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# Short note on global spatial regularity in elasto-plasticity with linear hardening

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**Abstract**

We study the global spatial regularity of solutions of elasto-plastic models with linear hardening. In order to point out the main idea, we consider a model problem on a cube, where we describe Dirichlet and Neumann boundary conditions on the top and the bottom, respectively, and periodic boundary conditions on the remaining faces. Under natural smoothness assumptions on the data we obtain $u \in \mathrm{L}^\infty((0,T);\mathrm{H}^{\frac{3}{2}-\delta}(\Omega))$ for the displacements and $z \in \mathrm{L}^\infty((0,T);\mathrm{H}^{\frac{1}{2}-\delta}(\Omega))$ for the internal variables. The proof is based on a difference quotient technique and a reflection argument.

# 1 Introduction

In this note we study the global spatial regularity of solutions of elasto-plastic models with linear hardening. The results are in particular applicable to elasto-plasticity with linear kinematic hardening. In order to keep the presentation as clear as possible and in order to point out the main idea, we consider a model problem on a cube $\Omega \subset \mathbb{R}^d$, where we describe Dirichlet and Neumann boundary conditions on the top and the bottom, respectively, and periodic boundary conditions on the remaining faces. In a forthcoming paper, we will extend the investigations to bounded domains with smooth boundaries and to more general rate independent models.

Let $u(t,x) \in \mathbb{R}^d$ be the displacement of the point $x \in \Omega$ at time $t$, $\sigma(t,x) \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ the Cauchy stress tensor and $z(t,x) \in \mathbb{R}^N$ the vector of the internal variables. Assuming small strains, the behavior of the body is described by the quasistatic balance of forces (1.1), Hooke's law (1.2), which relates the stress with the elastic part of the strain, and the principle of maximal plastic work, which determines the evolution law for the internal variable $z$ (1.3):

$$\operatorname{div}_x \sigma + f = 0 \qquad \text{in } (0,T)\times\Omega, \tag{1.1}$$

$$\sigma = \tilde{A}(\varepsilon(u) - \tilde{B}z) \qquad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\partial_t z \in \partial\chi_K(\tilde{B}^\top\sigma - Lz) \quad \text{in } (0,T)\times\Omega. \tag{1.3}$$

The convex set $K \subset \mathbb{R}^N$ is the set of admissible generalized stresses. These equations are completed with the initial condition

$$z(0,x) = z_0(x), \quad x \in \Omega \tag{1.4}$$

and with Dirichlet conditions on $\Gamma_D$, Neumann conditions on $\Gamma_N$ and periodic boundary conditions on the remaining faces $\Gamma_{\text{per}}$:

$$u\big|_{\Gamma_D} = h_D \quad \text{on } (0,T) \times \Gamma_D, \tag{1.5}$$

$$\sigma\vec{n} = h_N \quad \text{on } (0,T) \times \Gamma_N. \tag{1.6}$$

The functions $f$ and $h_N$ are given volume and surface force densities and $h_D$ prescribes the displacements on the Dirichlet boundary. The tensor $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top) \in \mathbb{R}^{d\times d}_{\text{sym}}$ is the linearized strain tensor, $\tilde{A} \in \text{Lin}(\mathbb{R}^{d\times d}_{\text{sym}}, \mathbb{R}^{d\times d}_{\text{sym}})$ the fourth order elasticity tensor and $\tilde{B} \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^{d\times d}_{\text{sym}})$ maps the vector $z$ of internal variables on the plastic strain $\varepsilon_p = \tilde{B}z$. Moreover, $L \in \text{Lin}(\mathbb{R}^N, \mathbb{R}^N)$ is a positive definite symmetric tensor describing the hardening properties.

The goal of the paper is to show that under natural smoothness assumptions on the volume forces and the boundary data, we obtain higher spatial regularity for the displacements and the internal variables. In particular we prove the following theorem:

**Theorem 1.1.** *Let $z_0 \in \mathrm{H}^1_{\Gamma_{per}}(\Omega)$, $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$, $h_D \in \mathrm{W}^{1,1}(S; \mathrm{H}^{\frac{3}{2}}_{per}(\Gamma_D))$ and $h_N \in \mathrm{W}^{1,1}(S; \mathrm{H}^{\frac{1}{2}}_{per}(\Gamma_N))$ and let $(u, \sigma, z)$ be a solution to* (1.1)–(1.6). *Then for every $\delta > 0$ we have*

$$u \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{3}{2}-\delta}(\Omega)), \quad \sigma \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{1}{2}-\delta}(\Omega)), \quad z \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{1}{2}-\delta}(\Omega)).$$

Let us give a short overview on regularity results in the literature. To the author's knowledge, the only global spatial regularity result for elasto-plastic models with linear hardening was derived recently by Alber and Nesenenko in [AN08]. Under similar assumptions on the data as in Theorem 1.1 they obtained for $\mathrm{C}^2$-smooth domains and with $\partial\Omega = \Gamma_D$ the regularity

$$u \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{4}{3}-\delta}(\Omega)), \quad \sigma \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{1}{3}-\delta}(\Omega)), \quad z \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{1}{3}-\delta}(\Omega)). \tag{1.7}$$

In a first step the authors of [AN08] proved a tangential result and showed that this implies that $u \in \mathrm{L}^\infty(S; \mathrm{H}^{\frac{5}{4}-\delta}(\Omega))$. By an iteration procedure they finally arrive at (1.7).

Local regularity results for elasto-plasticity with linear hardening and for variants of this model, like the Prandtl-Reuss model, were derived by several authors [BF96, Dem07, FL07, Shi99, Ser92]. Here, one typically finds

$$\sigma \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\text{loc}}(\Omega)).$$

Furthermore, global results are available for time discretized versions of (1.1)–(1.6) and variants of it, see e.g. [Rep96, KN08] and the references therein. Here, it is possible to prove for smooth domains

$$\sigma(t_k) \in \mathrm{H}^1(\Omega)$$

for every time step $t_k$. But up to now it is unknown how to derive a uniform bound of the type $\sup_{\text{time step } \tau>0, k\tau\le T} \|\sigma(k\tau)\|_{\mathrm{H}^1(\Omega)} \le c$, which would allow to carry over the result of the discretized model to the continuous one.

Let us finally remark that for the stationary Hencky model we have the global result $\sigma \in \mathrm{H}^{\frac{1}{2}-\delta}(\Omega)$ on Lipschitz domains, which satisfy additional conditions near those points, where the type of the boundary conditions changes, see [Kne06].

The paper is organized as follows. In Section 2 we introduce the notation and state the main regularity result, Theorem 2.2. We prove Theorems 1.1 and 2.2 in two steps. In the first step (Section 3) we study a pure periodic problem and derive two global regularity results depending on different smoothness assumptions on the data. The proof is carried out with a difference quotient technique and relies essentially on a priori estimates for solutions of the elasto-plastic model. In this step we apply techniques from [AN08, Nes06].

In the second step (Section 4) we prove first that the solution pair $(\nabla u, z)$ of the original model is differentiable in directions which are tangential to the Dirichlet and Neumann boundary (Theorem 4.1). This result refines slightly a result from [AN08]. The essential new idea in this paper is to use a reflection argument in order to obtain also a result concerning the differentiability of $(\nabla u, z)$ perpendicular to $\Gamma_D$ and $\Gamma_N$. We extend the problem described above by reflection to the periodic case and derive in this way Theorem 1.1 and Theorem 2.2 as special cases of the results for the pure periodic case. The right hand side of the extended problem contains extensions of the data $f$ and $z_0$ and additional terms, which include partial derivatives of $\nabla u$ and $z$ that are taken parallel to $\Gamma_D$ and $\Gamma_N$. Theorem 4.1 on tangential regularity of $(u, z)$ guarantees that the data of the extended problem is smooth enough such that the regularity Theorem 3.4 for purely periodic structures may be applied. We carry out these considerations for vanishing Dirichlet and Neumann data, first. In Section 4.3 we extend the results to the general case with non-zero boundary data.

