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Laguerre–Freud equations for the recurrence coefficients of D_ω semi-classical orthogonal polynomials of class one

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Abstract

The Laguerre–Freud equations giving the recurrence coefficients β_n, γ_n of orthogonal polynomials with respect to a D_ω semi-classical linear form are derived. D_ω is the difference operator. The limit when $\omega \to 0$ are also investigated recovering known results. Applications to generalized Meixner polynomials of class one are also treated. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $\mathscr{L}$ be a regular linear form in the vector space of all complex polynomials of one variable.

By 'regular linear form $\mathscr{L}$' [13] we mean that there exists $(P_n)_{n\geqslant 0}$ a sequence of monic orthogonal polynomials with respect to $\mathscr{L}$, i.e.,

$$\begin{aligned}&\text{degree of } P_n = n, \quad n \geqslant 0,\\ &\langle \mathscr{L}, P_n P_m\rangle = 0, \quad n \neq m, \qquad \langle \mathscr{L}, P_n P_n\rangle \neq 0, \quad n \geqslant 0,\end{aligned} \tag{1}$$

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where $\langle \mathcal{L}, P\rangle$ denotes the value of the linear form $\mathcal{L}$ applied to P. The three-term recurrence relation satisfied by the monic family $(P_n)_{n\geqslant 0}$ will be written as

$$\begin{aligned} &P_{n+1}(x)=(x-\beta_n)P_n(x)-\gamma_n P_{n-1}(x), \quad n\geqslant 1,\\ &P_0(x)=1, \qquad P_1(x)=x-\beta_0, \quad \gamma_n\neq 0,\ n\geqslant 1. \end{aligned} \tag{2}$$

The difference operator D_ω is defined by:

$$D_\omega P(x)=\frac{P(x+\omega)-P(x)}{\omega}, \quad \omega\neq 0,\ D_1=\Delta,\ D_{-1}=\nabla.$$

Definition 1 ([6, 12, 13]). The linear form $\mathcal{L}$ is said to be D_ω semi-classical if $\mathcal{L}$ is regular and there exist two polynomials ψ of degree $\geqslant 1$ and ϕ such that

$$D_\omega(\phi\mathcal{L})=\psi\mathcal{L}, \tag{3}$$

where

$$\langle\psi\mathcal{L},P\rangle=\langle\mathcal{L},\psi P\rangle, \qquad \langle D_\omega\mathcal{L},P\rangle=-\langle\mathcal{L},D_{-\omega}P\rangle. \tag{4}$$

Moreover, if $\mathcal{L}$ is D_ω semi-classical, the class of $\mathcal{L}$, denoted $\mathrm{cl}(\mathcal{L})$, is defined as [13]

$$\mathrm{cl}(\mathcal{L})=\min\{\max(-2+\text{degree of }\phi,-1+\text{degree of }\psi)\}, \tag{5}$$

where the minimum is taken over all pairs of polynomials ϕ, and ψ of degree at least one, satisfying

$$D_\omega(\phi\mathcal{L})=\psi\mathcal{L}.$$

The following characterization of the class of the D_ω semi-classical functionals [9, 13] follows from definition 1.

If $\mathcal{L}$ is a D_ω semi-classical functional satisfying (3), then $\mathcal{L}$ is said to be of class s if and only if for any root c of the polynomial ϕ, one of these two conditions is satisfied:
(i) $r_{c-\omega}\neq 0$,
(ii) $\langle\mathcal{L},\rho_{c-\omega}\rangle\neq 0$,
where

$$\begin{aligned} &s=\max\{-2+\text{degree of }\phi,-1+\text{degree of }\psi\},\\ &\phi(x)=(x-c)\phi_c(x),\\ &\psi(x)-\phi_c(x)=(x-c+\omega)\rho_{c-\omega}(x)+r_{c-\omega}. \end{aligned} \tag{6}$$

The two coupled algebraic equations giving β_n and γ_n are called the Laguerre–Freud equations, introduced for the first time in [2]. In [1], Belmehdi has given them explicitly for $\mathrm{D}(=\mathrm{d}/\mathrm{d}x)$ semi-classical polynomials of class $s=1$. Let us recall that when ϕ is of degree maximum 2 and ψ of degree 1, the corresponding polynomials are called (discrete) classical and of course the β_n,γ_n are well known [4, 10, 14].

