongst others to the variational problems for linear elasticity, the p-Laplace operator and for scalar and vectorial two-well potentials (compatible case).

Keywords: global stress regularity; convex variational problem; nonconvex variational problem; nonsmooth domain; difference quotient technique

AMS Subject Classification: 35J70, 35B65, 49N60, 74G40, 35D10.

1 Introduction

We investigate the global regularity of stress fields which are related to minimisers of convex, but not necessarily strictly convex, variational problems with mixed boundary conditions on domains with Lipschitz boundary. Furthermore, the results are carried over to stress fields of special nonconvex variational problems. The variational problems under consideration are of the following type: For a bounded domain $\Omega \subset \mathbb{R}^d$ we denote by $\Gamma_D \subset \partial\Omega$ the Dirichlet boundary, furthermore let $g \in W^{1,p}(\Omega)$, $f \in \big(W^{1,p}(\Omega)\big)'$ and $V := \{v \in W^{1,p}(\Omega) : v\big|_{\Gamma_D} = 0\}$. The variational problem is:

$$\textit{Find } u : \Omega \to \mathbb{R}^m,\ u \in g + V \textit{ such that for every } v \in g + V$$

$$I(u) \leqslant I(v) = \int_\Omega W(\nabla v(x))\,\mathrm{d}x - \langle f, v\rangle. \tag{1}$$

Here, $W : \mathbb{R}^{m\times d} \to \mathbb{R}$ is a given energy density. If u is a minimiser of I then the corresponding stress field σ is defined as

$$\sigma(x) := DW(\nabla u(x)),\ \ x \in \Omega,$$

where the notation $DW(A) = \left(\frac{\partial W(A)}{\partial A_{ik}}\right)_{\substack{1\leqslant i\leqslant m,\\ 1\leqslant k\leqslant d}} \in \mathbb{R}^{m\times d}$ is used for the derivative of W. In the first part of this paper the main assumption is that the energy density W is a $\mathcal{C}^1$ function and satisfies the following convexity inequality for every $A, B \in \mathbb{R}^{m\times d}$:

$$\left(W(A) - W(B) - DW(B) : (A - B)\right)(1 + |A|^s + |B|^s) \geqslant c\,|DW(A) - DW(B)|^r \tag{2}$$

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for some constants $c > 0$, $s \geqslant 0$ and $r > 1$. Inequality (2) implies that W is convex but not necessarily strictly convex. Examples for energy densities W with (2) will be given in this paper and include the energy densities of linear elastic materials, the p-Laplace equation and a model for Hencky elasto-plasticity with linear hardening.

It is well known for linear and quasilinear elliptic equations that the global regularity of weak solutions does not only depend on the smoothness of the right hand sides but also on the smoothness of the boundary of Ω. Global regularity results for weak solutions of quasilinear elliptic systems of p-structure were derived independently by C. Ebmeyer and J. Frehse [12, 14] (mixed boundary conditions, polyhedral domains) and G. Savaré [31] (pure Dirichlet or pure Neumann conditions on Lipschitz domains). Combining the geometrical assumptions from these articles, we describe here a rather general class of domains (Lipschitz domains with additional geometrical constraints near points with changing boundary conditions) for which we prove a global regularity result for the stress field σ. This result is derived with a difference quotient technique on the basis of convexity inequality (2) and describes the smoothness of σ in Nikolskii and Sobolev-Slobodeckij spaces. In the proof we extend ideas from [12, 14, 31] to our situation.

In the second part of the paper we discuss the regularity properties of the stress fields of a special class of nonconvex variational problems. Nonconvex problems need not have minimisers and in that case it is reasonable to study the relaxed variational problem

$$\textit{Find } u \in g + V \textit{ such that for every } v \in g + V$$

$$I^R(u) \leqslant I^R(v) = \int_\Omega W^R(\nabla v(x))\,\mathrm{d}x - \langle f, v\rangle. \tag{3}$$

Here, W^R is in general given by the quasiconvex envelope W^{qc} of the original energy density W [9, 29, 1]. It follows from the relaxation theory in the calculus of variations that if the original problem has a solution u then it is also a solution of the relaxed problem. Moreover, the stresses of both problems coincide: $\sigma = DW(\nabla u) = DW^{qc}(\nabla u)$. We use this relation in combination with the additional assumption that the quasiconvex envelope W^{qc} is equal to the convex envelope W^c of the original energy density W in order to carry over regularity results for stresses of convex problems to special nonconvex problems. Examples for such problems are scalar or vectorial two-well potentials (compatible case).

This paper and the examples herein are highly motivated by an article by C. Carstensen and S. Müller, where local and global stress regularity results for smooth domains are proved [5]. There, the main assumption is that the energy density W satisfies the following monotonicity inequality

$$\big((DW(A) - DW(B)) : (A - B)\big)\,(1 + |A|^s + |B|^s) \geqslant c\,|DW(A) - DW(B)|^r \tag{4}$$

with $c > 0$, $s \geqslant 0$ and $r > 1$. This condition follows directly from convexity inequality (2). In lemma 2.3 we describe sufficient conditions on W for which the monotonicity inequality and the convexity inequality are equivalent.

The paper is organised as follows: After a description of the assumptions on the energy density W and the geometry of the domain Ω, we formulate in section 2 the main result on the global regularity of stress fields of convex variational problems. The proof is based on a difference quotient technique. These results are then applied to convex examples from continuum mechanics. In section 3 we formulate a regularity theorem for the nonconvex case and illustrate it with further examples.

2 Regularity in the convex case

2.1 Notation

Let us first introduce some notation and general assumptions. For $m\times d$-matrices $A, B \in \mathbb{R}^{m\times d}$ the inner product is defined by $A : B = \operatorname{tr}\left(A^\top B\right) = \operatorname{tr}\left(B^\top A\right) = \sum_{i=1}^m \sum_{k=1}^d A_{ik}B_{ik}$ and $|A| = \sqrt{A : A}$ is the corresponding Frobenius norm.

If not otherwise stated it is assumed that $\Omega \subset \mathbb{R}^d$, $d \geqslant 2$, is a bounded domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. Γ_D and Γ_N are open and disjoint and denote the Dirichlet and Neumann boundary, respectively.

For $p \in (1, \infty)$ and $s > 0$ the spaces $W^{s,p}(\Omega)$ are the usual Sobolev-Slobodeckij spaces, see e.g. [2, 18]. Furthermore,

$$V = \{v \in W^{1,p}(\Omega) : v\big|_{\Gamma_D} = 0\}. \tag{5}$$

For the formulation of the boundary conditions we need the following trace space and its dual for and open subset $\Gamma \subset \partial\Omega$, $p \in (1, \infty)$:

$$W^{1-\frac{1}{p},p}(\Gamma) = \left\{u \in L^p(\Gamma) : \exists \hat{u} \in W^{1,p}(\Omega) \text{ such that } \hat{u}\big|_\Gamma = u\right\}, \tag{6}$$

$$\tilde{W}^{-\frac{1}{p'},p'}(\Gamma) = \left(W^{1-\frac{1}{p},p}(\Gamma)\right)'. \tag{7}$$

Throughout the whole paper p' is the conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, the dual pairing for elements u of a Banach space X and elements f of its dual X' is denoted by $\langle f, u\rangle = \langle f, u\rangle_X$. Besides the usual Sobolev spaces we deal also with Nikolskii spaces. Nikolskii spaces are very useful for proving regularity results with a difference quotient technique since their norms are based on difference quotients. For convenience we cite here the definition of Nikolskii spaces and an embedding theorem.

Definition 2.1 (Nikolskii space). [2, 26] Let $s = m + \delta$, where $m \geqslant 0$ is an integer and $0 < \delta < 1$. For $1 < p < \infty$ the Nikolskii spaces are defined as

$$\mathcal{N}^{s,p}(\Omega) := \left\{u \in L^p(\Omega) : \ \|u\|_{\mathcal{N}^{s,p}(\Omega)} < \infty\right\}$$

with

$$\|u\|^p_{\mathcal{N}^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0\\ h\in\mathbb{R}^d\\ 0<|h|<\eta}} \int_{\Omega_\eta} \frac{|D^\alpha u(x+h) - D^\alpha u(x)|^p}{|h|^{\delta p}} \,\mathrm{d}x \tag{8}$$

and $\Omega_\eta = \{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > \eta\}$.

Lemma 2.2. *[2, 26, 33, 34] Let s, p be as in definition 2.1 and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. The following embeddings are continuous for every $\epsilon > 0$:*

$$\mathcal{N}^{s+\epsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset \mathcal{N}^{s,p}(\Omega).$$

Lemma 2.2 is a consequence of [18, Thm. 1.4.1.3], [26, p. 381], [33, sections 1.3, 2.1.1, 2.2.9] and [34, sec. 2.3.2]. An equivalent norm is generated if the supremum in (8) is replaced by $\sup_{\substack{\eta>0, h=\eta e_j,\\ e_j\in\{e_1,\dots,e_d\}}}$, where $\{e_1, \dots, e_d\}$ is a basis of $\mathbb{R}^d$ [26, 23].

2.2 The convex minimisation problem

We study minimisation problems where the energy density $W : \mathbb{R}^{m\times d} \to \mathbb{R}$, $m, d \geqslant 1$, has the following properties:

H1 $W \in \mathcal{C}^1(\mathbb{R}^{m\times d}, \mathbb{R})$.

H2 There exist constants $p \in (1, \infty)$, $c_0, c_1, c_2, c_{21}, c_3 > 0$ such that for every $A \in \mathbb{R}^{m\times d}$

$$c_0 |A|^p - c_1 \leqslant W(A) \leqslant c_2 |A|^p + c_{21}, \tag{9}$$

$$|DW(A)| \leqslant c_3(1 + |A|^{p-1}). \tag{10}$$

H3 There exist constants $c > 0$, $r > 1$, $s \geqslant 0$ such that for every $A, B \in \mathbb{R}^{m\times d}$

$$\big(W(A) - W(B) - DW(B) : (A - B)\big)\,(1 + |A|^s + |B|^s) \geqslant c\,|DW(A) - DW(B)|^r. \tag{11}$$

Condition **H3** implies that the energy density W is convex but not necessarily strictly convex. The following lemma describes sufficient conditions on W for which convexity inequality (11) and monotonicity inequality (4) are equivalent.