Let us remark that the reflection technique applied to the elastic equation (1.1)–(1.2), only, and neglecting the coupling with the evolution equation (e.g. by assuming that $\tilde{B} = 0$) would lead to $u(t) \in \mathrm{H}^2(\Omega)$. We discuss this in more detail in Section 4.2. It remains an open question whether the result of Theorem 1.1 is optimal or whether one should expect $u \in \mathrm{L}^\infty(S; \mathrm{H}^2(\Omega))$.

## 2 Setting up of the model and main result

For $d > 1$ and $\ell > 0$ let $\Omega = (-\ell, \ell)^{d-1} \times (0, \ell) \subset \mathbb{R}^d$ be a half cube with side length $2\ell$. Throughout the paper we will use the notation $x = (x', x_d)$ for $x \in \mathbb{R}^d$ and define the boundary sets

$$\begin{aligned}\Gamma_0 &= \{\, x\in\mathbb{R}^d\,;\, x'\in(-\ell,\ell)^{d-1},\, x_d=0\,\},\\ \Gamma_1 &= \{\, x\in\mathbb{R}^d\,;\, x'\in(-\ell,\ell)^{d-1},\, x_d=\ell\,\},\\ \Gamma_{\mathrm{per}} &= \partial\Omega\backslash(\overline{\Gamma_0}\cup\overline{\Gamma_1}).\end{aligned}$$

We assume that periodic boundary conditions are prescribed on $\Gamma_{\mathrm{per}}$, while for the other parts of the boundary we assume that $\Gamma_D=\Gamma_1$ and $\Gamma_N=\Gamma_0$.

We denote by

$$\begin{aligned}\mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega,\mathbb{R}^n) = \{u\in \mathrm{H}^1(\Omega,\mathbb{R}^n)\,;\, \exists\tilde{u}\in \mathrm{H}^1_{\mathrm{loc}}(\mathbb{R}^{d-1}\times(0,\ell)) \text{ with } u=\tilde{u}\big|_\Omega\\ \text{and } \tilde{u}(y',x_d)=\tilde{u}(x',x_d)\ \forall y'\in x'{+}2\ell\mathbb{Z}^{d-1}\}\end{aligned}$$

the space of $\mathrm{H}^1$-functions which are periodic with respect to $\Gamma_{\mathrm{per}}$. Assuming vanishing Dirichlet data on $\Gamma_D$, the set of admissible displacements is given by

$$V=\{\, u\in \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega,\mathbb{R}^d)\,;\, u|_{\Gamma_D}{=}0\,\},$$

while $\mathcal{Z}=\mathrm{L}^2(\Omega,\mathbb{R}^N)$ denotes the space for the internal variables.

We will discuss the case of non-vanishing boundary data in section 4.3. The reduction of the model with non-zero boundary data to a model with vanishing data leads to more general force terms than those given in (1.1)–(1.6). We therefore study here an energy which already includes these additional force terms.

Given a volume force density $\ell\in \mathrm{W}^{1,1}([0,T];V')$ and $F\in \mathrm{W}^{1,1}(S;\mathrm{L}^2(\Omega))$, we consider the energy functional

$$\mathcal{E}(t,u,z)=\int_\Omega W(\nabla u,z)\,\mathrm{d}x-\langle \ell(t),u\rangle_V+\int_\Omega F(t)\cdot z\,\mathrm{d}x$$

for displacement fields $u\in V$, internal variables $z\in\mathcal{Z}$ and time $t\in S=[0,T]$.

The stored energy density $W:\mathbb{R}^{d\times d}\times\mathbb{R}^N\to\mathbb{R}$ is assumed to be quadratic with

$$W(D,z)=\tfrac12 A(D-Bz):(D-Bz)+\tfrac12 Lz\cdot z \tag{2.1}$$

for $D\in\mathbb{R}^{d\times d}$ and $z\in\mathbb{R}^N$. The tensors $A\in\mathrm{Lin}(\mathbb{R}^{d\times d},\mathbb{R}^{d\times d})$, $B\in\mathrm{Lin}(\mathbb{R}^N,\mathbb{R}^{d\times d})$ and $L\in\mathrm{Lin}(\mathbb{R}^N,\mathbb{R}^N)$ depend on the material properties. The term $Bz$ can be interpreted as the plastic strain tensor, while the term $\frac12 Lz\cdot z$ leads to an elasto-plastic model with linear hardening. The tensors $A$ and $L$ shall satisfy the following assumptions

**(A1)** $A=S^*\tilde{A}S$, where $S:\mathbb{R}^{d\times d}\to\mathbb{R}^{d\times d}_{\mathrm{sym}}$, $S(D)=\frac12(D+D^\top)$ maps the displacement gradient on the linearized strain tensor, $S^*$ is the adjoint operator and $\tilde{A}\in\mathrm{Lin}(\mathbb{R}^{d\times d}_{\mathrm{sym}},\mathbb{R}^{d\times d}_{\mathrm{sym}})$ is symmetric and positive definite.

**(A2)** $L\in\mathrm{Lin}(\mathbb{R}^N,\mathbb{R}^N)$ is symmetric and positive definite.

The tensor $\tilde{A}$ is the usual elasticity tensor.

Let $K \subset \mathbb{R}^N$ be nonempty, closed, convex with $0 \in K$. We denote by $\mathcal{K} \subset \mathcal{Z}$, $\mathcal{K} = \{\, z \in \mathcal{Z}\,;\, z(x) \in K \text{ a.e.} \,\}$ the set of the admissible generalized stresses. The characteristic functional with respect to $\mathcal{K}$ is given by $\chi_{\mathcal{K}}(z) = 0$ if $z \in \mathcal{K}$ and $\chi_{\mathcal{K}}(z) = \infty$ otherwise. Finally, $\partial\chi_{\mathcal{K}}$ is the subdifferential of $\chi_{\mathcal{K}}$ with respect to $\mathcal{Z}$ in the sense of convex analysis.

The elasto-plastic problem consists in determining a displacement field $u : S \to V$ and internal variables $z : S \to \mathcal{Z}$ which satisfy

$$z(0) = z_0, \tag{2.2}$$

$$\mathrm{D}_u\mathcal{E}(t, u(t), z(t)) = 0 \qquad \text{for all } t \in S, \tag{2.3}$$

$$\partial_t z(t) \in \partial\chi_{\mathcal{K}}(-\mathrm{D}_z\mathcal{E}(t, u(t), z(t))) \quad \text{for a.e. } t \in S. \tag{2.4}$$

Here, $\mathrm{D}_u\mathcal{E}$ and $\mathrm{D}_z\mathcal{E}$ denote the variational derivatives of $\mathcal{E}$ with respect to $V$ and $\mathcal{Z}$, respectively. Relations (2.2)–(2.4) may equivalently be written as

$$z(0) = z_0, \tag{2.5}$$

$$\int_\Omega A(\nabla u(t) - Bz(t)) : \nabla v \,\mathrm{d}x = \langle \ell(t), v\rangle_V \qquad \text{for all } v \in V,\ t \in S, \tag{2.6}$$

$$\partial_t z \in \partial\chi_{\mathcal{K}}(-Lz + B^\top A(\nabla u - Bz) - F(t)) \qquad \text{for a.e. } t \in S. \tag{2.7}$$

The stress tensor can be calculated via $\sigma = A(\nabla u - Bz)$. Relations (2.5)–(2.7) are a slightly more general version of the model (1.1)–(1.6), but with vanishing Dirichlet data. Altogether the relations (2.5)–(2.7) describe small-strain elasto-plasticity with linear hardening. This model comprises kinematic hardening, while pure isotropic hardening is excluded since in that case, the tensor $L$ is positive semidefinite, only.