The aim of this work is to give the Laguerre–Freud equations when the operator $\mathrm{D}=\mathrm{d}/\mathrm{d}x$ is replaced by the difference operator D_ω [4, 13]. D_ω semi-classical orthogonal polynomials appear already in [13], but β_n and γ_n have been computed only in the classical case.

We suppose that the linear form $\mathscr{L}$ is semi-classical and we present here the derivation of the Laguerre–Freud equations for the class $s=1$ which means that polynomials ϕ and ψ are restricted to

$$\psi(x)=a_0+a_1x+a_2x^2, \qquad \phi(x)=b_0+b_1x+b_2x^2+b_3x^3. \tag{7}$$

The Laguerre–Freud equations are deduced like in [2] from the two obvious relations deduced now from

$$D_\omega(\phi\mathscr{L})=\psi\mathscr{L}, \tag{8}$$

$$\langle D_\omega(\phi\mathscr{L}),P_nP_n\rangle=\langle\mathscr{L},\psi P_nP_n\rangle, \tag{9}$$

$$\langle D_\omega(\phi\mathscr{L}),P_nP_{n+1}\rangle=\langle\mathscr{L},\psi P_nP_{n-1}\rangle. \tag{10}$$

If the rules (4) are the same for D and D_ω, then the main differences (and difficulties) come from the more complicated product rule:

$$D_\omega[P(x)Q(x)]=P(x+\omega)D_\omega Q(x)+Q(x)D_\omega P(x), \tag{11}$$

introducing shifted polynomial $P(x+\omega)$.

Rules (4) and (11) transform the Eqs. (9) and (10) into

$$\langle\mathscr{L},P_n(x-\omega)\phi D_{-\omega}P_n(x)\rangle+\langle\mathscr{L},P_n\phi D_{-\omega}P_n\rangle=-\langle\mathscr{L},\psi P_nP_n\rangle, \tag{12}$$

$$\langle\mathscr{L},P_n(x-\omega)\phi D_{-\omega}P_{n+1}(x)\rangle+\langle\mathscr{L},P_{n+1}\phi D_{-\omega}P_n\rangle=-\langle\mathscr{L},\psi P_nP_{n+1}\rangle. \tag{13}$$

2. Basic iteration scheme

In order to express the coefficients of the orthogonal polynomial P_n in terms of the recurrence coefficients β_n and γ_n, let us compute them recursively from the iteration process given by the following lemma.

Lemma 2. *All basic coefficients $T_{n,i}$ in the expansion of*

$$P_n(x)=\sum_{i=0}^{n}T_{n,i}x^{n-i} \tag{14}$$

can be computed recursively from the relations:

$$T_{1,1}=-\beta_0,$$

$$T_{n,0}=1, \quad n\geqslant 0, \tag{15}$$

$$T_{n+1,1}=T_{n,1}-\beta_n, \quad n\geqslant 1, \tag{16}$$

$$T_{n+1,j}=T_{n,j}-\beta_nT_{n,j-1}-\gamma_nT_{n-1,j-2}, \quad 2\leqslant j\leqslant n, \tag{17}$$

$$T_{n+1,n+1}=-\beta_nT_{n,n}-\gamma_nT_{n-1,n-1}, \quad n\geqslant 1. \tag{18}$$

These basic relations are easily proved by identification from:

$$xP_n(x)=\sum_{i=0}^{n}T_{n,i}x^{n+1-i}=\sum_{i=0}^{n+1}T_{n-1,i}x^{n+1-i}+\beta_n\sum_{i=0}^{n}T_{n,i}x^{n-i}+\gamma_n\sum_{i=0}^{n-1}T_{n-1,i}x^{n-1-i}.$$

Use of Eqs. (15)–(17) gives

$$T_{n+1,1}=-\sum_{i=0}^{n}\beta_i,\quad n\geqslant 0, \tag{19}$$

$$T_{n+1,2}=\sum_{0\leqslant i<j\leqslant n}\beta_i\beta_j-\sum_{i=1}^{n}\gamma_i,\quad n\geqslant 1, \tag{20}$$

$$T_{n+1,3}=-\sum_{0\leqslant i<j<k\leqslant n}\beta_i\beta_j\beta_k+\sum_{1\leqslant i<j\leqslant n}(\gamma_i\beta_j+\beta_i\gamma_j)+\beta_0\sum_{i=1}^{n}\gamma_i$$
$$-\sum_{i=1}^{n}\beta_{i-1}\gamma_i,\quad n\geqslant 2. \tag{21}$$

All other terms can be computed in the same way but for class $s=1$, only these three terms will be used.