Lemma 2.3. *Let $W \in \mathcal{C}^1(\mathbb{R}^{m\times d}, \mathbb{R})$ satisfy (9) with $p > 1$ and let monotonicity inequality (4) be valid for $s \geqslant 0$ and $r > 1$. We denote by W^* the conjugate function of W and by $\partial W^*(\sigma)$ the subdifferential of W^* at $\sigma \in \mathbb{R}^{m\times d}$. Then*

1. *$W^* \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ and W^* is subdifferentiable on $\mathbb{R}^{m\times d}$. Furthermore, $\partial W^*(\sigma)$ is compact for every $\sigma \in \mathbb{R}^{m\times d}$.*

2. *If $s = 0$ then it holds for every $\sigma_1, \sigma_2 \in \mathbb{R}^{m\times d}$ and $A \in \partial W^*(\sigma_2)$ with the constant c from monotonicity inequality (4)*
$$W^*(\sigma_1) - W^*(\sigma_2) - A : (\sigma_1 - \sigma_2) \geqslant cr^{-1} |\sigma_1 - \sigma_2|^r . \tag{12}$$
Furthermore, it holds for every $A, B \in \mathbb{R}^{m\times d}$ with c from (4)
$$W(A) - W(B) - DW(B) : (A - B) \geqslant cr^{-1} |DW(A) - DW(B)|^r . \tag{13}$$

3. *Let $s \neq 0$ and assume in addition that (10) is satisfied. Then there exist constants $\kappa, \delta > 0$ such that it holds for every $\sigma_1, \sigma_2 \in \mathbb{R}^{m\times d}$ and $A_i \in \partial W^*(\sigma_i)$*
$$\begin{aligned} W^*(\sigma_1) - &W^*(\sigma_2) - A_2 : (\sigma_1 - \sigma_2) \\ &\geqslant \kappa \left(1 + |A_1|^s + |A_2|^s + \delta(|A_1|^{ps} + |A_2|^{ps})\right)^{-1} |\sigma_1 - \sigma_2|^r . \end{aligned} \tag{14}$$
Moreover, it holds for every $A_1, A_2 \in \mathbb{R}^{m\times d}$
$$\begin{aligned} W(A_1) - W(A_2) - DW(A_2) : (A_1 - A_2) & \\ \geqslant \kappa \left(1 + |A_1|^s + |A_2|^s + \delta(|A_1|^{ps} + |A_2|^{ps})\right)^{-1} &|DW(A_1) - DW(A_2)|^r . \end{aligned} \tag{15}$$
If $c_0 = c_2$ in (9) or if $|A| \leqslant c(1 + |DW(A)|^{\frac{1}{p-1}})$ for some $c > 0$ and every $A \in \mathbb{R}^{m\times d}$, then (14) and (15) hold with $\delta = 0$. This means that in these cases the monotonicity inequality (4) and convexity inequality (11) are equivalent.

Proof. Part *1.* of the lemma follows due to (9) from Corollaries 10.1.1, 13.3.1 and Theorems 12.2 and 23.4 in [28]. Furthermore, (13) and (15) follow from (12) and (14) via the relation $\sigma = DW(A) \Leftrightarrow W^*(\sigma) + W(A) = A : \sigma$ [28, Thm. 23.5]. For the proof of (12) and (14) let $\sigma_1 \neq \sigma_2 \in \mathbb{R}^{m\times d}$. We define $f(t) = W^*(\sigma_2 + t(\sigma_1 - \sigma_2))$, $t \in \mathbb{R}$ and
$$f'_+(t) := \lim_{\lambda \searrow 0} \lambda^{-1} \left(f(t + \lambda) - f(t)\right).$$
Theorem 23.1 in [28] guarantees that $f'_+(t) : \mathbb{R} \to \mathbb{R}$ is well defined. Moreover, it follows from [28, Thm. 23.4] that
$$f'_+(t) = \sup\{A : (\sigma_1 - \sigma_2);\ A \in \partial W^*(\sigma_2 + t(\sigma_1 - \sigma_2))\}. \tag{16}$$
Since $\partial W^*(\sigma)$ is compact for every $\sigma \in \mathbb{R}^{m\times d}$, there exists for every t an element $A_+(t) \in \partial W^*(\sigma_2 + t(\sigma_1 - \sigma_2))$ for which the supremum in (16) is attained. Taylor's expansion [28, Cor. 24.2.1] and monotonicity inequality (4) yield for every $A_2 \in \partial W^*(\sigma_2)$
$$\begin{aligned} W^*(\sigma_1) - W^*(\sigma_2) - A_2 : (\sigma_1 - \sigma_2) &= \int_0^1 f'_+(t) - (A_2 : (\sigma_1 - \sigma_2))\, \mathrm{d}t \\ &= \int_0^1 \frac{1}{t}(A_+(t) - A_2) : t(\sigma_1 - \sigma_2)\, \mathrm{d}t \\ &\overset{(4)}{\geqslant} c \int_0^1 t^{-1}(1 + |A_+(t)|^s + |A_2|^s)^{-1} |t(\sigma_1 - \sigma_2)|^r\, \mathrm{d}t. \end{aligned} \tag{17}$$

This proves (12) if $s = 0$. Assume now that $s > 0$. The next task is to find an upper bound for $|A_+(t)|$. If the estimate $|A| \leqslant c(1 + |DW(A)|^{\frac{1}{p-1}})$ is valid for every A, then analogous arguments as subsequent to (21) here below imply (14) with $\delta = 0$. If this estimate does not hold then direct calculations show (use (9) and the definition of W^*) that for every $\sigma \in \mathbb{R}^{m \times d}$:

$$q^{-1}(c_2 p)^{\frac{-1}{p-1}} |\sigma|^q - c_{21} \leqslant W^*(\sigma) \leqslant c_1 + q^{-1}(c_0 p)^{\frac{-1}{p-1}} |\sigma|^q , \tag{18}$$

where c_0, c_1, c_2 are the constants from (9) and $\frac{1}{q} + \frac{1}{p} = 1$. The convexity of W^* and (18) imply for every $\sigma \in \mathbb{R}^{m \times d}$ and $A \in \partial W^*(\sigma)$, $A \neq 0$,

$$\begin{aligned} |A| = A : (|A|^{-1} A) &\leqslant W^*(\sigma + |A|^{-1} A) - W^*(\sigma) \\ &\overset{(18)}{\leqslant} d_1 + d_0 \left|\sigma + |A|^{-1} A\right|^q - d_2 |\sigma|^q , \end{aligned} \tag{19}$$

where $d_0 = q^{-1}(c_0 p)^{\frac{-1}{p-1}}$, $d_1 = c_1 + c_{21}$ and $d_2 = q^{-1}(c_2 p)^{\frac{-1}{p-1}}$. Furthermore, Taylor's expansion yields for $\sigma, \tau \in \mathbb{R}^{m \times d}$

$$|\sigma + \tau|^q - |\sigma|^q \leqslant q \int_0^1 (|\sigma| + t |\tau|)^{q-1} |\tau| \, \mathrm{d}t \leqslant q(|\sigma| + |\tau|)^{q-1} |\tau| . \tag{20}$$

Combining inequalities (19) and (20) leads to

$$|A| \leqslant d_1 + (d_0 - d_2) |\sigma|^q + d_0 q(|\sigma| + 1)^{q-1} \tag{21}$$

for every $A \in \partial W^*(\sigma)$. Thus, it follows for $t \in (0, 1)$ and $\sigma(t) = \sigma_2 + t(\sigma_1 - \sigma_2)$ together with (10) that

$$\begin{aligned} |A_+(t)| &\leqslant d_1 + (d_0 - d_2) |\sigma(t)|^q + d_0 q(|\sigma(t)| + 1)^{q-1} \\ &\leqslant d_1 + (d_0 - d_2)(|\sigma_1| + |\sigma_2|)^q + d_0 q(|\sigma_1| + |\sigma_2| + 1)^{q-1} \\ &\overset{(10)}{\leqslant} d_1 + c(d_0 - d_2)(1 + |A_1|^p + |A_2|^p) + c(1 + |A_1| + |A_2|). \end{aligned} \tag{22}$$

Here, $c > 0$ is a constant and $A_1 \in \partial W^*(\sigma_1)$ and $A_2 \in \partial W^*(\sigma_2)$ are arbitrary. Furthermore we have used that $(|A| + |B|)^\alpha \leqslant c_\alpha(|A|^\alpha + |B|^\alpha)$ for $\alpha > 0$, see e.g. [22]. Together with (17) we obtain finally

$$\begin{aligned} &W^*(\sigma_1) - W^*(\sigma_2) - A_2 : (\sigma_1 - \sigma_2) \\ &\qquad \geqslant \kappa \left(1 + (d_0 - d_2)(|A_1|^{ps} + |A_2|^{ps}) + |A_1|^s + |A_2|^s\right)^{-1} |\sigma_1 - \sigma_2|^r \end{aligned} \tag{23}$$

for every $\sigma_1, \sigma_2 \in \mathbb{R}^{m \times d}$ and every $A_i \in \partial W^*(\sigma_i)$ with a constant $\kappa > 0$ which is independent of σ_i and A_i. This proves (14) with $\delta = d_0 - d_2 \geqslant 0$. If $c_0 = c_2$, then $\delta = d_0 - d_2 = 0$. □

The existence of minimisers of problem (1) follows with standard arguments from the direct method in the calculus of variations, see e.g. [9].

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and assume that the energy density $W : \mathbb{R}^{m \times d} \to \mathbb{R}$ satisfies* **H1-H3** *with $p \in (1, \infty)$. Furthermore, let $g \in W^{1,p}(\Omega)$, $f \in L^{p'}(\Omega)$ and $h \in \tilde{W}^{-\frac{1}{p'},p'}(\Gamma_N)$. If $\Gamma_D = \emptyset$, we require in addition that f and h satisfy the solvability condition*

$$\int_\Omega f v \, \mathrm{d}x + \langle h, v \rangle_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0 \tag{24}$$

for every constant $v \in \mathbb{R}^m$. The minimisation problem

$$\begin{gathered} \text{Find } u : \Omega \to \mathbb{R}^m,\ u \in g + V \text{ such that for every } v \in g + V \\ I(u) \leqslant I(v) = \int_\Omega W(\nabla v(x)) \, \mathrm{d}x - \int_\Omega f v \, \mathrm{d}x - \langle h, v \rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)} \end{gathered} \tag{25}$$

has a solution u. Moreover, $\sigma = DW(\nabla u) \in L^{p'}(\Omega)$. If u_1 and u_2 are two minimisers, then $\sigma_1 = DW(\nabla u_1) = DW(\nabla u_2) = \sigma_2$. The functional I is Fréchet-differentiable and the minimisation problem is equivalent to solving the weak Euler-Lagrange equations

$$\text{Find } u \in g + V \text{ such that for every } v \in V$$
$$\int_\Omega DW(\nabla u(x)) : \nabla v(x)\,\mathrm{d}x = \int_\Omega f v\,\mathrm{d}x + \langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)}. \tag{26}$$

Remark 2.5.
1. Even though minimisers of I need not be unique, the stress field is unique. This is due to the lower bound $|DW(A) - DW(B)|^r$ in **H3**.
2. The theorem remains true if the energy I in (25) is defined via the linearised strain tensor $\varepsilon(v)$. More precisely, let $m = d \in \{2,3\}$, $DW(A) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ for $A \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and let in the definition of I the gradient ∇v be replaced by the linearised strain tensor $\varepsilon(v) = \frac{1}{2}\left(\nabla v + (\nabla v)^\top\right)$. In this case, the solvability condition has to be replaced by the condition $\int_\Omega f r\,\mathrm{d}x + \langle h, r\rangle_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0$ for every $r \in \mathcal{R} = \{r : \Omega \to \mathbb{R}^d : r(x) = a + Bx,\, a \in \mathbb{R}^d, B \in \mathbb{R}^{d\times d},\, B + B^\top = 0\} = \ker \varepsilon$. Now, the stress σ is defined as $\sigma = DW(\varepsilon(u))$.

2.3 Admissible domains

It is known from the regularity theory for weak solutions of linear elliptic equations that the global regularity does not only depend on the smoothness of the data but also on the geometry of the domain Ω. In this section we describe geometrical assumptions on Ω which enable us to apply the difference quotient technique for the derivation of global regularity results for σ. The geometrical assumptions depend on the boundary conditions. We give first an abstract definition of admissible domains. In lemma 2.8 we then describe two and three dimensional examples.

Definition 2.6 (Cone). A set $\mathcal{K} \subset \mathbb{R}^d$ is a cone with vertex in $x_0 \in \mathbb{R}^d$ if there exists a simply connected, open and nonempty set $\mathcal{C} \subset \partial B_1(0) = \{x \in \mathbb{R}^d : |x| = 1\}$ such that $\mathcal{K} = \{x \in \mathbb{R}^d \backslash \{x_0\} : (x - x_0)/|x - x_0| \in \mathcal{C}\}$.