It is shown in [Mie05], see also [Ste08], that an equivalent formulation for (2.6)–(2.7) is to find a displacement field $u : S \to V$ and internal variables $z : S \to \mathcal{Z}$ with $z(0) = z_0$, which for every $t \in [0, T]$ satisfy the following global stability condition (S) and the energy balance (E)

$$\begin{aligned} &\text{(S)} \quad \mathcal{E}(t, u(t), z(t)) \le \mathcal{E}(t, v, \zeta) + \mathcal{R}(\zeta - z(t)) \quad \text{for all } v \in V,\ \zeta \in \mathcal{Z}, \\ &\text{(E)} \quad \mathcal{E}(t, u(t), z(t)) + \textstyle\int_0^t \mathcal{R}(\dot z(\tau))\,\mathrm{d}\tau = \mathcal{E}(0, z_0) + \textstyle\int_0^t \partial_t\mathcal{E}(\tau, u(\tau), z(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Here, the dissipation pseudo potential $\mathcal{R}$ is defined through $\mathcal{R}(\eta) = \chi^*_{\mathcal{K}}(\eta)$, where $\chi^*_{\mathcal{K}}$ is the functional related with $\chi_{\mathcal{K}}$ by convex conjugation in $\mathcal{Z}$.

For the applied forces $\ell$ and $F$ and for the initial value $z_0$ of the internal variables we assume

**(A3)** $\ell \in \mathrm{W}^{1,1}(S; V')$, $F \in \mathrm{W}^{1,1}(S; \mathcal{Z})$, $z_0 \in \mathcal{Z}$ and there exists $u_0 \in V$ such that $\mathrm{D}_u\mathcal{E}(0, u_0, z_0) = 0$ and $-\mathrm{D}_z\mathcal{E}(0, u_0, z_0) \in \mathcal{K}$.

Since, by the assumptions on the coefficients $A$, $B$ and $L$ and the choice of $V$ the functional $\mathcal{E} : V \times \mathcal{Z} \to \mathbb{R}$ is strictly convex, coercive and strongly continuous and since the dissipation potential $\mathcal{R} : \mathcal{Z} \to \mathbb{R}$ is convex, lower semi continuous and homogeneous of degree one, the following existence theorem is a standard result [DL72, Joh78, HHLN88, HR99, AC04, Bré73]

**Theorem 2.1.** *If (A1), (A2) and (A3) are satisfied, then there exists a unique solution pair* $(u, z) \in \mathrm{W}^{1,1}(S; V) \times \mathrm{W}^{1,1}(S; \mathcal{Z})$*, which solves* (2.5)–(2.7) *and (S) & (E).*

In order to obtain higher regularity of the solution, more regularity is required for the data. We assume that the force term $\ell \in \mathrm{W}^{1,1}(S; V')$ is of the special structure

$$\langle \ell(t), v \rangle_V \equiv \int_\Omega f(t) \cdot v + H(t) : \nabla v \, \mathrm{d}x \tag{2.8}$$

for $v \in V$. A sufficient condition for (A3) to hold is

**(A3')** $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$, $H \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$, $F \in \mathrm{W}^{1,1}(S; \mathcal{Z})$, $z_0 \in \mathcal{Z}$ and there exists $u_0 \in V$ such that $\mathrm{D}_u \mathcal{E}(0, u_0, z_0) = 0$ and $-\mathrm{D}_z \mathcal{E}(0, u_0, z_0) \in \mathcal{K}$.

The next theorem is the main result of this paper.

**Theorem 2.2.** *Assume that (A1), (A2) and (A3') are satisfied. If in addition* $z_0 \in \mathrm{H}^1_{\Gamma_{per}}(\Omega, \mathbb{R}^N)$, $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$, $H \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$ *and* $F \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$, *then the solution* $(u, z)$ *to* (2.5)–(2.7) *with* $\ell(t)$ *as in* (2.8) *satisfies for every* $\delta > 0$

$$u \in \mathrm{L}^\infty(\mathrm{B}^{\frac{3}{2}}_{2,\infty}(\Omega)) \subset \mathrm{L}^\infty(S; \mathrm{H}^{\frac{3}{2}-\delta}(\Omega)), \tag{2.9}$$

$$z \in \mathrm{L}^\infty(\mathrm{B}^{\frac{1}{2}}_{2,\infty}(\Omega)) \subset \mathrm{L}^\infty(S; \mathrm{H}^{\frac{1}{2}-\delta}(\Omega)). \tag{2.10}$$

*Moreover,* $\partial_i u \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$, $\partial_i z \in \mathrm{L}^\infty(S; \mathrm{L}^2(\Omega))$ *for* $1 \le i \le d-1$.

Here, $\mathrm{B}^s_{p,q}(\Omega)$ are Besov spaces and we refer to [Tri83] for a precise definition. Note that for $s > 0$, $s \notin \mathbb{N}$ and for every $\delta > 0$ the following continuous embeddings are valid

$$\mathrm{H}^s(\Omega) \subset \mathrm{B}^s_{2,\infty}(\Omega) \subset \mathrm{H}^{s-\delta}(\Omega).$$

Furthermore, we recall that $v \in \mathrm{B}^s_{2,\infty}(\Omega)$ for $s \in (0,1)$ if and only if $v \in \mathrm{L}^2(\Omega)$ and

$$\sup_{\widetilde{\Omega} \Subset \Omega,\, h \in \mathbb{R}\setminus\{0\},\, i \in \{1,\ldots,d\}} |h|^{-s} \, \|\triangle_{he_i} v\|_{\mathrm{L}^2(\widetilde{\Omega})} < \infty.$$

Here, $e_i$ is the $i$-th coordinate vector and $\triangle_{he_i} v(x) = v(x + he_i) - v(x)$. This characterization of $\mathrm{B}^s_{2,\infty}$ gives the link between estimates of difference quotients and regularity properties.

As already discussed in the introduction, the global results (2.9)–(2.10) seem to be new, whereas the tangential result is known for Dirichlet boundaries (see [Nes06]). We give it here for completeness and since it is the basis for the global result.

For proving Theorem 2.2 we derive in a first step a regularity result for a purely periodic situation by estimating difference quotients of $\nabla u$ and $z$. These considerations will be carried out in Section 3. In the second step, Section 4, we extend the problem described above by reflection to the periodic case and derive in this way Theorem 2.2. The right hand side of the extended problem contains extensions of $f$ and $z_0$ and additional terms, which include partial derivatives of $\nabla u$ and $z$ that are taken parallel to $\Gamma_D$ and $\Gamma_N$. The regularity of tangential derivatives guarantees that the data of the extended problem is smooth enough such that the regularity theorem for purely periodic structures may be applied.

# 3 A pure periodic model problem

In the whole section we assume that conditions (A1) and (A2) are satisfied.