Let us emphasize that the two terms: $T_{n,1}$ and $T_{n,2}$ are already given in [4]; the computation of the higher order coefficients allows to generate Laguerre–Freud equations for any arbitrary class $s>1$. These coefficients play the role (but in a simpler way) of the Turán determinants introduced in [2] showing the interest in Laguerre–Freud equations.

3. Intermediate coefficients

The structure constants [1, 13] will first be obtained in terms of the $T_{n,i}$ and the coefficients of polynomials ϕ and ψ defining the linear form $\mathscr{L}$.

In order to do that, we first need to expand the polynomials $P_n(x+\omega)$ and $D_\omega P_n$ in terms of the $T_{n,i}$ and also to control the action of the linear form $\mathscr{L}$ on polynomial $x^{n+k}P_n$ via the coefficients $T_{n,i}$.

3.1. Coefficients $A_{n,i}$ and $A^*_{n,i}$

These coefficients appear in the following expansions:

$$P_n(x+\omega)=\sum_{i=0}^{n}A_{n,i}(\omega)x^{n-i}, \tag{22}$$

$$D_\omega P_n(x)=\sum_{i=1}^{n}A^*_{n,i}(\omega)x^{n-i}. \tag{23}$$

Both coefficients are related to $T_{n,i}$ via the relation

$$A^*_{n,i}(\omega) = \frac{A_{n,i}(\omega) - T_{n,i}}{\omega}, \tag{24}$$

and are explicitly given by

$$A_{n,i}(\omega) = \sum_{k=0}^{i} \binom{n-k}{i-k} \omega^{i-k} T_{n,k}, \quad 0 \leqslant i \leqslant n, \tag{25}$$

$$A^*_{n,i}(\omega) = \sum_{k=0}^{i-1} \binom{n-k}{i-k} \omega^{i-k-1} T_{n,k}, \quad 1 \leqslant i \leqslant n. \tag{26}$$

3.2. Coefficients B_n^k

The coefficients B_n^k appear from the action of the linear form $\mathscr{L}$ on $x^{n+k}P_n(x)$:

$$B_n^k = \langle \mathscr{L}, x^{n+k} P_n \rangle. \tag{27}$$

Expansion of $x^{n+k}P_n(x)$ in terms of the $T_{n,i}$ and use of the basic norm:

$$I_{0,n} = \langle \mathscr{L}, P_n P_n \rangle, \tag{28}$$

taking care that

$$B_n^0 = \langle \mathscr{L}, x^n P_n \rangle = \langle \mathscr{L}, P_n P_n \rangle = I_{0,n}, \tag{29}$$

and using the orthogonality property give the relations easily derived:

$$B_n^1 = -T_{n+1,1} I_{0,n}, \tag{30}$$

$$B_n^2 = (T_{n+1,1} T_{n+2,1} - T_{n+2,2}) I_{0,n}, \tag{31}$$

$$B_n^3 = [T_{n+1,1}(T_{n+3,2} - T_{n+2,1} T_{n+3,1}) + T_{n+3,1} T_{n+2,2} - T_{n+3,3}] I_{0,n}, \tag{32}$$

$$B_n^k = -\sum_{i=1}^{k} T_{n+k,i} B_n^{k-i}. \tag{33}$$

All other coefficients B_n^k can be computed in the same way.