Definition 2.7 (Admissible domain). Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ where Γ_D and Γ_N are open (possibly empty) and disjoint.

1. Case, $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$: Ω is an admissible domain if it has a Lipschitz boundary.

2. Case, $\overline{\Gamma_D} \cap \overline{\Gamma_N} \neq \emptyset$: Ω is an admissible domain if it has a Lipschitz boundary and if in addition there exists a finite number of open balls $B_{R_j}(x_j)$ with radius R_j and centre $x_j \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ and a finite number of cones $\mathcal{K}_j \subset \mathbb{R}^d$ with vertex in 0 such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} \subset \cup_{j=1}^J B_{R_j}(x_j)$. Furthermore, for every j there exist nonempty domains $\Omega_D^j, \Omega_N^j \subset B_{R_j}(x_j)$ with $\Omega_D^j \cap \Omega_N^j = \emptyset$ and

$$\overline{B_{R_j}(x_j)\backslash\Omega} = \overline{\Omega_D^j \cup \Omega_N^j},\;\; \Gamma_D \cap B_{R_j}(x_j) \subset \partial\Omega_D^j,\;\; \Gamma_N \cap B_{R_j}(x_j) \subset \partial\Omega_N^j, \tag{27}$$
$$\Big(\big(B_{R_j}(x_j)\backslash\overline{\Omega_N^j}\big) + \mathcal{K}_j\Big) \cap \Omega_N^j = \emptyset, \tag{28}$$
$$(\Omega_D^j + \mathcal{K}_j) \cap \Big(B_{R_j}(x_j)\backslash\overline{\Omega_D^j}\Big) = \emptyset, \tag{29}$$

see also figure 1 (left, the index j is omitted). Here, the notation $\Omega + \mathcal{K} = \{y \in \mathbb{R}^d : y = x + h, x \in \Omega, h \in \mathcal{K}\}$ is used.

The next lemma describes some examples of admissible domains for $d = 2, 3$. The proof of this lemma is technical and is given in the appendix.

Lemma 2.8. *1. Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz-polygon. Ω is admissible if and only if the interior opening angle at those points, where $\overline{\Gamma_D}$ and $\overline{\Gamma_N}$ intersect, is strictly less than π, $\measuredangle(\Gamma_D, \Gamma_N) < \pi$.*

Figure 1: Examples for admissible domains

2. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz-polyhedron where at most three faces intersect in the neighbourhood of those points, where the type of the boundary conditions changes. Assume in addition that the interior opening angle between the Dirichlet and Neumann boundary is strictly less than π. Then Ω is an admissible domain, see figure 1 (right) for an example.*

2.4 Stress regularity in the convex case

We are now ready to formulate the main result on the global regularity of stress fields of convex minimisation problems with mixed boundary conditions on admissible domains. In addition to the assumptions on the domain we also have to require higher smoothness of the right hand sides f, g, h in (26). In particular we assume that the Neumann datum h can be written as $h = H\big|_{\Gamma_N}\vec{n}$ on Γ_N, where $\vec{n}$ is the exterior unit normal vector on Γ_N and the function $H : \Omega \to \mathbb{R}^{m\times d}$ is specified in the following theorem.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and assume that $W : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies* **H1**-**H3** *for $r, p > 1$, $s \geqslant 0$ with $\frac{rp}{p+s} > 1$. Let $\hat{\Omega} \supset\supset \Omega$ be an arbitrary domain and assume further that $f \in L^{p'}(\Omega)$, $g \in W^{2,p}(\hat{\Omega})$, $\nabla g \in L^\infty(\hat{\Omega})$ and $H \in W^{1,p'}(\hat{\Omega}, \mathbb{R}^{m\times d}) \cap L^\infty(\hat{\Omega})$. Let $u \in W^{1,p}(\Omega)$ be a minimiser of problem (25) with $u\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ and $h = H\big|_{\Gamma_N}\vec{n}$ on Γ_N.*

If $\nabla u \in L^\alpha(\Omega)$ for some $\alpha \geqslant p$ and if $\sigma = DW(\nabla u) \in L^\gamma(\Omega)$ with $\gamma = \max\{p', \frac{\alpha r}{\alpha+s}\}$, then σ has the following global regularity for every $\delta > 0$:

$$\sigma = DW(\nabla u) \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega) \subset W^{\frac{1}{r}-\delta,\tau}(\Omega). \tag{30}$$

Here, $\tau = \frac{\alpha r}{\alpha+s} \geqslant \frac{pr}{p+s} > 1$.

Remark 2.10. 1. The theorem remains true if in the minimisation problem (25) ∇u is replaced by $\varepsilon(u)$ and if the assumptions of remark 2.5 hold.

2. In [5] C. Carstensen and S. Müller obtained the local regularity $\sigma \in W^{1,\tau}_{\mathrm{loc}}(\Omega)$ with $\tau = \frac{pr}{p+s}$.

3. In the next section we discuss the optimality of theorem 2.9 for linear elasticity and for equations of power-law type.

Proof. We apply a difference quotient technique to deduce estimates for the stress fields in Nikolskii norms. For the derivation of these estimates the domain Ω is covered by a finite number of balls and the estimates are proved for each of these balls separately. The estimates are obtained by inserting suitable differences of weak solutions and shifted weak solutions into the weak formulation and by applying the convexity inequality. The main difficulty is that weak solutions have to be extended across the boundary of Ω in such a way that differences of weak solutions and shifted weak solutions are admissible test functions for the weak formulation. Due to the assumptions on the domain Ω it is possible to define such extensions. We partially take up the ideas from [13] in the proof.

Let $\Omega \subset \mathbb{R}^d$ be an admissible domain. In particular, Ω is a Lipschitz domain and satisfies therefore the uniform interior and exterior cone condition [18]. It follows together with part *2.* of definition 2.7 that there exists a finite number of balls $B_{R_j}(x_j)$ and cones $\mathcal{K}_j$ with vertices in 0 such that $\overline{\Omega} \subset \cup_{j=1}^J B_{R_j}(x_j)$ and each of the pairs $(B_{R_j}(x_j), \mathcal{K}_j)$ satisfies one of the following four cases:

1. $\overline{B_{R_j}(x_j)} \subset \Omega$.
2. $\overline{(B_{R_j}(x_j) \cap \partial\Omega)} \subset \Gamma_D$ and for every $x \in B_{R_j}(x_j) \cap \Gamma_D$ it holds $((x+\mathcal{K}_j) \cap B_{R_j}(x_j)) \cap \overline{\Omega} = \emptyset$.
3. $\overline{(B_{R_j}(x_j) \cap \partial\Omega)} \subset \Gamma_N$ and for every $x \in B_{R_j}(x_j) \cap \overline{\Omega}$ it holds $((x + \mathcal{K}_j) \cap B_{R_j}(x_j)) \subset \Omega$.
4. $x_j \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ and the pair $(B_{R_j}(x_j), \mathcal{K}_j)$ satisfies (27)-(29) of definition 2.7 with suitable domains Ω_D^j and Ω_N^j.

Note that there exists $\theta > 0$ such that the balls $B_{R_j-\theta}(x_j)$ still cover $\overline{\Omega}$. We prove now that

$$\sigma\big|_{\Omega\cap B_{R_j-\theta}(x_j)} \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega \cap B_{R_j-\theta}(x_j))$$

for every j and consider the fourth case in detail. The remaining cases can be treated similarly. In order to simplify the notation we omit the index j in the following.

Let B_R be a ball, $\mathcal{K}$ a cone with vertex in 0 and $\Omega_D, \Omega_N \subset B_R$ domains such that (27)-(29) of definition 2.7 hold. Let $u \in W^{1,p}(\Omega)$ be a weak solution of minimisation problem (25) with $f \in L^{p'}(\Omega)$, $g \in W^{2,p}(\hat{\Omega})$, $\nabla g \in L^\infty(\hat{\Omega})$ and $H \in W^{1,p'}(\hat{\Omega}, \mathbb{R}^{m\times d}) \cap L^\infty(\hat{\Omega})$. Note that the Neumann term in (25) can be rewritten as

$$\langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)} = \langle H\vec{n}, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)} = \int_\Omega (\operatorname{div} H)v \,\mathrm{d}x + \int_\Omega H : \nabla v \,\mathrm{d}x \tag{31}$$

for $v \in V$. Let $\Omega_0 = \operatorname{int}\left(\overline{\Omega \cap B_R} \cup \overline{\Omega}_D\right) = B_R \backslash \overline{\Omega}_N$ and assume that $\Omega_0 \subset \hat{\Omega}$, see figure 1. We extend u to Ω_D as follows

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ g(x), & x \in \Omega_0. \end{cases} \tag{32}$$

Since $u\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ it follows that $\tilde{u} \in W^{1,p}(\Omega \cup \Omega_0)$. Choose $\eta \in \mathcal{C}_0^\infty(B_R)$ with $\eta = 1$ on $B_{R-\theta}$ and define for $x \in \Omega$ and $h \in \mathcal{K}$ with $|h| < h_0 = \frac{1}{2}\operatorname{dist}(\operatorname{supp}\eta, \partial B_R)$:

$$\begin{aligned} v(x) &= \eta^2(x)\left(\tilde{u}(x+h) - g(x+h) - (\tilde{u}(x) - g(x))\right) \\ &= \eta^2(x)\triangle_h(\tilde{u}(x) - g(x)). \end{aligned} \tag{33}$$

Here, we use the notation $\triangle_h w(x) = w(x+h) - w(x)$ for $h \in \mathbb{R}^d$. Note that $v \in W_0^{1,p}(\Omega)$ and therefore v is an admissible test function for the weak formulation (26). Assume that $\nabla u \in L^\alpha(\Omega)$ for some $\alpha \geqslant p$ and let $\tau = \frac{\alpha r}{\alpha+s}$. It follows from convexity inequality (11) with $A = \nabla\tilde{u}(x+h)$, $B = \nabla\tilde{u}(x)$ and Hölder's inequality with $\frac{r}{\tau} \geqslant 1$ that

$$\begin{aligned} \int_\Omega \eta^{\frac{4\tau}{r}} \left|\triangle_h DW(\nabla\tilde{u})\right|^\tau \mathrm{d}x &\overset{(11)}{\leqslant} c\int_\Omega \eta^{\frac{4\tau}{r}} \left(1 + |\nabla\tilde{u}|^s + |\nabla\tilde{u}(x+h)|^s\right)^{\frac{\tau}{r}} \\ &\qquad \times \left(\triangle_h W(\nabla\tilde{u}) - DW(\nabla\tilde{u}) : \triangle_h\nabla\tilde{u}\right)^{\frac{\tau}{r}} \mathrm{d}x \\ &\leqslant c\left(\int_\Omega \eta^{\frac{2\tau}{r-\tau}} \left(1 + |\nabla\tilde{u}|^s + |\nabla\tilde{u}(x+h)|^s\right)^{\frac{\tau}{r-\tau}} \mathrm{d}x\right)^{\frac{r-\tau}{r}} \\ &\qquad \times \left(\int_\Omega \eta^2 \left(\triangle_h W(\nabla\tilde{u}) - DW(\nabla\tilde{u}) : \triangle_h\nabla\tilde{u}\right) \mathrm{d}x\right)^{\frac{\tau}{r}} \\ &= cI_1I_2. \end{aligned} \tag{34}$$