## 3.1 Definition of the pure periodic model

Let $\Omega_P = (-\ell, \ell)^d$. We denote by $\mathrm{H}^1_{\mathrm{per}}(\Omega_P)$ the space of $\mathrm{H}^1$-functions which are periodic with respect to $\Omega_P$. The space of admissible displacements is given by

$$V_P = \{\, u \in \mathrm{H}^1_{\mathrm{per}}(\Omega_P)\,;\ \int_{\Omega_P} u \,\mathrm{d}x = 0\,\}.$$

We consider the same energy as in the previous section, namely

$$\mathcal{E}_P(t, v, \zeta) = \int_{\Omega_P} W(\nabla v, \zeta)\,\mathrm{d}x - \int_{\Omega_P} f(t)\cdot v + H : \nabla v \,\mathrm{d}x + \int_{\Omega_P} F(t)\cdot \zeta \,\mathrm{d}x$$

for $v \in V_P$, $\zeta \in \mathcal{Z} = \mathrm{L}^2(\Omega_P; \mathbb{R}^N)$ and with $W$ from (2.1). The periodic problem under consideration is to find a displacement field $u : S \to V_P$ and internal variables $z : S \to \mathcal{Z}$ with $z(0) = z_0$ and satisfying for a.e. $t \in S$ and all $v \in \mathrm{H}^1_{\mathrm{per}}(\Omega_P)$ the relations

$$\int_{\Omega_P} A(\nabla u - Bz) : \nabla v \,\mathrm{d}x = \int_{\Omega_P} f(t)\cdot v + H(t) : \nabla v \,\mathrm{d}x, \tag{3.1}$$

$$\partial_t z(t) \in \partial\chi_{\mathcal{K}}(-\mathrm{D}_z \mathcal{E}_P(t, u(t), z(t))). \tag{3.2}$$

This problem is again equivalent to (S) & (E) formulated with $\mathcal{E}_P$ and $V_P$ instead of $\mathcal{E}$ and $V$. We assume that

**(A4)** $z_0 \in \mathcal{Z}$, $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega_P))$ with $\int_{\Omega_P} f \,\mathrm{d}x = 0$; $H \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega_P, \mathbb{R}^{d\times d}))$; $F \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega_P))$ and there exists $u_0 \in V_P$ such that $\mathrm{D}_u \mathcal{E}_P(0, u_0, z_0) = 0$ and $-\mathrm{D}_z \mathcal{E}(0, u_0, z_0) \in \mathcal{K}$.

As before, we have the following existence result

**Theorem 3.1.** *If (A1), (A2) and (A4) are satisfied, then there exists a unique solution pair* $(u, z)$ *for problem* (3.1)–(3.2) *with* $u \in \mathrm{W}^{1,1}(S; V_P)$ *and* $z \in \mathrm{W}^{1,1}(S; \mathcal{Z})$.

### 3.2 The reduced problem and a-priori estimates

We introduce the linear, elliptic operator

$$\mathcal{A} : V_P \to V_P' \text{ with } \langle \mathcal{A}(u), v\rangle_{V_P} = \int_{\Omega_P} A\nabla u : \nabla v \, \mathrm{d}x$$

for every $u$, $v \in V_P$. Note that $\mathcal{A}$ is self adjoint and that there exists a constant $c_A > 0$ such that for all $u \in V_P$ we have $\langle \mathcal{A}(u), u\rangle_{V_P} \geq c_A \|u\|^2_{\mathrm{H}^1(\Omega_P)}$. Thus $\mathcal{A}$ is an isomorphism and we can define the linear and bounded operator

$$\mathcal{L} : \mathcal{Z} \to \mathcal{Z}, \quad \mathcal{L}(z) = Lz + B^\top A\big(\nabla \mathcal{A}^{-1} \operatorname{div} ABz + Bz\big),$$

where the operator $\operatorname{div} : \mathrm{L}^2(\Omega_P, \mathbb{R}^{d\times d}) \to V_P'$ is given by

$$\langle \operatorname{div} \eta, v\rangle = -\int_{\Omega_P} \eta : \nabla v \, \mathrm{d}x,$$

for $\eta \in \mathrm{L}^2(\Omega_P, \mathbb{R}^{\mathrm{d}\times d})$ and $v \in V_P$.

Let $\hat{\mathcal{E}}(u, z) = \int_{\Omega_P} W(\nabla u, z) \, \mathrm{d}x$. The operator $\mathcal{L}$ is the Schur complement operator associated with

$$\mathrm{D}_{(u,z)}\hat{\mathcal{E}}(u, z) = \begin{pmatrix} \mathcal{A} & \operatorname{div} AB(\cdot) \\ -B^\top A\nabla(\cdot) & L + B^\top AB \end{pmatrix} \begin{pmatrix} u \\ z \end{pmatrix}.$$

The properties of $\hat{\mathcal{E}}$ imply therefore that $\mathcal{L}$ is self adjoint and coercive with respect to $\mathcal{Z}$, i.e. we have

$$\langle \mathcal{L}z_1, z_2\rangle_{\mathcal{Z}} = \langle \mathcal{L}z_2, z_1\rangle_{\mathcal{Z}}, \quad \langle \mathcal{L}z, z\rangle_{\mathcal{Z}} \geq c_L \|z\|^2_{\mathcal{Z}} \tag{3.3}$$

for every $z$, $z_1$, $z_2 \in \mathcal{Z}$ and some constant $c_L > 0$. Finally we define

$$\widetilde{F}(t) = F(t) + B^\top A\nabla \mathcal{A}^{-1}(\operatorname{div} H(t) - f(t)). \tag{3.4}$$

Note that $f(t)$, $H(t)$ and $F(t)$ in $\mathrm{L}^2(\Omega_P)$ imply that $\widetilde{F}(t) \in \mathrm{L}^2(\Omega_P)$.

With these definitions, problem (3.1)–(3.2) is equivalent to

$$\mathcal{A}(u(t)) = f(t) - \operatorname{div} H - \operatorname{div} ABz(t), \tag{3.5}$$

$$\partial_t z(t) \in \partial\chi_{\mathcal{K}}(-\mathcal{L}(z(t)) - \widetilde{F}(t)), \quad z(0) = z_0. \tag{3.6}$$

We will focus the discussion on the reduced equation (3.6). Let us remark that one possibility to prove the existence Theorem 3.1 is to apply Prop. 3.4 of [Bré73] to relation (3.6).

The following a-priori estimates are the basis for proving our regularity results.

**Lemma 3.2** (A-priori estimates)**.** *Assume that for $i \in \{1,2\}$ the functions $z_0^i \in \mathcal{Z}$, $\widetilde{F}_i \in \mathrm{L}^\infty(S; \mathcal{Z})$ and $z_i \in \mathrm{W}^{1,1}(S; \mathcal{Z})$ satisfy for a.e. $t \in S$ the relation*

$$\partial_t z_i(t) \in \partial\chi_{\mathcal{K}}(-\mathcal{L}z_i(t) - \widetilde{F}_i(t)), \quad z_i(0) = z_0^i. \tag{3.7}$$

*Then there exists a constant $c > 0$, which is independent of the data, such that for every $t \in S$ we have*

$$c\,\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2 \le \|z_0^1 - z_0^2\|_{\mathcal{Z}}^2 + \|z_1 - z_2\|_{\mathrm{W}^{1,1}(S;\mathcal{Z})}\,\|\widetilde{F}_1 - \widetilde{F}_2\|_{\mathrm{L}^\infty((0,t);\mathcal{Z})}. \tag{3.8}$$

*If furthermore $\widetilde{F}_i \in \mathrm{W}^{1,1}(S;\mathcal{Z})$, then there exists $c > 0$ such that*

$$\|z_1 - z_2\|_{\mathrm{L}^\infty(S;\mathcal{Z})} \le c\big(\|z_0^1 - z_0^2\|_{\mathcal{Z}} + \|\widetilde{F}_1 - \widetilde{F}_2\|_{\mathrm{W}^{1,1}(S;\mathcal{Z})}\big). \tag{3.9}$$