The connection between the B_n^k and the coefficients $C_{j,k}^k$ introduced in [2]:

$$x^{n+k} P_n(x) = \sum_{j=0}^{2n+k} C_{j,n}^{n+k} P_j(x), \tag{34}$$

is obviously

$$B_n^k = C_{0,n}^{n+k} I_{0,0}. \tag{35}$$

3.3. Structure relation

As any semi-classical orthogonal family, P_n satisfies a structure relation [13] which for the class $s=1$, and taking into account Definition 1, reduces to

$$\phi(x)D_{-\omega}P_n(x)=\sum_{i=n-2}^{n+2}\lambda_{n,i}(\omega)P_i(x). \tag{36}$$

The aim is again to represent the structure constants $\lambda_{n,i}$ in terms of the $T_{n,i}$. Using the orthogonality rules, Eqs. (28) and (36), we obtain:

$$\lambda_{n,i}(\omega)I_{0,i}=\langle\mathscr{L},P_i\phi D_{-\omega}P_n\rangle. \tag{37}$$

Use of the orthogonality rules in Eq. (37) gives

$$\lambda_{n,i}(\omega)I_{0,i}=\left\langle\mathscr{L},P_i(x)\phi(x)\left[\sum_{k=1}^{n-i+3}A^*_{n,k}(-\omega)x^{n-k}\right]\right\rangle. \tag{38}$$

Using Eqs. (7), (19)–(21), (26), (27), (30)–(32) and (38), we compute the structure relation constants $\lambda_{n,n+2}$, $\lambda_{n,n+1}$, $\lambda_{n,n}$, $\lambda_{n,n-1}$ as

$$\lambda_{n,n+2}(\omega)=nb_3, \tag{39}$$

$$\lambda_{n,n+1}(\omega)=nb_2+\left[-\binom{n}{2}\omega+n(\beta_n+\beta_{n+1})+\sum_{i=0}^{n-1}\beta_i\right]b_3, \tag{40}$$

$$\begin{aligned}\lambda_{n,n}(\omega)={}&nb_1+\left[-\binom{n}{2}\omega+n\beta_n+\sum_{i=0}^{n-1}\beta_i\right]b_2\\&+\left\{\binom{n}{3}\omega^2-\binom{n}{2}\omega\beta_n-(n-1)\omega\sum_{i=0}^{n-1}\beta_i\right\}b_3\\&+\left\{n\beta_n^2+n(\gamma_n+\gamma_{n+1})+\sum_{i=0}^{n-1}\beta_i^2+\beta_n\sum_{i=0}^{n-1}\beta_i+2\sum_{i=1}^{n-1}\gamma_i\right\}b_3,\end{aligned} \tag{41}$$

$$\begin{aligned}\lambda_{n,n-1}(\omega)={}&\sum_{i=0}^{n-1}\phi(\beta_i)+\left[2\sum_{i=1}^{n-1}\gamma_i+n\gamma_n\right]b_2\\&+\left[3\sum_{i=1}^{n-1}\gamma_i(\beta_{i-1}+\beta_i)+\gamma_n\sum_{i=0}^{n-1}\beta_i+n(\beta_{n-1}+\beta_n)\gamma_n\right]b_3\\&-\binom{n}{2}\omega b_1+\left[-(n-1)\omega\sum_{i=0}^{n-1}\beta_i+\binom{n}{3}\omega^2\right]b_2\\&-\omega\left[\sum_{0\leqslant i<j\leqslant n-1}\beta_i\beta_j+(n-1)\sum_{i=0}^{n-1}\beta_i^2+(2n-3)\sum_{i=1}^{n-1}\gamma_i+\binom{n}{2}\gamma_n\right]b_3\\&+\left[\binom{n-1}{2}\omega^2\sum_{i=0}^{n-1}\beta_i-\binom{n}{4}\omega^3\right]b_3.\end{aligned} \tag{42}$$

The aim of the following is to avoid the computation of $T_{n,4}$ in order to have simple expression for $\lambda_{n,n-2}$ and $\lambda_{n,n-1}$.