It is $s\tau(r-\tau)^{-1} = \alpha$ and since $\nabla u \in L^\alpha(\Omega)$ and $\nabla g \in L^\infty(\hat{\Omega})$, the factor I_1 is bounded independently of $h \in \mathcal{K}$. Therefore, there exists a constant $c > 0$ such that for every $h \in \mathcal{K}$ with $|h| < h_0$

$$c \left\| \eta^{\frac{4}{r}} \left| \triangle_h DW(\nabla \tilde{u}) \right| \right\|_{L^\tau(\Omega)}^r \leqslant \int_\Omega \eta^2 \triangle_h W(\nabla \tilde{u}) \, \mathrm{d}x - \int_\Omega \eta^2 DW(\nabla \tilde{u}) : \triangle_h \nabla \tilde{u} \, \mathrm{d}x$$
$$= I_{21} + I_{22}. \tag{35}$$

We prove now that $|I_{21}| + |I_{22}| \leqslant c\,|h|$ for a constant $c > 0$ which is independent of $h \in \mathcal{K}$. Due to the product rule for differences, $\triangle_h(f(x)g(x)) = f(x)\triangle_h g(x) + g(x+h)\triangle_h f(x)$, we obtain for I_{21}

$$\begin{aligned} I_{21} &= \int_\Omega \triangle_h \left(\eta^2 W(\nabla \tilde{u})\right) \, \mathrm{d}x - \int_\Omega W(\nabla \tilde{u}(x+h)) \triangle_h \eta^2 \, \mathrm{d}x. \\ &= I_{211} + I_{212}. \end{aligned} \tag{36}$$

Note that $(\operatorname{supp}\eta \cup \operatorname{supp}\eta(\cdot + h)) \subset B_R$ for $h \in \mathcal{K}$ with $|h| < h_0$ and therefore we get taking into account the definition of $\tilde{u}$

$$\begin{aligned} I_{211} &= \int_{\Omega \cap B_R} \eta^2(x+h) W(\nabla \tilde{u}(x+h)) \, \mathrm{d}x - \int_\Omega \eta^2 W(\nabla \tilde{u}) \, \mathrm{d}x \\ &= \int_{(\Omega+h) \cap B_R} \eta^2 W(\nabla \tilde{u}) \, \mathrm{d}x - \int_{\Omega \cap B_R} \eta^2 W(\nabla \tilde{u}) \, \mathrm{d}x \\ &= \int_{((\Omega+h)\setminus\Omega) \cap B_R} \eta^2 W(\nabla g) \, \mathrm{d}x - \int_{((\Omega\setminus(\Omega+h)) \cap B_R} \eta^2 W(\nabla u) \, \mathrm{d}x. \end{aligned} \tag{37}$$

From $\nabla g \in L^\infty(\hat{\Omega})$ and assumption **H2**, inequality (9), we obtain

$$\begin{aligned} I_{211} &\leqslant |((\Omega+h)\setminus\Omega) \cap B_R| \left\| \eta^2 W(\nabla g) \right\|_{L^\infty(B_R)} - \int_{(\Omega\setminus(\Omega+h)) \cap B_R} \eta^2 \left(c_0 \left|\nabla u\right|^p - c_1\right) \, \mathrm{d}x \\ &\leqslant c\,|h| \left(\left\| \eta^2 W(\nabla g) \right\|_{L^\infty(B_R)} + c_1\right) - c_0 \int_{(\Omega\setminus(\Omega+h)) \cap B_R} \eta^2 \left|\nabla u\right|^p \, \mathrm{d}x \end{aligned} \tag{38}$$

and the constant c is independent of $h \in \mathcal{K}$. Since $\eta \in \mathcal{C}_0^\infty(B_R)$, there exists $c(\eta) > 0$ such that

$$\left| \eta^2(x+h) - \eta^2(x) \right| \leqslant c(\eta)\,|h|$$

for every $h \in \mathbb{R}^d$ and $x \in B_R$. Thus the term I_{212} can be estimated as follows

$$|I_{212}| \leqslant c(\eta)\,|h| \int_{(\Omega \cup \Omega_D) \cap B_R} \left|W(\nabla \tilde{u})\right| \, \mathrm{d}x \leqslant c\,|h|\,. \tag{39}$$

We obtain finally from (38) and (39) that

$$I_{21} = I_{211} + I_{212} \leqslant c\,|h| - c_0 \int_{(\Omega\setminus(\Omega+h)) \cap B_R} \eta^2 \left|\nabla u\right|^p \, \mathrm{d}x, \tag{40}$$

where c is independent of $h \in \mathcal{K}$ and c_0 is the constant from (9). Applying the product rule to I_{22} and taking into account (31) and that $v = \eta^2 \triangle_h(\tilde{u} - g)$ is an admissible test function for the weak formulation (26) we obtain

$$
\begin{aligned}
I_{22} &= -\int_\Omega \eta^2 DW(\nabla\tilde u) : \triangle_h \nabla \tilde u \,\mathrm{d}x \\
&= -\int_\Omega DW(\nabla\tilde u) : \nabla\left(\eta^2\triangle_h(\tilde u - g)\right)\,\mathrm{d}x - \int_\Omega DW(\nabla\tilde u) : \nabla(\eta^2\triangle_h g)\,\mathrm{d}x \\
&\quad + \int_\Omega DW(\nabla\tilde u) : (\triangle_h\tilde u \otimes \nabla\eta^2)\,\mathrm{d}x \\
&= -\int_\Omega \eta^2(f + \operatorname{div} H)\triangle_h(\tilde u - g)\,\mathrm{d}x - \int_\Omega H : \nabla(\eta^2\triangle_h\tilde u)\,\mathrm{d}x + \int_\Omega H : \nabla(\eta^2\triangle_h g)\,\mathrm{d}x \\
&\quad + \int_\Omega DW(\nabla\tilde u) : (\triangle_h\tilde u \otimes \nabla\eta^2)\,\mathrm{d}x - \int_\Omega DW(\nabla\tilde u) : \nabla(\eta^2\triangle_h g)\,\mathrm{d}x \\
&= I_{221} + \ldots + I_{225}.
\end{aligned} \tag{41}
$$

Here, $(\triangle_h\tilde u \otimes \nabla\eta^2)_{ij} = (\partial_i\eta)(\triangle_h\tilde u_j)$ and $\tilde u_j$ is the j-th component of $\tilde u$. Lemma 7.23 in [17] implies for $\tilde u, g \in W^{1,p}(\Omega\cup\Omega_0)$, and $h \in \mathcal{K}$ that

$$
\|\eta\triangle_h(\tilde u - g)\|_{L^p((\Omega\cup\Omega_0)\cap B_R)} \leqslant |h|\,\|u-g\|_{L^p(\Omega\cap B_R)}\,\|\eta\|_{L^\infty(B_R)}\,.
$$

Therefore, there exists a constant $c > 0$ which is independent of $h \in \mathcal{K}$ such that

$$
\begin{aligned}
|I_{221}| + |I_{224}| &\leqslant \|f + \operatorname{div} H\|_{L^{p'}(\Omega)}\,\|\eta\triangle_h(\tilde u - g)\|_{L^p((\Omega\cup\Omega_0)\cap B_R)} \\
&\quad + \|DW(\nabla u)\|_{L^{p'}(\Omega)}\,c(\eta)\,\|\eta\triangle_h\tilde u\|_{L^p((\Omega\cup\Omega_0)\cap B_R)} \leqslant c\,|h|\,.
\end{aligned} \tag{42}
$$

Since $g \in W^{2,p}(\hat\Omega)$, similar arguments show that

$$
|I_{223}| + |I_{225}| \leqslant c\,|h|\,. \tag{43}
$$

In order to estimate I_{222} we apply again the product rules for differences and derivatives:

$$
\begin{aligned}
I_{222} &= -\int_\Omega H : \nabla(\eta^2\triangle_h\tilde u)\,\mathrm{d}x \\
&= -\int_\Omega H : \left(\triangle_h\tilde u \otimes \nabla\eta^2\right)\,\mathrm{d}x - \int_\Omega \triangle_h\left(\eta^2 H : \nabla\tilde u\right)\,\mathrm{d}x + \int_\Omega (\triangle_h(\eta^2 H)) : \nabla\tilde u(x+h)\,\mathrm{d}x.
\end{aligned} \tag{44}
$$

The first term can be treated similarly to I_{224}, the third term similarly to I_{221}. The second term can be transformed as follows (compare also (37))

$$
\int_\Omega \triangle_h(\eta^2 H : \nabla\tilde u)\,\mathrm{d}x = \int_{(\Omega+h\backslash\Omega)\cap B_R} \eta^2 H : \nabla g\,\mathrm{d}x - \int_{(\Omega\backslash(\Omega+h))\cap B_R} \eta^2 H : \nabla u\,\mathrm{d}x. \tag{45}
$$

Since $\nabla g, H \in L^\infty(\hat\Omega, \mathbb{R}^{m\times d})$, we obtain

$$
\left|\int_{(\Omega+h\backslash\Omega)\cap B_R} \eta^2 H : \nabla g\,\mathrm{d}x\right| \leqslant \left|\left((\Omega+h)\backslash\Omega\right)\cap B_R\right|\,\left\|\eta^2 H : \nabla g\right\|_{L^\infty(\hat\Omega)} \leqslant c\,|h| \tag{46}
$$

and c is independent of $h \in \mathcal{K}$. By Hölder's and Young's inequality we get for the second term in (45) for every $\epsilon > 0$

$$
\left|\int_{(\Omega\backslash(\Omega+h))\cap B_R} (\epsilon^{-1}\eta^{\frac{2}{p'}} H) : (\epsilon\eta^{\frac{2}{p}}\nabla u)\,\mathrm{d}x\right| \leqslant \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \frac{1}{p'}\left|\epsilon^{-1}\eta^{\frac{2}{p'}} H\right|^{p'}\,\mathrm{d}x + \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \frac{\epsilon^p}{p}\eta^2\,|\nabla u|^p\,\mathrm{d}x \tag{47}
$$

and the first term can be estimated by $c\,|h|$, where c depends on $\epsilon > 0$. Estimates (44)-(47) imply that

$$|I_{222}| \leqslant c\,|h| + \frac{\epsilon^p}{p}\int_{(\Omega\setminus\Omega+h)\cap B_R} \eta^2\,|\nabla u|^p\;\mathrm{d}x. \tag{48}$$

Collecting the above estimates (inequalities (40), (42), (43), (48)) we obtain finally that there exists a constant $c > 0$ such that it holds for every $h \in \mathcal{K}$, $|h| < h_0$

$$\begin{aligned}\left\|\eta^{\frac{4}{r}}\,|\triangle_h DW(\nabla\tilde{u})|\right\|^r_{L^\tau(\Omega)} &\leqslant I_{21} + I_{221} + I_{222} + I_{223} + I_{224} + I_{225} \\ &\leqslant c\,|h| + \left(p^{-1}\epsilon^p - c_0\right)\int_{(\Omega\setminus(\Omega+h))\cap B_R} \eta^2\,|\nabla u|^p\;\mathrm{d}x.\end{aligned}$$

Choosing $0 < \epsilon < (pc_0)^{\frac{1}{p}}$ and taking into account that $\eta\big|_{B_{R-\theta}} = 1$ we get

$$\|\triangle_h DW(\nabla u)\|^r_{L^\tau(\Omega\cap B_{R-\theta})} \leqslant c\,|h|$$

for every $h \in \mathcal{K}$, $|h| < h_0$, with a constant c which is independent of h. This implies that $\sigma = DW(\nabla u) \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega\cap B_{R-\theta}) \subset W^{\frac{1}{r}-\delta,\tau}(\Omega\cap B_{R-\theta})$ for every $\delta > 0$, see the definition of the Nikolskii norm and lemma 2.2. $\square$

2.5 Examples for the convex case

As examples for energy densities which satisfy the convexity inequality (11) we consider the energy densities of linear elastic materials, of a variational functional from the deformation theory of plasticity and the energy density which corresponds to the p-Laplace equation.