**Proof.** Let $z_0^i$, $\widetilde{F}_i$ and $z_i$ be given as in Lemma 3.2. Then relation (3.7) implies that for a.e. $t \in S$ we have

$$\langle \partial_t(z_1(t) - z_2(t)), \mathcal{L}(z_1(t) - z_2(t))\rangle_{\mathcal{Z}} \le -\langle \partial_t(z_1(t) - z_2(t)), \widetilde{F}_1(t) - \widetilde{F}_2(t)\rangle_{\mathcal{Z}}.$$

Integration with respect to $t$ and using the properties of $\mathcal{L}$ from (3.3), we see that there is a constant $c > 0$ such that

$$c\,\|z_1(t) - z_2(t)\|_{\mathcal{Z}}^2 \le \|z_0^1 - z_0^2\|_{\mathcal{Z}}^2 - \int_0^t \langle \partial_t(z_1(s) - z_2(s)), \widetilde{F}_1(s) - \widetilde{F}_2(s)\rangle_{\mathcal{Z}}\,\mathrm{d}s.$$

Partial integration in the last term, Young's inequality and the properties of traces of $\mathrm{W}^{1,1}(S;\mathcal{Z})$-functions lead to estimate (3.9), while Hölder's inequality applied to the last term results in inequality (3.8). □

### 3.3 Regularity in the pure periodic case

Before we state the regularity result in the periodic case, we introduce some further notation. For a function $v : \Omega_P \to \mathbb{R}^n$ we denote by $\tilde{v} : \mathbb{R}^d \to \mathbb{R}^n$ the periodic extension of $v$ to $\mathbb{R}^d$, i.e. $\tilde{v}(x + y) = v(x)$ for all $x \in \Omega_P$ and $y \in (2\ell\mathbb{Z})^d$. Moreover, for $h \in \mathbb{R}^d$ we define the shift by $h$ as $v_h : \Omega_P \to \mathbb{R}^n$, $v_h(x) = \tilde{v}(x + h)$ and set $\triangle_h v(x) = v_h(x) - v(x)$. Finally

$$\mathcal{F}_i = \{\, v \in \mathrm{L}^2(\Omega_P)\,;\ \partial_i v \in \mathrm{L}^2(\Omega_P) \text{ and } v \text{ is periodic w.r. to the } i\text{-th coordinate}\,\}$$

with $\|v\|_{\mathcal{F}_i} = \|v\|_{\mathrm{L}^2(\Omega_P)} + \|\partial_i v\|_{\mathrm{L}^2(\Omega_P)}$. For every $h \in \mathbb{R}$ the mapping $\mathcal{F}_i \to \mathcal{F}_i$ defined via $v \mapsto v_{he_i}$, where $e_i$ is a unit vector of the $i$-th coordinate direction, is an isometric isomorphism. Moreover, $\mathrm{H}^1_{\mathrm{per}}(\Omega_P) = \cap_{1 \le i \le d}\mathcal{F}_i$.

**Theorem 3.3.** *Let $z_0 \in \mathcal{F}_i$ and $\widetilde{F} \in \mathrm{L}^\infty(S;\mathcal{F}_i)$ for some $i \in \{1, \dots, d\}$ and assume that $z \in \mathrm{W}^{1,1}(S;\mathcal{Z})$ satisfies (3.6). Then there exists a constant $c > 0$ such that*

$$\sup_{t \in S,\, h \in \mathbb{R}\setminus\{0\}} |h|^{-\frac{1}{2}}\,\|z(t, \cdot + he_i) - z(t, \cdot)\|_{\mathrm{L}^2(\Omega_P)} \le c. \tag{3.10}$$

*If furthermore $\widetilde{F} \in \mathrm{W}^{1,1}(S;\mathcal{F}_i)$, then*

$$z \in \mathrm{L}^\infty(S;\mathcal{F}_i). \tag{3.11}$$

This theorem shows that the time regularity of $\widetilde{F}$ has a strong influence on the spatial regularity of $z$.

**Proof.** We prove Theorem 3.3 by estimating difference quotients of $z$.

Let $z \in \mathrm{W}^{1,1}(S;\mathcal{Z})$ satisfy (3.6) with data $z_0 \in \mathcal{F}_i$ and $\widetilde{F} \in \mathrm{L}^\infty(S;\mathcal{F}_i)$. Let furthermore $e_i$ denote the corresponding unit vector. Due to the periodicity assumptions on the data it holds that for every $h \in \mathbb{R}$ the shifted functions $z_{0,he_i}$ and $\widetilde{F}_{he_i}$ have the same regularity as $z_0$ and $\widetilde{F}$. Moreover, $z$ satisfies (3.6) with respect to $z_0$ and $\widetilde{F}$ if and only if $z_{he_i}$ satisfies (3.6) with respect to the data $z_{0,he_i}$ and $\widetilde{F}_{he_i}$. Inequality (3.8) with $z_1 = z$ and $z_2 = z_{he_i}$ implies that there exists a constant $c > 0$, which is independent of $h$ and $t$, such that

$$c\,\|\triangle_{he_i} z(t)\|_{\mathcal{Z}}^2 \le \|\triangle_{he_i} z_0\|_{\mathcal{Z}}^2 + \|\triangle_{he_i} z\|_{\mathrm{W}^{1,1}(S;\mathcal{Z})}\,\big\|\triangle_{he_i}\widetilde{F}\big\|_{\mathrm{L}^\infty(S;\mathrm{L}^2(\Omega_P))}.$$

From the regularity assumptions on the data we conclude that the right hand side can further be estimated as (see e.g. [GT77])

$$r.h.s. \le |h|^2\,\|z_0\|_{\mathcal{F}_i}^2 + 2\,|h|\,\|z\|_{\mathrm{W}^{1,1}(S;\mathcal{Z})}\,\big\|\widetilde{F}\big\|_{\mathrm{L}^\infty(S;\mathcal{F}_i)}.$$

This proves estimate (3.10).

Assume now that $\widetilde{F}$ has the additional temporal regularity $\widetilde{F} \in \mathrm{W}^{1,1}(S;\mathcal{F}_i)$. Now, estimate (3.9) implies that

$$\|\triangle_{he_i} z\|_{\mathrm{L}^\infty(S;\mathcal{Z})} \le c\,|h|\,\big(\,\|z_0\|_{\mathcal{F}_i} + \big\|\widetilde{F}\big\|_{\mathrm{W}^{1,1}(S;\mathcal{F}_i)}\big).$$

From Lemma 7.24 in [GT77] it follows that $z \in \mathrm{L}^\infty(S;\mathcal{F}_i)$. $\square$

As a conclusion we obtain the following result for the full periodic problem.