One easily shows, using the derivative rule:

$$P_i(x)D_{-\omega}P_n(x) = D_{-\omega}[P_i(x+\omega)P_n(x)] - P_n(x)D_\omega P_i(x). \tag{43}$$

Using Eq. (37) and taking into account Eqs. (4), (8) and (43), we obtain:

$$\lambda_{n,i}(\omega)I_{0,i} = -\langle\psi(x)\mathscr{L}, P_i(x+\omega)P_n(x)\rangle - \langle\phi(x)\mathscr{L}, P_n(x)D_\omega P_i(x)\rangle. \tag{44}$$

Using Eqs. (7), (19)–(21), (26), (27), (30)–(32) and (44), we compute the structure relation constants $\lambda_{n,n-1}, \lambda_{n,n-2}$ as

$$\lambda_{n,n-1}(\omega) = -a_1\gamma_n - \gamma_n[(n-1)\omega + \beta_{n-1} + \beta_n]a_2 - (n-1)\gamma_n b_2$$
$$- \left[(n-2)(\beta_{n-1}+\beta_n) + \sum_{i=0}^{n}\beta_i + \binom{n-1}{2}\omega\right]\gamma_n b_3, \tag{45}$$
$$\lambda_{n,n-2}(\omega) = -[a_2 + (n-2)b_3]\gamma_{n-1}\gamma_n. \tag{46}$$

This representation of the structure relation transforms Eqs. (12) and (13), respectively, into

$$\lambda_{n,n-2}(\omega)\langle\mathscr{L}, P_n(x-\omega)P_{n-2}(x)\rangle + \lambda_{n,n-1}(\omega)\langle\mathscr{L}, P_n(x-\omega)P_{n-1}(x)\rangle$$
$$+2\lambda_{n,n}(\omega)I_{0,n} = -\langle\psi\mathscr{L}, P_nP_n\rangle, \tag{47}$$
$$\lambda_{n-1,n-1}(\omega)\langle\mathscr{L}, P_n(x-\omega)P_{n-1}(x)\rangle + \lambda_{n+1,n}(\omega)I_{0,n} + \lambda_{n,n+1}(\omega)\gamma_{n+1}I_{0,n}$$
$$= -\langle\psi\mathscr{L}, P_nP_{n+1}\rangle. \tag{48}$$

The right-hand side of each equation containing ψ is the same as in the continuous case $s=1$, already given in [2], i.e.:

$$\langle\psi\mathscr{L}, P_nP_n\rangle = \psi(\beta_n)I_{0,n} + a_2(\gamma_n + \gamma_{n+1})I_{0,n}, \tag{49}$$
$$\langle\psi\mathscr{L}, P_nP_{n+1}\rangle = [a_1 + a_2(\beta_n + \beta_{n+1})]\gamma_{n+1}I_{0,n}. \tag{50}$$

4. Final form of the Laguerre–Freud equations

The D_ω term in both Laguerre–Freud equations in the form (12) and (13) are now eliminated and replaced by $\lambda_{n,i}P_i$ terms. The remaining quantities to be computed are therefore

$$\langle\mathscr{L}, P_n(x-\omega)P_i(x)\rangle = \left\langle\mathscr{L}, P_i\sum_{k=0}^{n-i}A_{n,k}(-\omega)x^{n-k}\right\rangle, \tag{51}$$

all quantities and all constants being written in terms of the $T_{n,i}$.

Replacing the structure constants $\lambda_{n,i}$ $(n-2 \leqslant i \leqslant n)$ given by Eqs. (41), (45) and (46), Eq. (47), and using also Eqs. (25)–(27) and (49)–(51), the first Laguerre–Freud equation reads

$$\psi(\beta_n) + 4b_3 \sum_{i=1}^{n-1} \gamma_i + 2\sum_{i=0}^{n-1} \theta_{\beta_n}\phi(\beta_i) + \omega \sum_{i=0}^{n-1} \theta_{\beta_n}\psi(\beta_i) + 2\binom{n}{3}\omega^2 b_3 + \binom{n}{2}\omega^2 a_2 = -(a_2 + 2nb_3)(\gamma_n + \gamma_{n+1}), \tag{52}$$

with

$$\theta_a\phi(x) = \frac{\phi(x) - \phi(a)}{x - a}.$$

Replacing the structure constants $\lambda_{n+1,n-1}$, $\lambda_{n+1,n}$ and $\lambda_{n,n+1}$ given by Eqs. (40), (42) and (46), Eq. (48), and using also Eqs. (25)–(27) and (49)–(51), the second Laguerre–Freud equation reduces to