2.5.1 Linear elasticity

The energy density for linear elastic materials with elasticity tensor $\mathbf{C} \in \mathbb{R}^{(d\times d)\times(d\times d)}$, symmetric and positive definite, is given by $W(\varepsilon) = \frac{1}{2}\mathbf{C}\varepsilon : \varepsilon$ for $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Obviously, it holds due to the positive definiteness of $\mathbf{C}$ that

$$W(\varepsilon_1) - W(\varepsilon_2) - DW(\varepsilon_2) : (\varepsilon_1 - \varepsilon_2) \geqslant c\,|\mathbf{C}\varepsilon_1 - \mathbf{C}\varepsilon_2|^2$$

and thus $p = 2$, $s = 0$, $r = 2$ in (11).

Corollary 2.11. *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let u be a solution of the minimisation problem for linear elasticity. It follows for every $\delta > 0$ that $\sigma = \mathbf{C}\varepsilon(u) \in \mathcal{N}^{\frac{1}{2},2}(\Omega) \subset W^{\frac{1}{2}-\delta,2}(\Omega)$.*

This result is well known for boundary value problems with pure Dirichlet or pure Neumann conditions [10]. For polyhedral domains Ω, the behaviour of displacement and stress fields near corners and edges can be characterised completely by asymptotic expansions [21, 11, 24]. Let $\Omega \subset \mathbb{R}^2$ be a polygon with mixed boundary conditions and suppose that $\mathbf{C}$ describes an isotropic material. It is shown in [25, 27] by a careful study of the asymptotic expansions that $\sigma \in W^{\frac{1}{2},2}(\Omega)$ if Ω is an admissible domain, i.e. if $\measuredangle(\Gamma_D, \Gamma_N) < \pi$ at every point $S \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. Moreover, if $S \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ with $\measuredangle(\Gamma_D, \Gamma_N) > \pi$, then weak solutions exist with $\sigma \in W^{\alpha-\delta,2}(\Omega)$ for an appropriate $0 < \alpha < \frac{1}{2}$ and every $\delta > 0$ but not for $\delta = 0$. The parameter α depends on the material parameters and the opening angle at S. This example shows the optimality of corollary 2.11 for admissible domains.

2.5.2 Hencky elasto-plasticity with linear hardening

For $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we define as in [16]

$$W(\varepsilon) = \frac{1}{2}\kappa_0(\operatorname{tr}\varepsilon)^2 + g_0(\left|\varepsilon^D\right|), \tag{49}$$

where $\kappa_0 > 0$ and $\varepsilon^D = \varepsilon - \frac{1}{d} \operatorname{tr} \varepsilon I$ is the deviatoric part of ε. It is assumed that $g_0 \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\backslash\{t_0\})$ for some $t_0 > 0$ and the left and right limits of g_0'' exist at t_0. The quantity $g_0'(t_0)$ may be interpreted in this context as yield stress. Furthermore, we suppose that there exist constants $\kappa_1, \kappa_2 > 0$ such that for every $t \in \mathbb{R}$

$$\kappa_1 \leqslant \min\{g_0''(t), t^{-1} g_0'(t)\} \leqslant \max\{g_0''(t), t^{-1} g_0'(t)\} \leqslant \kappa_2. \tag{50}$$

It follows with Taylor's expansion that $c_0 t^2 - c_1 \leqslant g_0(t) \leqslant c_2(1 + t^2)$ for every t and constants $c_i > 0$. Note that the energy density

$$W_\eta(\varepsilon) = \frac{1}{2}\mathbf{C}\varepsilon : \varepsilon - \frac{1}{4\mu(1+\eta)} \max\{0, \left|(\mathbf{C}\varepsilon)^D\right| - \sigma_y\}^2, \ \varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \tag{51}$$

from [5] is a special case of (49). Here, $\mathbf{C}$ is the elasticity tensor for isotropic materials, $\mu > 0$ a Lamé constant, $\sigma_y > 0$ the yield stress and $\eta > 0$ the modulus of hardening. The variational problem related to energy density (49) is:

Find $u \in W^{1,2}(\Omega)$ with $u\big|_{\Gamma_D} = g\big|_{\Gamma_D}$ such that for every $v \in W^{1,2}(\Omega)$ with $v\big|_{\Gamma_D} = g\big|_{\Gamma_D}$

$$I(u) \leqslant I(v) = \int_\Omega \frac{1}{2}\kappa_0 (\operatorname{tr} \varepsilon(v))^2 + g_0(\left|\varepsilon^D(v)\right|) \,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x - \langle h, v\rangle_{W^{\frac{1}{2},2}(\Gamma_N)}. \tag{52}$$

Functionals of this type describe in the framework of deformation theory of plasticity the behaviour of materials with linear hardening. The local regularity of stress fields corresponding to minimisers of (52) is studied in [16, 32].

Lemma 2.12. *Energy density W from (49) satisfies the convexity inequality (11) on $\mathbb{R}^{d\times d}_{sym}$ with $s = 0$ and $r = 2$.*

Proof. Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\varepsilon_1 \neq \varepsilon_2$ and $\theta(s) = \varepsilon_2 + s(\varepsilon_1 - \varepsilon_2)$, $s \in [0, 1]$. Note that there are at most two elements $s_i \in [0, 1]$ with $|\theta(s_i)| = t_0$. Therefore we may apply Taylor's expansion at least piecewise on $[0, 1]$ and obtain

$$\begin{aligned}
&W(\varepsilon_1) - W(\varepsilon_2) - DW(\varepsilon_2) : (\varepsilon_1 - \varepsilon_2) \\
&\quad= \frac{1}{2}\kappa_0 \left|\operatorname{tr}(\varepsilon_1 - \varepsilon_2)\right|^2 + \int_0^1 (1-s)\frac{g_0''(\left|\theta^D(s)\right|)}{\left|\theta^D(s)\right|^2} \left(\theta^D(s) : (\varepsilon_1 - \varepsilon_2)^D\right)^2 \mathrm{d}s \\
&\qquad+ \int_0^1 (1-s)\frac{g_0'(\left|\theta^D(s)\right|)}{\left|\theta^D(s)\right|} \left(\left|\varepsilon_1^D - \varepsilon_2^D\right|^2 - \frac{(\theta^D(s) : (\varepsilon_1^D - \varepsilon_2^D))^2}{\left|\theta^D(s)\right|^2}\right) \mathrm{d}s \\
&\quad\overset{(50)}{\geqslant} \frac{1}{2}\kappa_0 \left|\operatorname{tr}(\varepsilon_1 - \varepsilon_2)\right|^2 + \int_0^1 (1-s)\kappa_1 (\theta^D(s) : (\varepsilon_1^D - \varepsilon_2^D))^2 \left|\theta^D(s)\right|^{-2} \mathrm{d}s \\
&\qquad+ \int_0^1 (1-s)\kappa_1 \left(\left|\varepsilon_1^D - \varepsilon_2^D\right|^2 - (\theta^D(s) : (\varepsilon_1^D - \varepsilon_2^D))^2 \left|\theta^D(s)\right|^{-2}\right) \mathrm{d}s \\
&\quad= \frac{1}{2}\left(\kappa_0 \left|\operatorname{tr}(\varepsilon_1 - \varepsilon_2)\right|^2 + \kappa_1 \left|\varepsilon_1^D - \varepsilon_2^D\right|^2\right).
\end{aligned} \tag{53}$$

In a similar way it follows again by (50) that there exists a constant $c > 0$ with

$$\begin{aligned}
\left|DW(\varepsilon_1) - DW(\varepsilon_2)\right| &\leqslant \int_0^1 \left|D^2 W(\theta(s))(\varepsilon_1 - \varepsilon_2)\right| \mathrm{d}s \\
&\leqslant c(\left|\operatorname{tr}(\varepsilon_1 - \varepsilon_2)\right| + \left|\varepsilon_1^D - \varepsilon_2^D\right|).
\end{aligned} \tag{54}$$

Combining (53) and (54) finishes the proof. ∎

Corollary 2.13. *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let $u \in W^{1,2}(\Omega)$ be a minimiser of (52) with data $f, g, h = H\vec{n}$ as in theorem 2.9 ($p = 2$). Then $\sigma = DW(\varepsilon(u)) \in \mathcal{N}^{\frac{1}{2},2}(\Omega) \cap W^{\frac{1}{2}-\delta,2}(\Omega)$ for every $\delta > 0$.*

2.5.3 The p-Laplace equation

The energy density corresponding to the p-Laplace equation is given by

$$W(A) = \frac{1}{p} |A|^p , \quad A \in \mathbb{R}^d. \tag{55}$$

If $p \geqslant 2$, then energy density W satisfies monotonicity inequality (4) with $r = 2$ and $s = p - 2$ [5]. Lemma 2.3 implies that the convexity inequality holds also. Furthermore, the following global regularity result is available for weak solutions $u : \Omega \to \mathbb{R}$, $u \in W^{1,p}(\Omega)$ of the p-Laplace equation on admissible domains ($p \geqslant 2$) [31, 13]:

$$u \in W^{1+\frac{1}{p}-\delta,p}(\Omega)$$

for every $\delta > 0$. By the Sobolev embedding theorems we get $\nabla u \in L^{\alpha-\delta}(\Omega)$ with $\alpha = \frac{dp}{d-1} > p$.

Corollary 2.14. *Let $m = 1$, $p \geqslant 2$ and $u \in W^{1,p}(\Omega)$ be a minimiser of (25) with energy density (55) on an admissible domain $\Omega \subset \mathbb{R}^d$. Let the data $f, g, h = H\vec{n}$ be given according to theorem 2.9 with $\alpha = \frac{dp}{d-1}$. Then $\sigma = DW(\nabla u) \in W^{\frac{1}{2}-\delta,\tau}(\Omega)$ for every $\delta > 0$ and $\tau = \frac{2\alpha}{\alpha+p-2}$.*

Remark 2.15. Let $\Omega \subset \mathbb{R}^2$ be an admissible polygon and assume that $\sigma = DW(\nabla u)$ is a weak solution of the p-Laplace equation with $p \geqslant 2$ of the form $\sigma = r^\gamma \sigma_0(\varphi)$, where (r, φ) are polar coordinates with respect to a corner point S and $\sigma_0 \not\equiv 0$. By [30, Lemma 2.3.1] we obtain for δ, τ as in corollary 2.14 that $\sigma \in W^{\frac{1}{2}-\delta,\tau}(\Omega)$ if and only if $\gamma \geqslant \frac{1-p}{p}$. In [3] a weak solution u is constructed for a domain with a crack and smooth right hand sides, where $\gamma = \frac{1-p}{p}$. This indicates the optimality of theorem 2.9 also for nonlinear elliptic equations of p-structure.