**Theorem 3.4** (Regularity in the periodic case)**.**

*(a) Let the pair* $(u,z) \in \mathrm{L}^\infty(S;V_P) \times \mathrm{W}^{1,1}(S;\mathcal{Z})$ *satisfy* (3.1)–(3.2) *with* $z_0 \in \mathcal{F}_i$, $f \in \mathrm{W}^{1,1}(S;\mathrm{L}^2(\Omega_P))$, $\int_{\Omega_P} f(t)\,\mathrm{d}x = 0$, $H \in \mathrm{W}^{1,1}(S;\mathcal{F}_i)$ *and* $F \in \mathrm{W}^{1,1}(S;\mathcal{F}_i)$ *for some* $i \in \{1,\dots,d\}$*. Then*

$$\partial_i u \in \mathrm{L}^\infty(S;\mathrm{H}^1_{per}(\Omega_P)), \qquad z \in \mathrm{L}^\infty(S;\mathcal{F}_i).$$

*(b) Assume that the pair* $(u,z) \in \mathrm{L}^\infty(S;V_P) \times \mathrm{W}^{1,1}(S;\mathcal{Z})$ *satisfies* (3.1)–(3.2) *with* $z_0 \in \mathrm{B}^{\frac{1}{2}}_{2,\infty}(\Omega_P)$, $f \in \mathrm{L}^\infty(S;\mathrm{L}^2(\Omega_P))$ *with* $\int_{\Omega_P} f(t)\,\mathrm{d}x = 0$, $H \in \mathrm{L}^\infty(S;\mathcal{F}_i)$ *and* $F \in \mathrm{L}^\infty(S;\mathcal{F}_i)$ *for some* $i \in \{1,\dots,d\}$*. Then there exists a constant* $c > 0$ *such that*

$$\sup_{t\in S,\,h\in\mathbb{R}\setminus\{0\}} |h|^{-\frac{1}{2}}\,\|\triangle_{he_i}\nabla u(t)\|_{\mathrm{L}^2(\Omega_P)} \le c,$$
$$\sup_{t\in S,\,h\in\mathbb{R}\setminus\{0\}} |h|^{-\frac{1}{2}}\,\|\triangle_{he_i} z(t)\|_{\mathrm{L}^2(\Omega_P)} \le c.$$

*Here,* $e_i$ *is the unit vector of the* $i$*-th coordinate direction.*

If the assumptions of part (b) are satisfied for every $i \in \{1, \ldots, d\}$, then

$$u \in \mathrm{L}^\infty(S; \mathrm{B}^{\frac{3}{2}}_{2,\infty}(\Omega_P)), \quad z \in \mathrm{L}^\infty(S; \mathrm{B}^{\frac{1}{2}}_{2,\infty}(\Omega_P)).$$

Part (a) of the previous Theorem is closely related to a result in [Nes06] for periodic structures.

**Proof.** Assume that the regularity assumptions of part (a) are valid. Elliptic regularity theory implies that the function $\widetilde{F}$ defined in (3.4) belongs to $\mathrm{W}^{1,1}(S; \mathcal{F}_i)$. Thus it follows from Theorem 3.3 that $z \in \mathrm{L}^\infty(S; \mathcal{F}_i)$. The result for $u$ follows again by elliptic regularity theory on the basis of relation (3.5).

If the data has the regularity described in Theorem 3.4(b), then $\widetilde{F} \in \mathrm{L}^\infty(S; \mathcal{F}_i)$. Theorem 3.3 and elliptic regularity results lead to the results for $z$ and $u$. $\square$

# 4 Proof of the regularity properties of the original problem

In this section we prove regularity theorem 2.2. The tangential regularity of $u$ and $z$ described in Theorem 2.2 follows in the same way as part (a) of Theorem 3.3 and we reformulate the result in Section 4.1. The essential new idea in this paper is the proof of the higher differentiability in directions orthogonal to the Dirichlet and Neumann boundary. This is carried out in section 4.2, where we extend the elasto-plastic model by reflection with respect to the Dirichlet and Neumann boundary to a problem which is periodic in the $e_d$ direction. For deriving the regularity properties with respect to the $e_d$ direction, we apply part (b) of Theorem 3.4. The tangential regularity result from Theorem 4.1 guarantees that the extended data satisfy the assumptions of part (b) of Theorem 3.4.

## 4.1 Regularity tangential to $\Gamma_0$ and $\Gamma_1$

**Theorem 4.1** (Tangential regularity)**.** *Assume that (A1), (A2) and (A3') are satisfied and that* $z_0 \in \mathrm{H}^1_{\Gamma_{per}}(\Omega)$*,* $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$*,* $H \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$ *and* $F \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$*. Then the solution* $(u, z)$ *of the partially periodic problem* (2.5)–(2.7) *satisfies for* $1 \leq i \leq d$

$$\partial_i u \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega)), \quad z \in \mathrm{L}^\infty(S; \cap_{1 \leq j \leq d-1} \mathcal{F}_j(\Omega)).$$

**Proof.** Like in Section 3.2 one can derive a reduced formulation for the partially periodic problem and prove Theorem 4.1 in the same way as Theorem 3.4(a). $\square$

## 4.2 Proof of higher regularity in normal direction

Let $(u, z) \in \mathrm{W}^{1,1}(S; V \times \mathcal{Z})$ be the solution to (2.5)–(2.7) with data according to Theorem 2.2. We recall that $u\big|_{\Gamma_1} = 0$.

Let $\varphi \in \mathrm{C}^\infty([0,\ell])$ with $\varphi(x_d) = 1$ in a neighborhood of 0, $\varphi(x_d) = 0$ in a neighborhood of $\ell$ and $0 \le \varphi \le 1$. By $\gamma_0$ we denote the trace operator from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Gamma_0)$ and define

$$\hat{u}(t,x) := \varphi(x_d)(\gamma_0 u(t))(x') \quad \text{for } x = (x', x_d) \in \Omega,\ t \in S.$$

The tangential regularity of $u$ leads to the following regularity for $\hat{u}$:

**Lemma 4.2.** *It holds $\hat{u} \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$ and $\partial_d \hat{u} \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega))$.*

**Proof.** Since $u \in \mathrm{W}^{1,1}(S;V) \subset \mathrm{L}^\infty(S;\mathrm{H}^1(\Omega))$, the trace theorem implies that $\hat{u} \in \mathrm{L}^\infty(S;\mathrm{L}^2(\Omega))$. Moreover, from Theorem 4.1 we conclude that for all $h \in \mathbb{R}$, for all $i \in \{1,\dots,d-1\}$ and a.e. $t \in S$ it holds

$$\begin{aligned}\|\triangle_{he_i}\hat{u}(t)\|_{\mathrm{L}^2(\Omega)} &\le \sqrt{2\ell}\,\|\triangle_{he_i}\gamma_0 u(t)\|_{\mathrm{L}^2(\Gamma_0)}\\ &\le c\,\|\triangle_{he_i}u(t)\|_{\mathrm{H}^1(\Omega)} \le c\,|h|\,\|\partial_i u(t)\|_{\mathrm{H}^1(\Omega)}\,.\end{aligned}$$

In the second inequality we have used the continuity of the trace operator $\gamma_0$. Since all the estimates are uniform with respect to $h$, it follows that $\hat{u} \in \mathrm{L}^\infty(S; \cap_{1\le i\le d-1}\mathcal{F}_i(\Omega))$. Obviously, for $k \in \mathbb{N}$ it holds $\partial_d^{(k)}\hat{u}(t,x) = \varphi^{(k)}(x_d)(\gamma_0 u(t))(x')$, and the previous considerations show that $\partial_d^{(k)}\hat{u} \in \mathrm{L}^\infty(S;\mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega))$, which finishes the proof. $\square$

Let $R = \mathbb{I} - 2e_d \otimes e_d \in \mathbb{R}^{d\times d}$ describe the reflection at the boundary $\Gamma_0$. For functions $v : \Omega \to \mathbb{R}^d$ we will use the notation

$$\nabla' v(x) := (\partial_1 v(x), \dots, \partial_{d-1}v(x), 0) = \tfrac{1}{2}(\nabla v(x))(R+\mathbb{I}) \in \mathbb{R}^{d\times d}.$$

We consider the following extensions to $\Omega_P = (-\ell,\ell)^d$:

$$u_P(t,x) = \begin{cases} u(t,x) - \hat{u}(t,x) & x \in \Omega_+ = \Omega \\ -u(t,Rx) + \hat{u}(t,Rx) & x \in \Omega_- = R\Omega \end{cases}.$$