$$\begin{aligned}
&\sum_{i=0}^{n} \phi(\beta_i) + \left[(2n+1)\gamma_{n+1} + 2\sum_{i=1}^{n}\gamma_i\right] b_2 \\
&\quad + 3b_3 \sum_{i=1}^{n} \gamma_i(\beta_{i-1} + \beta_i) + 2\gamma_{n+1}\left(n\beta_n + \sum_{i=0}^{n}\beta_i\right) b_3 \\
&\quad + n\omega a_2\gamma_{n+1} - \binom{n+1}{2}\omega b_1 + \left[-n\omega\sum_{i=0}^{n}\beta_i + \binom{n+1}{3}\omega^2\right] b_2 \\
&\quad - \omega\left[\sum_{0\leqslant i<j\leqslant n} \beta_i\beta_j + n\sum_{i=0}^{n}\beta_i\beta_i + (2n-1)\sum_{i=1}^{n}\gamma_i + n\gamma_{n+1}\right] b_3 \\
&\quad + \left[\binom{n}{2}\omega^2\sum_{i=0}^{n}\beta_i - \binom{n+1}{4}\omega^3\right] b_3 + [a_1 + a_2\beta_n]\gamma_{n+1} \\
&= -[a_2 + (2n+1)b_3]\beta_{n+1}\gamma_{n+1}.
\end{aligned} \tag{53}$$

Let us emphasize that we can also obtain the second Laguerre–Freud equation by identification of the two expressions of $\lambda_{n,n-1}$ given by Eqs. (42) and (45).

The first equation gives γ_{n+1} linearly in terms of γ_j and β_j $(1 \leqslant j \leqslant n)$.The second equation is used in order to compute β_{n+1} from the previous γ_{n+1} via the two non linear terms

$$(2n+1)\gamma_{n+1}\beta_{n+1}b_3 \quad \text{and} \quad -\gamma_{n+1}\beta_{n+1}a_2.$$

The nonlinearities in the second equation only exemplify the fundamental barrier between semi-classical and classical situation in which both a_2 and b_3 are zero. From the nonlinearities, both relations must therefore be used simultaneously starting with

$$\beta_0 = \frac{M_1}{M_0}, \qquad a_2\gamma_1 = -\psi(\beta_0), \tag{54}$$

M_0 and M_1 are the moments of order 0 and of 1 of the linear form $\mathscr{L}$. In the classical case, the equations can be decoupled [13].

5. Limiting situation and particular cases

1. The starting D_ω Laguerre–Freud equations reduce to the D Laguerre–Freud equations already considered in [2] as ω tends to 0. The D_ω Laguerre–Freud equations (52) and (53) reduce exactly to the equations given in [2] relabelling the coefficients of ϕ and ψ in a proper way:

$$\psi(\beta_n) + 4b_3 \sum_{i=1}^{n-1} \gamma_i + 2\sum_{i=0}^{n-1} \theta_{\beta_n}\phi(\beta_i) = -(a_2 + 2nb_3)(\gamma_n + \gamma_{n+1}),$$

$$\begin{aligned}&\sum_{i=0}^{n} \phi(\beta_i) + 3b_3 \sum_{i=1}^{n} \gamma_i(\beta_{i-1} + \beta_i) + \left[(2n+1)\gamma_{n+1} + 2\sum_{i=1}^{n} \gamma_i\right] b_2 \\ &\quad +2\gamma_{n+1}\left(n\beta_n + \sum_{i=0}^{n} \beta_i\right) b_3 + [a_1 + a_2\beta_n]\gamma_{n+1} \\ &\quad = -[a_2 + (2n+1)b_3]\beta_{n+1}\gamma_{n+1}. \end{aligned} \tag{55}$$

The ω dependence is of order 2 in the first equation (52) and of order 3 in the second one (53), and does not affect the nonlinear term $\gamma_{n+1}\beta_{n+1}$ already mentioned.