3 Regularity for stresses of some nonconvex variational problems

Nonconvex variational problems may fail to have minimisers and a relaxed problem is studied instead. This relaxed problem is in general defined through an energy density which is the quasi-convex envelope of the nonconvex energy density. Weak cluster points of infimising sequences of the nonconvex problem are minimisers of the relaxed problem [9, 1, 15]. Moreover, if the nonconvex problem has a minimiser, then this minimiser is also a minimiser of the relaxed problem and the corresponding stress fields coincide under suitable assumptions on the energy densities. This relation is the key for carrying over regularity results from the convex case to minimisers of nonconvex problems. After a short description of these relations we formulate the regularity theorem and illustrate it with some examples.

3.1 Regularity for stress fields of nonconvex variational problems

Let $W \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ be an energy density satisfying growth condition (9) for some $p > 1$ and let I be the energy functional related to W, see (25). By W^{qc} and W^c we denote the quasi-convex and convex envelope of W, respectively, i.e. for $A \in \mathbb{R}^{m\times d}$

$$W^{qc}(A) = \sup\{g(A) : \; g \leqslant W \text{ and } g \text{ is quasi-convex}\}$$

and similar for W^c. For a definition of quasi-convexity we refer to B. Dacorogna's book [9]. Furthermore, we define for $v \in W^{1,p}(\Omega)$

$$I^{qc}(v) = \int_\Omega W^{qc}(\nabla v)\,\mathrm{d}x - \int_\Omega f v \,\mathrm{d}x - \langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)},$$

where f, h are given as in theorem 2.4; I^c is analogously defined. The following well-known theorem describes the relation between minimisers of I^{qc} and infimising sequences of I. For convenience we formulate it here for our situation.

Theorem 3.1. *[9, 1, 15] Let $W \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ satisfy* (9) *for $p > 1$, $g \in W^{1,p}(\Omega)$, $f \in L^{p'}(\Omega)$, $h \in W^{-\frac{1}{p'},p'}(\Gamma_N)$ and assume that f, h satisfy solvability condition* (24) *if $\Gamma_D = \emptyset$. Then the minimisation problem for I^{qc} on $g + V$, where V is the space defined in* (5)*, has a minimiser $u^{qc} \in g + V$ and it holds*

$$\inf_{v\in g+V} I(v) = I^{qc}(u^{qc}).$$

Furthermore, every weak cluster point of infimising sequences of I is a minimiser of I^{qc}. In particular, every minimiser u of I is also a minimiser of I^{qc}.

This theorem follows immediately from [9, Thm. 5.2.1]. Note that it is still valid for functionals I with energy density $W(\varepsilon(u))$ instead of $W(\nabla u)$. In this case, $W^{qc}(\varepsilon) = \tilde{W}^{qc}(\varepsilon)$, where $\tilde{W}^{qc}$ is the quasiconvex envelope of $\tilde{W}(A) = W(\frac{1}{2}(A + A^\top))$, $A \in \mathbb{R}^{d\times d}$, [9, Thm. 5.1.1, Thm. 5.2.1]. Due to theorem 3.1 we have

Lemma 3.2. *Let the assumptions of theorem 3.1 be satisfied and suppose that $u \in g + V$ is a minimiser of I. Then $W(\nabla u) = W^{qc}(\nabla u)$ almost everywhere in Ω. Furthermore, let $\mathcal{M} = \{A \in \mathbb{R}^{m\times d} : W(A) = W^{qc}(A)\}$ and assume that W and W^{qc} are differentiable on an open neighbourhood of $\mathcal{M}$. Then*

$$DW(\nabla u) = DW^{qc}(\nabla u) \quad \text{a.e. in } \Omega.$$

Proof. Let u be a minimiser of I. Due to theorem 3.1, u is also a minimiser of I^{qc} and $I(u) = I^{qc}(u)$. Taking into account the definition of W^{qc} we obtain immediately the first assertion of lemma 3.2. In order to show the second assertion we only have to prove that $DW(A) = DW^{qc}(A)$ for every $A \in \mathcal{M}$. Let $A \in \mathcal{M}$, $H \in \mathbb{R}^{m\times d}$ be arbitrary. Then

$$\begin{aligned} DW(A) : H &= \lim_{t\searrow 0} t^{-1}(W(A + tH) - W(A)) = \lim_{t\searrow 0} t^{-1}(W(A + tH) - W^{qc}(A)) \\ &\geqslant \lim_{t\searrow 0} t^{-1}(W^{qc}(A + tH) - W^{qc}(A)) = DW^{qc}(A) : H \end{aligned}$$

and in the same way

$$DW(A) : H = \lim_{t\nearrow 0} t^{-1}(W(A + tH) - W(A)) \leqslant DW^{qc}(A) : H.$$

Since H is arbitrary we obtain $DW(A) = DW^{qc}(A)$. □

Combining lemma 3.2 and regularity theorem 2.9 implies the following regularity theorem for stress fields in the nonconvex case:

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let $W \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ satisfy* (9) *for $p > 1$. Moreover, let W be differentiable on a neighbourhood of $\mathcal{M}$ with $\mathcal{M}$ as in lemma 3.2. Assume that the data f, g, H is given as in regularity theorem* (2.9)*. Furthermore, we suppose that the convex envelope and the quasi-convex envelope of W coincide, $W^{qc} = W^c$, and that W^c satisfies* **H1**, **H2** *and convexity inequality* **H3** *with $s \geqslant 0$, $r > 1$ and $\tau = \frac{rp}{p+s} > 1$. Let u be a minimiser of the minimisation problem for I and assume that $DW^c(\nabla u) \in L^\gamma(\Omega)$ with $\gamma = \max\{p', \tau\}$. Then $DW(\nabla u) = DW^c(\nabla u)$ and for every $\delta > 0$*

$$\sigma = DW(\nabla u) \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega) \subset W^{\frac{1}{r}-\delta,\tau}(\Omega).$$

Furthermore, σ is unique, which means that $DW(\nabla u_1) = DW(\nabla u_2)$ for any two minimisers u_i of I.

The assumption $W^{qc} = W^c$ is automatically satisfied if $m = 1$ or $d = 1$. In the next section we give an example with $W^{qc} = W^c$ and $\min\{m, d\} \geqslant 2$. Note that theorem 3.3 holds also if ∇u is replaced with $\varepsilon(u)$ in the definition of I.

3.2 Nonconvex examples

We show in this section that the nonconvex examples from [5] are covered by the global regularity theorem 3.3. To this end we only have to verify that the convex envelopes of the energy densities satisfy the convexity inequality.

3.2.1 Scalar two-well potential, $m = 1$

The energy density of the scalar two-well potential reads for $A \in \mathbb{R}^d$ and fixed $A_1 \neq A_2 \in \mathbb{R}^d$

$$W(A) = |A - A_1|^2 |A - A_2|^2 .$$

Since $m = 1$, the convex and the quasi-convex envelopes W^c and W^{qc} coincide and [6]

$$W^c(A) = \max\left\{|A - F|^2 - |G|^2, 0\right\}^2 + 4\left(|G|^2 |A - F|^2 - (G \cdot (A - F))^2\right),$$

where $G = (A_2 - A_1)/2$ and $F = (A_1 + A_2)/2$. It is shown in [6] that W^c satisfies monotonicity inequality (4) with $p = 4$ and $r = s = 2$.

Lemma 3.4. *There exists a constant $c > 0$ such that it holds for every $A \in \mathbb{R}^d$*

$$|A| \leqslant c(1 + |DW^c(A)|^{\frac{1}{3}}). \tag{56}$$

Therefore, W^c satisfies convexity inequality (11) *with $p = 4$ and $r = s = 2$ due to lemma 2.3.*

Proof. For $A \neq F$ it holds

$$\begin{aligned} |DW^c(A)| &= \sup_{H \in \mathbb{R}^d \setminus \{0\}} DW^c(A) : H |H|^{-1} \geqslant DW^c(A) : (A - F) |A - F|^{-1} \\ &\geqslant 4 \max\left\{|A - F|^2 - |G|^2, 0\right\} |A - F| . \end{aligned}$$

Assume now that $|A - F| \geqslant |G|$. Young's inequality yields for every $\delta > 0$

$$|DW^c(A)| \geqslant 4 |A - F|^3 - \delta^{-1} 4 |G|^2 \delta |A - F| \geqslant \left(4 - \frac{\delta^3}{3}\right) |A - F|^3 - \frac{16}{3} |G|^3 \delta^{-\frac{3}{2}}.$$

For $\delta_0 = 2^{\frac{2}{3}}$ it holds $c_1 := 4 - \frac{\delta_0^3}{3} > 0$. Moreover, it follows for $|A - F| \leqslant |G|$ that

$$\left(4 - \frac{\delta_0^3}{3}\right) |A - F|^3 - \frac{16}{3} \delta_0^{-\frac{3}{2}} |G|^3 \leqslant 0.$$

Therefore

$$|DW^c(A)| \geqslant c_1 |A - F|^3 - c_1$$

for every $A \in \mathbb{R}^d$. With $|A - F| \geqslant |A| - |F|$ and applying once more Young's inequality we obtain finally (56). $\square$

Corollary 3.5. *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let u be a minimiser of I or I^c. Then it follows for the corresponding stress field for every $\delta > 0$: $\sigma \in \mathcal{N}^{\frac{1}{2}, \frac{4}{3}}(\Omega) \subset W^{\frac{1}{2} - \delta, \frac{4}{3}}(\Omega)$.*

3.2.2 A vectorial two-well potential, $m = d$

For $\varepsilon_1 \neq \varepsilon_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we consider the following energy densities

$$W_i(\varepsilon) = \frac{1}{2}\mathbf{C}(\varepsilon - \varepsilon_i) : (\varepsilon - \varepsilon_i) + W_i^0, \quad \varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ i = 1, 2, \tag{57}$$

where $\mathbf{C}$ is the elasticity tensor for linear elastic materials and $W_i^0 \in \mathbb{R}$. Let

$$W(\varepsilon) = \min\{W_1(\varepsilon), W_2(\varepsilon)\}, \quad \varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{58}$$

The nonconvex function W describes in a geometrically linear framework the elastic strain energy density of a two-phase material with stress-free strains ε_i, see e.g. [20, 19]. It is assumed that both phases have identical elasticity tensors. If the strains ε_1 and ε_2 are compatible, i.e. if there exist $a, b \in \mathbb{R}^d$ with

$$\varepsilon_1 - \varepsilon_2 = \frac{1}{2}(a \otimes b + b \otimes a),$$

then the convex and quasi-convex envelopes of W coincide and are given by [20]

$$W^c(\varepsilon) = \begin{cases} W_2(\varepsilon) & \text{if } W_2(\varepsilon) + \gamma \leqslant W_1(\varepsilon), \\ W_3(\varepsilon) & \text{if } |W_1(\varepsilon) - W_2(\varepsilon)| \leqslant \gamma, \\ W_1(\varepsilon) & \text{if } W_1(\varepsilon) + \gamma \leqslant W_2(\varepsilon), \end{cases} \tag{59}$$

where $\gamma = \frac{1}{2}\mathbf{C}(\varepsilon_1 - \varepsilon_2) : (\varepsilon_1 - \varepsilon_2)$ and

$$W_3(\varepsilon) = \frac{1}{2}(W_2(\varepsilon) + W_1(\varepsilon)) - \frac{1}{4\gamma}(W_2(\varepsilon) - W_1(\varepsilon))^2 - \frac{\gamma}{4}. \tag{60}$$

It is shown by C. Carstensen and P. Plecháč in [7] that W^c satisfies the monotonicity inequality with $p = r = 2$ and $s = 0$. We prove directly that W^c satisfies also the convexity inequality.