For the internal variable we use an even extension:

$$z_P(t,x) = \begin{cases} z(t,x) & x \in \Omega_+ \\ z(t,Rx) & x \in \Omega_- \end{cases}, \qquad z_{0,P}(x) = \begin{cases} z_0(x) & x \in \Omega_+ \\ z_0(Rx) & x \in \Omega_- \end{cases}.$$

The extended volume forces are defined as

$$f_P(t,x) = \begin{cases} f(t,x) & x \in \Omega_+ \\ -f(t,Rx) - \big(\mathrm{div}\,\big((A(\nabla u - Bz) - H)(R+\mathbb{I})\big)\big)\circ R & x \in \Omega_- \end{cases}.$$

Note that in the definition of $f_P$ only tangential derivatives of $\nabla u$, $z$ and $H$ are involved. Finally, we define

$$\theta_P(t,x) = \begin{cases} \nabla\hat{u}(t,x) & x \in \Omega_+ \\ -\nabla(\hat{u}\circ R) + 2\nabla'(u\circ R) & x \in \Omega_- \end{cases}$$

and set

$$H_P(t,x) = -A\theta_P(t,x) + \begin{cases} H(t,x) & x \in \Omega_+ \\ H(t,Rx) & x \in \Omega_- \end{cases},$$

$$F_P(t,x) = -B^\top A\theta_P(t,x) + \begin{cases} F(t,x) & x \in \Omega_+ \\ F(t,Rx) & x \in \Omega_- \end{cases}.$$

**Lemma 4.3.** *Under the assumptions of Theorem 2.2 the above defined functions have the following regularity*

$$\begin{aligned} &u_P \in \mathrm{L}^\infty(S; V_P(\Omega_P)), \\ &z_P \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega_P)), \quad z_{0,P} \in \mathrm{H}^1_{per}(\Omega_P), \\ &f_P \in \mathrm{L}^\infty(S; \mathrm{L}^2(\Omega_P)), \quad \int_{\Omega_P} f_P(t)\,\mathrm{d}x = 0, \\ &\theta_P, H_P,\, F_P \in \mathrm{L}^\infty(S; \mathcal{F}_d). \end{aligned}$$

**Proof.** The assertions for $z_P$ and $z_{0,P}$ are obvious. It follows from Lemma 4.2 that $u_P\big|_{\Omega_\pm} \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega_\pm))$. Moreover, we have $\gamma_0(u_P\big|_{\Omega_+}) = 0 = \gamma_0(u_P\big|_{\Omega_-})$ and therefore $u_P \in \mathrm{L}^\infty(S; V_P(\Omega_P))$.

The higher tangential regularity of $u$ and $z$, see Theorem 4.1, guarantees that $f_P \in \mathrm{L}^\infty(S; \mathrm{L}^2(\Omega_P))$. By partial integration we conclude that

$$\int_{\Omega_+} \operatorname{div}\big((A(\nabla u - Bz) - H)(R + \mathbb{I})\big)\,\mathrm{d}x = 0$$

for a.e. $t \in S$, since on $\Gamma_0 \cup \Gamma_1$ it holds $(R+\mathbb{I})\vec{n} = 0$ and on $\Gamma_{\mathrm{per}}$ we may use the periodicity conditions. Thus, $\int_{\Omega_P} f_P(t)\,\mathrm{d}x = 0$ for a.e. $t \in S$.

Lemma 4.2 and Theorem 4.1 imply that $\partial_d \theta_P\big|_{\Omega_\pm} \in \mathrm{L}^\infty(S; \mathrm{L}^2(\Omega_\pm))$ and we only have to check whether the traces on $\Gamma_0$ coincide and whether $\theta_P\big|_{\Gamma_1} = \theta_P\big|_{R\Gamma_1}$. Straight forward calculations, using integration by parts on $\Omega_+$ and $\Omega_-$ separately and taking into account that $u\big|_{\Gamma_1} = 0$, show that for every $\psi \in \mathrm{C}^\infty_{\mathrm{per}}(\Omega_P)$ it holds

$$\int_{\Omega_P} \theta_P : \partial_d \psi\,\mathrm{d}x = \int_{\Omega_-} \partial_d(2\nabla'(u \circ R) - \nabla(\hat{u} \circ R)) : \psi\,\mathrm{d}x + \int_{\Omega_+} \partial_d \nabla \hat{u} : \psi\,\mathrm{d}x.$$

This proves the assertions on $\theta_P$ and finally on $H_P$ and $F_P$. □

**Lemma 4.4.** *For almost all $t \in S$ and every $v \in H^1_{per}(\Omega_P)$ the above defined functions $u_P \in \mathrm{L}^\infty(S; V_P)$ and $z_P \in \mathrm{W}^{1,1}(S; \mathcal{Z})$ satisfy $z_P(0) = z_{0,P}$ and*

$$\int_{\Omega_P} A(\nabla u_P(t) - Bz_P(t)) : \nabla v\,\mathrm{d}x = \int_{\Omega_P} f_P(t) \cdot v + H_P(t) : \nabla v\,\mathrm{d}x, \tag{4.1}$$

$$\partial_t z_P(t) \in \partial\chi_{\mathcal{K}}(-Lz_P(t) + B^\top A(\nabla u_P(t) - Bz_P(t)) - F_P(t)). \tag{4.2}$$

*Thus,* $\sup_{t\in S, h\in\mathbb{R}\setminus\{0\}} |h|^{-\frac{1}{2}} \big( \|\triangle_{he_d} \nabla u_P(t)\|_{\mathrm{L}^2(\Omega_P)} + \|\triangle_{he_d} z_P(t)\|_{\mathrm{L}^2(\Omega_P)} \big) \le c$ .

**Proof.** Relations (4.1)–(4.2) show that the extended functions $u_P$ and $z_P$ are solutions of a pure periodic model. The regularity of $u_P$ and $z_P$ therefore follows from part (b) of Theorem 3.4. Due to Lemma 4.3 the extended functions $f_P$, $F_P$, $H_P$ and $z_{0,P}$ have the required regularity properties for $i = d$.

We prove now that (4.1) and (4.2) are valid.

Relation (4.2) follows immediately from the definitions taking into account that the convex set $K$ does not depend on $x$.

Relation (4.1) can be verified as follows: for every $v \in V_P$ it holds

$$\begin{aligned}\int_{\Omega_P} A(\nabla u_P - Bz_P) : \nabla v \,\mathrm{d}x = &\int_{\Omega_+} A(\nabla u - Bz) : \nabla(v - v{\circ}R)\,\mathrm{d}x \\ &- \int_{\Omega_+} A\nabla\hat{u} : \nabla v\,\mathrm{d}x \\ &+ \int_{\Omega_+} A(\nabla\hat{u}R - \nabla u(R+\mathbb{I})) : \nabla(v{\circ}R)R\,\mathrm{d}x \\ &+ \int_{\Omega_+} \big(A(\nabla u - Bz)\big)(R+\mathbb{I}) : \nabla(v{\circ}R)\,\mathrm{d}x. \end{aligned} \tag{4.3}$$