Lemma 3.6. *W^c satisfies convexity inequality* (11) *with $p = r = 2$ and $s = 0$.*

Proof. Let $A \neq B \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ and $\varphi(t) = W^c(B + t(A - B))$ for $t \in \mathbb{R}$. Then $\varphi \in \mathcal{C}^1(\mathbb{R})$ and φ' is piecewise continuously differentiable. Moreover, the interval $(0, 1)$ can be split into at most three open, disjoint sub-intervals $I_i = (t_1^i, t_2^i)$ such that $\varphi(t) = W_i(B + t(A - B))$ for $t \in I_i$. Taylor's expansion yields

$$\begin{aligned} W^c(A) - W^c(B) - DW^c(B) : (A - B) &= \int_0^1 (1 - t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(t)\,\mathrm{d}t \\ &= \sum_{i=1}^{2} \int_{I_i} (1 - t)\,\mathrm{d}t\,|A - B|^2_{\mathbf{C}} \\ &\quad + \int_{I_3} (1 - t)\,\mathrm{d}t \left(|A - B|^2_{\mathbf{C}} - \frac{1}{2\gamma}\left(\mathbf{C}(\varepsilon_1 - \varepsilon_2) : (A - B)\right)^2 \right), \end{aligned} \tag{61}$$

where we use the notation $|A|^2_{\mathbf{C}} = \mathbf{C}A : A$. On the other hand it follows again with Taylor's expansion that

$$\begin{aligned} DW^c(A) - DW^c(B) &= (|I_1| + |I_2|)\mathbf{C}(A - B) \\ &\quad + |I_3| \left(\mathbf{C}(A - B) - \frac{1}{2\gamma}\left(\mathbf{C}(\varepsilon_1 - \varepsilon_2) : (A - B)\right)\mathbf{C}(\varepsilon_1 - \varepsilon_2) \right) \end{aligned}$$

and thus, by Young's inequality,

$$\begin{aligned} |DW^c(A) - DW^c(B)|^2_{\mathbf{C}^{-1}} &\leqslant 2(|I_1| + |I_2|)^2\,|A - B|^2_{\mathbf{C}} \\ &\quad + 2\,|I_3|^2 \left|(A - B) - \frac{1}{2\gamma}\left(\mathbf{C}(\varepsilon_1 - \varepsilon_2) : (A - B)\right)(\varepsilon_1 - \varepsilon_2)\right|^2_{\mathbf{C}}. \end{aligned} \tag{62}$$

The second term in (62) reduces to

$$\left|(A-B)-\frac{1}{2\gamma}\left(\mathbf{C}(\varepsilon_1-\varepsilon_2):(A-B)\right)(\varepsilon_1-\varepsilon_2)\right|_{\mathbf{C}}^2$$
$$=|A-B|_{\mathbf{C}}^2-\frac{1}{2\gamma}\left(\mathbf{C}(A-B):(\varepsilon_1-\varepsilon_2)\right)^2. \quad (63)$$

Note finally that $|I_i|^2=(t_2^i-t_1^i)^2\leqslant 2(t_2^i-t_1^i)=4\int_{I_i}(1-t)\,\mathrm{d}t$ for $0\leqslant t_1^i\leqslant t_2^i\leqslant 1$. Combining (61) and (62) finishes the proof of lemma 3.6. □

Corollary 3.7. *Let $\Omega\subset\mathbb{R}^d$ be an admissible domain and u a minimiser of I or I^c with energy density W from (58) and W^c from (59), respectively. Assume that the data f,g,H is given according to theorem 2.9 with $p=2$. Then the corresponding stress field is independent of u. Furthermore, $\sigma\in\mathcal{N}^{\frac{1}{2},2}(\Omega)\subset W^{\frac{1}{2}-\delta,2}(\Omega)$ for every $\delta>0$.*

3.2.3 A special case of the Ericksen James energy

The last example deals with a special case of the two dimensional Ericksen James energy function [8]. Let $\kappa_1,\kappa_2>0$. For $A\in\mathbb{R}^{2\times 2}$ and $C=A^\top A$ we consider the function

$$W(A)=\kappa_1(\operatorname{tr}C-2)^2+\kappa_2c_{12}^2=\kappa_1(|A|^2-2)^2+\frac{\kappa_2}{4}(a(A,A))^2, \quad (64)$$

where the bilinear form $a(\cdot,\cdot)$ is defined as

$$a(A,B)=a_{11}b_{12}+a_{12}b_{11}+a_{21}b_{22}+a_{22}b_{21},\quad A,B\in\mathbb{R}^{2\times 2}. \quad (65)$$

Note that $2c_{12}=a(A,A)$. The complete Ericksen James energy has the additional term

$$\kappa_3\left(\frac{1}{4}\left(a_{11}^2+a_{21}^2-a_{12}^2-a_{22}^2\right)^2-\varepsilon^2\right)^2,\ \kappa_3>0$$

and is applied to model crystalline microstructure, see [8] and the references therein. In this context, $u:\Omega\to\mathbb{R}^2$ is the deformation field, $W(\nabla u)$ the stored energy function of a two dimensional crystal and $C=\nabla u^\top\nabla u$ the right Cauchy-Green strain tensor. Let us emphasise that we consider here only the case $\kappa_3=0$. It is shown by M. Bousselsal and B. Brighi in [4] that the convex and the quasiconvex envelopes W^{qc} and W^c of W from (64) coincide and have the form

$$W^c(A)=\Phi_i(A)\quad\text{for } A\in\mathcal{M}_i,\ 1\leqslant i\leqslant 4, \quad (66)$$

where $\mathbb{R}^{2\times 2}=\cup_{i=1}^4\mathcal{M}_i$ with

$$\begin{aligned}
\mathcal{M}_1&=\{A\in\mathbb{R}^{2\times 2}:\ |a(A,A)|\leqslant 2-|A|^2\},\\
\mathcal{M}_2&=\{A\in\mathbb{R}^{2\times 2}:\ \kappa_2\,|a(A,A)|\leqslant 4\kappa_1(|A|^2-2)\},\\
\mathcal{M}_3&=\{A\in\mathbb{R}^{2\times 2}:\ \kappa_2a(A,A)\geqslant 4\kappa_1(|A|^2-2)\geqslant 0\text{ or }a(A,A)\geqslant 2-|A|^2\geqslant 0\},\\
\mathcal{M}_4&=\{A\in\mathbb{R}^{2\times 2}:\ -\kappa_2a(A,A)\geqslant 4\kappa_1(|A|^2-2)\geqslant 0\text{ or }-a(A,A)\geqslant 2-|A|^2\geqslant 0\}
\end{aligned}$$

and

$$\Phi_1(A)=0,\quad\Phi_2(A)=W(A),$$
$$\Phi_3(A)=\Phi_4(A)=\frac{\kappa_1\kappa_2}{4\kappa_1+\kappa_2}\left(|A|^2-2+|a(A,A)|\right)^2.$$

Lemma 3.8. *W^c from (66) satisfies convexity inequality (11) with $p=4$, $r=2$, $s=2$.*

Corollary 3.9. *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and u a minimiser of I or I^c with energy density $W(\nabla u)$ from (64) and W^c from (66), respectively. Assume that the data f, g, H is given according to theorem 2.9 with $p = 4$. Then the stress field σ satisfies $\sigma \in \mathcal{N}^{\frac{1}{2},\frac{4}{3}}(\Omega) \subset W^{\frac{1}{2}-\delta,\frac{4}{3}}(\Omega)$ for every $\delta > 0$. Moreover, σ is unique.*

Proof of lemma 3.8. The proof of lemma 3.8 is quite technical and we split it into two parts. In the first step we show that Φ_i satisfies the convexity inequality for every $A, B \in \mathcal{M}_i$, $1 \leqslant i \leqslant 4$. Putting these estimates together we show in the second step that W^c satisfies the convexity inequality for arbitrary $A, B \in \mathbb{R}^{2\times 2} = \cup_{i=1}^4 \mathcal{M}_i$.

Let $i = 2$ and $A, B \in \mathcal{M}_2$, $A \neq B$. It follows

$$
\begin{aligned}
&\Phi_2(A) - \Phi_2(B) - D\Phi_2(B) : (A - B) \\
&\quad = \kappa_1(|A|^2 - |B|^2)^2 + \frac{\kappa_2}{4}(a(A, A) - a(B, B))^2 \\
&\qquad + 2\kappa_1(|B|^2 - 2)\,|A - B|^2 + \frac{\kappa_2}{2}a(B, B)a(A - B, A - B) \\
&\quad = s_1 + \ldots + s_4,
\end{aligned} \tag{67}
$$

where $a(\cdot,\cdot)$ is defined in (65). Let $T(A) = \left(\begin{smallmatrix} a_{12} & a_{11} \\ a_{22} & a_{21}\end{smallmatrix}\right)$ and note that $T(A) : B = a(A, B)$ and $D_A a(A, A) = 2T(A)$. Young's inequality yields

$$
\begin{aligned}
|D\Phi_2(A) - D\Phi_2(B)|^2 &= \big|4\kappa_1(|A|^2 - |B|^2)A + \kappa_2(a(A, A) - a(B, B))T(A) + \\
&\qquad + 4\kappa_1(|B|^2 - 2)(A - B) + \kappa_2 a(B, B)T(A - B)\big|^2 \\
&\leqslant c\big((|A|^2 - |B|^2)^2\,|A|^2 + (a(A, A) - a(B, B))^2\,|A|^2\big) \\
&\qquad + c\big|4\kappa_1(|B|^2 - 2)(A - B) + \kappa_2 a(B, B)T(A - B)\big|^2 \\
&= c(t_1 + t_2) + ct_3.
\end{aligned} \tag{68}
$$

Obviously, there exists a constant $c > 0$ such that

$$
t_1 + t_2 \leqslant c(1 + |A|^2 + |B|^2)(s_1 + s_2). \tag{69}
$$

It remains to show that

$$
t_3 \leqslant c(1 + |A|^2 + |B|^2)(s_3 + s_4). \tag{70}
$$

If $a(B, B)a(A - B, A - B) \geqslant 0$, then

$$
(1 + |A|^2 + |B|^2)a(B, B)a(A - B, A - B) \geqslant (|B|^2 - 2)a(B, B)a(A - B, A - B), \tag{71}
$$

$$
(1 + |A|^2 + |B|^2)(|B|^2 - 2)\,|A - B|^2 \geqslant (|B|^2 - 2)^2\,|A - B|^2 \overset{B\in\mathcal{M}_2}{\geqslant} \frac{\kappa_2^2}{16\kappa_1^2}a^2(B, B)\,|A - B|^2. \tag{72}
$$

Evaluating t_3 and taking into account estimates (71) and (72) finally implies (70).

If $a(B, B)a(A - B, A - B) < 0$, then

$$
\begin{aligned}
s_3 + s_4 &= \big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B, B)|\big)\,|A - B|^2 \\
&\qquad + \frac{\kappa_2}{2}\,|a(B, B)|\,\big(|A - B|^2 - |a(A - B, A - B)|\big)
\end{aligned} \tag{73}
$$

and both terms are nonnegative. On the other hand

$$
\begin{aligned}
t_3 &= 4\big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B, B)|\big)^2\,|A - B|^2 \\
&\qquad + 8\kappa_1\kappa_2(|B|^2 - 2)\,|a(B, B)|\,\big(|A - B|^2 - |a(A - B, A - B)|\big)
\end{aligned} \tag{74}
$$

and since $B \in \mathcal{M}_2$, we have

$$\Big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B,B)|\Big)^2 \leqslant c\,\Big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B,B)|\Big)\,(1 + |A|^2 + |B|^2) \tag{75}$$

for a constant $c > 0$ which is independent of A, B. Combining (73)-(75) results in (70) and convexity inequality (11) is proofed for Φ_2 on $\mathcal{M}_2$ with $r = s = 2$.

Let $i = 3$. For $A, B \in \mathcal{M}_3$, $A \neq B$, it holds

$$\begin{aligned}
&(4\kappa_1 + \kappa_2)(\kappa_1\kappa_2)^{-1}\,(\Phi_3(A) - \Phi_3(B) - D\Phi_3(B) : (A - B)) \qquad (76)\\
&= (|A|^2 - |B|^2 + a(A,A) - a(B,B))^2 + 2\big(|B|^2 - 2 + a(B,B)\big)\big(|A - B|^2 + a(A - B, A - B)\big)
\end{aligned}$$

and both summands are nonnegative. On the other hand, by Young's inequality, there exists a constant $c > 0$ such that

$$\begin{aligned}
c\,|D\Phi_3(A) - D\Phi_3(B)|^2 &\leqslant \big(|A|^2 - |B|^2 + a(A,A) - a(B,B)\big)^2\,|A + T(A)|^2\\
&\quad + \big(|B|^2 - 2 + a(B,B)\big)^2\big(2\,|A - B|^2 + a(A - B, A - B)\big). \qquad (77)
\end{aligned}$$

$B \in \mathcal{M}_3$ implies $0 \leqslant |B|^2 - 2 + a(B,B) \leqslant 2(1 + |A|^2 + |B|^2)$ and therefore, combining (76) and (77), it follows that Φ_3 satisfies the convexity inequality on $\mathcal{M}_3$. The case $i = 4$ can be treated in the same way.

In order show that the convexity inequality is valid for every $A, B \in \mathbb{R}^{2\times 2}$ note first that there exists a $J_0 \in \mathbb{N}$ such that it holds for every $A, B \in \mathbb{R}^{2\times 2}$: there exist real numbers $0 = t_0 < t_1 \ldots < t_J = 1$, $J \leqslant J_0$, and numbers $i_0, \ldots, i_{J-1} \in \{1, \ldots, 4\}$ such that $F(t) = B + t(A - B) \in \mathcal{M}_{i_j}$ for $t \in [t_{i_j}, t_{i_{j+1}}]$. We obtain

$$\begin{aligned}
&W^c(A) - W^c(B) - DW^c(B) : (A - B)\\
&\quad = \sum_{j=1}^{J} W^c(F(t_j)) - W^c(F(t_{j-1})) - DW^c(F(t_{j-1})) : (F(t_j) - F(t_{j-1}))\\
&\quad\quad + \sum_{j=1}^{J} \big(DW^c(F(t_{j-1})) - DW^c(F(0))\big) : (F(t_j) - F(t_{j-1}))\\
&\quad = s_1 + s_2. \qquad (78)
\end{aligned}$$

Since W^c is convex, the derivative DW^c is a monotone function and thus

$$s_2 = \sum_{j=2}^{J} \frac{t_j - t_{j-1}}{t_{j-1}}\big(DW^c(F(t_{j-1}) - DW^c(F(0))\big) : (F(t_{j-1}) - F(0)) \geqslant 0.$$

Moreover, $F(t_{j-1}), F(t_j) \in \mathcal{M}_{i_{j-1}}$ and therefore the convexity inequality may be applied to every summand of s_1 separately due to the first part of this proof:

$$s_1 \geqslant c \sum_{j=1}^{J} (1 + |F(t_j)|^2 + |F(t_{j-1})|^2)^{-1}\,|DW^c(F(t_j)) - DW^c(F(t_{j-1}))|^2\,.$$

Note that $(1 + |F(t_j)|^2 + |F(t_{j-1})|^2)^{-1} \geqslant \frac{1}{4}(1 + |A|^2 + |B|^2)^{-1}$ and $\sum_{j=1}^{J} |B_j|^2 \geqslant J^{-1}\left|\sum_{j=1}^{J} B_j\right|^2$ for $B_j \in \mathbb{R}^{2\times 2}$ and thus, since $J \leqslant J_0$,

$$s_1 \geqslant \frac{c}{4J_0}(1 + |A|^2 + |B|^2)^{-1}\,|DW^c(A) - DW^c(B)|^2\,.$$

This finishes the proof. $\square$

A Appendix: Proof of lemma 2.8

The two dimensional case

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz-polygon and assume that $0 \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. Then there exists $R > 0$ such that $B_R(0) \cap \partial\Omega$ does not contain any further corner point of $\partial\Omega$. Assume further that $\Gamma_N \cap B_R(0)$ is a subset of the positive x_1-axis and that there exists $\Phi > 0$ such that $\Omega \cap B_R(0) = \{x \in B_R(0) : 0 < \varphi < \Phi\}$ (polar coordinates, $x = |x|\,(\cos\varphi, \sin\varphi)^\top$).

1. Case: Let $\measuredangle(\Gamma_D, \Gamma_N) < \pi$, i.e. $\Phi < \pi$. Choose $\Omega_N = \{x \in B_R(0) : \pi < \varphi < 2\pi\}$, $\Omega_D = \{x \in B_R(0) : \Phi < x < \pi\}$ and $\mathcal{K} = \{x \in \mathbb{R}^2 : \Phi < x < \pi\}$. Then conditions (28)-(29) of definition 2.7 are satisfied and thus Ω is an admissible domain.

2. Case: Assume that Ω is admissible. We have to show that $\Phi < \pi$. Let $\mathcal{K}, \Omega_D$ and Ω_N be the cone and domains of (28)-(29) in definition 2.7 corresponding to the corner $0 \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. It follows from $\Gamma_N \cap B_R(0) \subset (\partial\Omega_N \cap B_R(0)) \subset$ positive x_1-axis together with (28) that $\mathcal{K}$ is completely contained in the upper half plane, i.e. $\mathcal{K} = \{x \in \mathbb{R}^2 : \Phi_1 < \varphi < \Phi_2\}$ and $0 \leqslant \Phi_1 < \Phi_2 \leqslant \pi$. Furthermore, (29) together with $\Gamma_D \cap B_R(0) \subset \{x \in \mathbb{R}^2 : \varphi = \Phi\}$ implies $\Phi \leqslant \Phi_1$ and thus $\Phi < \pi$.

The three dimensional case

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz-polyhedron according to part 2 of lemma 2.8 and let $x_0 \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. There exists $R > 0$ and a polyhedral cone $\tilde{\mathcal{K}}$ with vertex in x_0 such that Ω coincides with $\tilde{\mathcal{K}}$ on $B_R(x_0)$:

$$\Omega \cap B_R(x_0) = \tilde{\mathcal{K}} \cap B_R(x_0).$$

We assume that $\tilde{K}$ has exactly three faces Γ_i, $1 \leqslant i \leqslant 3$, which intersect at x_0 and which satisfy $(\Gamma_1 \cup \Gamma_2) \cap B_R(x_0) \subset \Gamma_N$ and $\Gamma_3 \cap B_R(x_0) \subset \Gamma_D$. Furthermore we assume that $\measuredangle(\Gamma_1, \Gamma_2) \neq \pi$. The remaining cases can be treated similarly. Let n_i be the exterior unit normal vector on Γ_i and denote by $H_i = \{x \in \mathbb{R}^3 : (x - x_0)n_i < 0\}$ the "interior" half space with respect to Γ_i and n_i. Due to the assumption $\measuredangle(\Gamma_D, \Gamma_N) < \pi$ it follows that $(\Omega \cup \Gamma_N) \cap B_R(x_0) \subset H_3$. Therefore we have exactly the following two cases for $\tilde{\mathcal{K}}$:

$$\tilde{\mathcal{K}} = H_1 \cap H_2 \cap H_3 \ \text{ or } \ \tilde{\mathcal{K}} = (H_1 \cup H_2) \cap H_3.$$

In order to show that Ω is an admissible domain we have to construct domains Ω_D, Ω_N and a cone $\mathcal{K}$ according to (28)-(29) of definition 2.7. We define

$$\Omega_D = \{x \in B_R(x_0) : (x - x_0)n_3 > 0\}, \qquad \Omega_N = B_R(x_0) \backslash (\overline{\Omega_D} \cup \overline{\Omega}).$$

Since $\measuredangle(\Gamma_D, \Gamma_N) < \pi$ it follows that $\Omega_D \cap \Omega = \emptyset$ and $\Omega_N \neq \emptyset$. Let e_1 be tangential to $\overline{\Gamma_1} \cap \overline{\Gamma_3}$, e_2 be tangential to $\overline{\Gamma_2} \cap \overline{\Gamma_3}$ and e_3 tangential to $\overline{\Gamma_1} \cap \overline{\Gamma_2}$. The the orientation of the vectors e_i is chosen in such a way that

$$e_1 n_2 < 0, \ e_2 n_1 < 0, \ e_3 n_3 > 0. \tag{79}$$

This choice is always possible since $\measuredangle(\Gamma_1, \Gamma_2) \neq \pi$ and $\measuredangle(\Gamma_N, \Gamma_D) < \pi$. Note that $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$. We define the cone $\mathcal{K}$ by

$$\mathcal{K} = \{v \in \mathbb{R}^3 : v = \sum_{i=1}^{3} \lambda_i e_i, \ \lambda_i \geqslant 0\}.$$

Then Ω_D, Ω_N and $\mathcal{K}$ satisfy (28)-(29) of definition 2.7, which can be seen as follows:

Choose $x \in \Omega_D$ and $v = \sum_i \lambda_i e_i \in \mathcal{K}$ such that $x + v \in B_R(x_0)$. Since $e_1 n_3 = 0$, $e_2 n_3 = 0$ we get from the definition of Ω_D and (79) that

$$(x + v - x_0)n_3 = (x - x_0)n_3 + \lambda_3 e_3 n_3 > 0$$

and therefore $x + v \in \Omega_D$ and (29) is proved. For the proof of (28) choose $x \in \Omega \cap B_R(x_0) = \tilde{\mathcal{K}} \cap B_R(x_0)$ and $v = \sum_i \lambda_i e_i \in \mathcal{K}$ such that $x + v \in B_R(x_0)$. If $\lambda_3 \geqslant -(x - x_0)/e_3 n_3$ then

$(x + v - x_0)n_3 \geqslant 0$ which yields $x + v \in \overline{\Omega_D}$ and (28) holds for this case. If $\lambda_3 < -(x - x_0)/e_3 n_3$ then $x + v \in H_3$ and we have to show that $x + v \in \tilde{\mathcal{K}}$ in order to verify (28).

1. Case: $\tilde{K} = H_1 \cap H_2 \cap H_3$. It follows for $i, j \in \{1, 2\}$ with $i \neq j$ from the definitions of $\mathcal{K}$, H_i and from (79) that

$$(x + v - x_0)n_i = (x - x_0)n_i + \lambda_j e_j n_i < 0$$

and therefore $x + v \in H_1 \cap H_2 \cap H_3$.

2. Case: $\tilde{K} = (H_1 \cup H_2) \cap H_3$. It follows for $i, j \in \{1, 2\}$, $i \neq j$ as before that

$$(x + v - x_0)n_i = (x - x_0)n_i + \lambda_j e_j n_i.$$

Since $x \in H_1 \cup H_2$ we have $(x - x_0)n_1 < 0$ or $(x - x_0)n_2 < 0$. Together with $\lambda_j e_j n_i \leqslant 0$ we obtain finally $(x + v - x_0)n_1 < 0$ or $(x + v - x_0)n_2 < 0$ which shows that $x + v \in H_1 \cup H_2$.

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