Note that $(v - v{\circ}R)\big|_{\Omega_+} \in V(\Omega_+)$ and therefore we may use relations (2.6) and (2.8) to replace the first term on the right hand side of (4.3). This yields

$$\begin{aligned}\int_{\Omega_P} A(\nabla u_P - Bz_P) : \nabla v\,\mathrm{d}x = &\int_{\Omega_+} f\cdot(v - v{\circ}R)\,\mathrm{d}x \\ &+ \int_{\Omega_+} \big(A(\nabla u - Bz) - H\big)(R+\mathbb{I}) : \nabla(v{\circ}R)\,\mathrm{d}x \\ &+ \int_{\Omega_+} H : (\nabla v + \nabla(v{\circ}R)R)\,\mathrm{d}x \\ &- \int_{\Omega_+} A\nabla\hat{u} : \nabla v\,\mathrm{d}x \\ &+ \int_{\Omega_-} A(\nabla(\hat{u}{\circ}R) - \nabla(u{\circ}R)(R+\mathbb{I})) : \nabla v\,\mathrm{d}x.\end{aligned}$$

Transforming the terms with $v{\circ}R$ back to $\Omega_-$ and applying the Gauss Theorem to the second term on the right hand side finally proves relation (4.1). Note that the boundary terms vanish due to the periodicity properties on $\Gamma_{\mathrm{per}}$ and since $(R{+}\mathbb{I})\vec{n} = 0$ on $\Gamma_0{\cup}\Gamma_1$. $\square$

The main regularity theorem, Theorem 2.2, is now an immediate consequence of the previous lemma:

**Corollary 4.5.** *Let the assumptions of Theorem 2.2 be satisfied.*
*Then* $u \in \mathrm{L}^\infty(S; \mathrm{B}^{\frac{3}{2}}_{2,\infty}(\Omega))$ *and* $z \in \mathrm{L}^\infty(S; \mathrm{B}^{\frac{1}{2}}_{2,\infty}(\Omega))$.

**Proof.** Let $u_P$ and $z_P$ be the extensions of $u$ and $z$ as defined above. By Lemma 4.4 we have

$$\sup_{t\in S,\,\widetilde{\Omega}\Subset\Omega,\,h\in\mathbb{R}\setminus\{0\}} |h|^{-\frac{1}{2}} \left( \|\triangle_{he_d}\nabla u(t)\|_{\mathrm{L}^2(\widetilde{\Omega})} + \|\triangle_{he_d} z(t)\|_{\mathrm{L}^2(\widetilde{\Omega})} \right) \le c.$$

Combining this estimate with the tangential regularity in Theorem 4.1 proves Corollary 4.5 and Theorem 2.2. $\square$

Assume that $B = 0$. Then the problem (2.5)–(2.7) decouples into an elliptic equation (2.6) for $u$ and an evolution inclusion (2.7) for $z$. If $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$ and $H \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega))$, then standard results for linear elliptic systems guarantee that $u \in \mathrm{W}^{1,1}(S; \mathrm{H}^2(\Omega))$. This result is usually obtained by first proving a tangential result like in Theorem 4.1 and then by solving the equation for the missing derivatives: $\partial_d(A\nabla u)_d = -f - \operatorname{div} H - \frac{1}{2}\operatorname{div}((A\nabla u)(R + \mathbb{I}))$. Due to the tangential regularity, the right hand side belongs to $\mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$, and thus $\partial_d^2 u(t) \in \mathrm{L}^2(\Omega)$. For pure elliptic systems this argument is equivalent to the reflection argument which we applied in the proof of Lemma 4.4.

Note finally that in the decoupled case, i.e. $B = 0$, and under the assumptions of Theorem 2.2 the internal variable has the regularity $z \in \mathrm{L}^\infty(S; \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega))$. This follows since for $B = 0$ the extended function $F_P$ belongs to $\mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$ and not only to $\mathrm{L}^\infty(S; \mathrm{L}^2(\Omega))$, and therefore part (a) of Theorem 3.4 can be applied.

### 4.3 Non-zero boundary conditions

We consider now the case with non-vanishing Dirichlet datum $h_D \in \mathrm{W}^{1,1}(S; \mathrm{H}^{\frac{1}{2}}_{\mathrm{per}}(\Gamma_1))$ and Neumann datum $h_N \in \mathrm{W}^{1,1}(S; (\mathrm{H}^{\frac{1}{2}}_{\mathrm{per}}(\Gamma_0))')$. The task is to find a pair $(u, z) \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega) \times \mathrm{L}^2(\Omega))$ with $u(t)\big|_{\Gamma_1} = h_D(t)$ and $z(0) = z_0$ such that for all $v \in V$ and a.e. $t \in S$ we have

$$\int_\Omega A(\nabla u(t) - Bz(t)) : \nabla v \,\mathrm{d}x = \int_\Omega f(t) \cdot v \,\mathrm{d}x - \langle h_N(t), v\rangle_{\Gamma_0}, \tag{4.4}$$

$$\partial_t z(t) \in \partial\chi_{\mathcal{K}}(-Lz(t) + B^\top A(\nabla u(t) - Bz(t))). \tag{4.5}$$

Assume that

$$h_D \in \mathrm{W}^{1,1}(S; \mathrm{H}^{\frac{3}{2}}_{\mathrm{per}}(\Gamma_1)), \qquad h_N \in \mathrm{W}^{1,1}(S; \mathrm{H}^{\frac{1}{2}}_{\mathrm{per}}(\Gamma_0)). \tag{4.6}$$

By the trace theorem there exists $u_D \in \mathrm{W}^{1,1}(S; \mathrm{H}^2_{\Gamma_{\mathrm{per}}}(\Omega))$ with $u_D\big|_{\Gamma_1} = h_D$. Moreover, there exists $u_N \in \mathrm{W}^{1,1}(S; V \cap \mathrm{H}^2_{\Gamma_{\mathrm{per}}}(\Omega))$ with

$$\int_\Omega A\nabla u_N : \nabla v \,\mathrm{d}x = \langle h_N(t), v\rangle_{\Gamma_0}$$

for all $v \in V$. It follows that the pair $(u, z)$ solves (4.4)–(4.5) if and only if $u = u_0 + u_D$ with $u_0(t) \in V$ and the pair $(u_0, z)$ satisfies for every $v \in V$

$$\int_\Omega A(\nabla u_0(t) - Bz(t)) : \nabla v \,\mathrm{d}x = \int_\Omega f(t) \cdot v + A(\nabla u_N(t) - \nabla u_D(t)) : \nabla v \,\mathrm{d}x, \tag{4.7}$$

$$\partial_t z(t) \in \partial\chi_{\mathcal{K}}(-Lz(t) + B^\top A(\nabla u_0(t) - Bz(t)) + B^\top A\nabla u_D(t)). \tag{4.8}$$

From (4.6) we conclude that

$$H := A(\nabla u_N - \nabla u_D) \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega)),$$
$$F := -B^\top A \nabla u_D \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{\mathrm{per}}}(\Omega)).$$

Thus, the next theorem and Theorem 1.1 follow immediately from Theorem 2.2.

**Theorem 4.6** (Non-vanishing boundary data)**.** *Let* $z_0 \in \mathrm{H}^1_{\Gamma_{per}}(\Omega)$*,* $f \in \mathrm{W}^{1,1}(S; \mathrm{L}^2(\Omega))$*,* $h_D$ *and* $h_N$ *with* (4.6) *and assume that the pair* $(u, z) \in \mathrm{W}^{1,1}(S; \mathrm{H}^1_{\Gamma_{per}}(\Omega) \times \mathrm{L}^2(\Omega))$ *satisfies* (4.4)*–*(4.5)*. Then*

$$u \in \mathrm{L}^\infty(S; \mathrm{B}^{\frac{3}{2}}_{2,\infty}(\Omega)), \qquad z \in \mathrm{L}^\infty(S; \mathrm{B}^{\frac{1}{2}}_{2,\infty}(\Omega)).$$

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