2. The Laguerre–Freud equations obtained in (52) and (53) contain obviously the classical cases when $a_2 = b_3 = 0$. In the other cases, let us use the notation of [13] so that we can compare more easily with the results in [13]:

$$\phi(x) = ax^2 + bx + c \quad \text{and} \quad \psi(x) = px + q. \tag{56}$$

Eqs. (52) and (53) reduce to

$$\psi(\beta_n) + 2a\sum_{i=0}^{n-1} \beta_i + 2nb + 2na\beta_n = -n\omega p, \tag{57}$$

$$\begin{aligned}&\sum_{i=0}^{n} \phi(\beta_i) + \left[(2n+1)\gamma_{n+1} + 2\sum_{i=1}^{n} \gamma_i\right] a - \binom{n+1}{2}\omega b \\ &\quad + \left[-n\omega\sum_{i=0}^{n} \beta_i + \binom{n+1}{3}\omega^2\right] a = -p\gamma_{n+1}. \end{aligned} \tag{58}$$

Rewriting the second equation with $n \to n-1$ and subtracting we get:

$$\begin{aligned}&\phi(\beta_n) + [p + (2n+1)a]\gamma_{n+1} - [p + (2n-3)a]\gamma_n \\ &\quad -n\omega b - an\omega\beta_n - a\omega\sum_{i=0}^{n-1} \beta_i + a\omega^2\binom{n}{2} = 0. \end{aligned} \tag{59}$$

Using symbolic computation with Maple V.4 [3], we have checked positively that for classical discrete orthogonal polynomials, coefficients β_n and γ_n, given explicitely in terms of polynomials ϕ and ψ appearing in the discrete Pearson equation [7, 8], are solutions of Eqs. (57) and (58) (with $\omega = 1$).

Eqs. (57) and (59) are exactly the equations derived in the thesis [13] taking into account the D_ω derivative of the linear form given by Eq. (4) and the one used in [13]. Let us remark, however, that in [13] the γ_n equation is obtained using the so-called D_ω representation, expanding a classical orthogonal polynomials P_n as a sum of (maximum three) $D_\omega P_i$ ($i=n+1,n,n-1$). This technique cannot be extended to the class 1, because of the nonexistence of such a representation for semi-classical orthogonal polynomials of class $s>0$.

6. Applications

6.1. Generalized Meixner

These polynomials with q parameters were introduced in [11] in order to show the quasi-orthogonality character of the D_ω derivative (with $\omega=1$). The weight ρ is given by

$$\rho(i)=\frac{\mu^i}{(i!)^q}\prod_{j=1}^{q}\Gamma(i+\alpha_j), \quad (0<\mu<1, \alpha_j>0), \; i=0,1,2,\dots . \tag{60}$$

These polynomials are denoted by $m_n^{(\boldsymbol{\alpha},\mu)}$ where $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_q)$, which reduce of course to the well-known classical Meixner polynomials when $\boldsymbol{\alpha}$ is the scalar α ($q=1$).

When $q=2$, $\alpha_1\neq 1$ and $\alpha_2\neq 1$, this family is semi-classical of class $s=1$ with

$$\phi(x)=x^2 \quad \text{and} \quad \psi(x)=(\mu-1)x^2+(\alpha_1+\alpha_2)\mu x+\mu\alpha_1\alpha_2. \tag{61}$$

Of course when $\alpha_1=\alpha_2=1$, the class reduces to 0 and the polynomials, a particular case of the Meixner polynomials, are called discrete Laguerre polynomials [4]:

$$la_n(x)=m_n^{(1,\mu)}(x). \tag{62}$$

We have checked, positively, the Laguerre–Freud equations when $\omega\to 1$ with the known β_n,γ_n of the discrete Laguerre polynomials. Let us also emphasize that for $q=2$ and for arbitrary positive α_1 and α_2, the weight given by Eq. (60), is not a polynomial modification of the Meixner weight, except when α_1 or α_2 is an integer.

Replacing in Eqs. (52) and (53) ω by one and polynomials ϕ and ψ given by Eq. (61), we obtain Laguerre–Freud equations for generalized Meixner polynomial of class $s=1$:

$$\begin{aligned}(1-\mu)(\gamma_n+\gamma_{n+1})=(\mu-1)\left(\binom{n}{2}+\beta_n^2\right)&+((1+\mu)n\\ &+\mu(\alpha_1+\alpha_2))\beta_n+(1+\mu)\sum_{i=0}^{n-1}\beta_i\\ &+\mu(\alpha_1+\alpha_2)n+\mu\alpha_1\alpha_2,\end{aligned} \tag{63}$$

$$\begin{aligned}(1-\mu)(\beta_n+\beta_{n+1})\gamma_{n+1} &= -n\sum_{i=0}^{n}\beta_i+((1+\mu)n+\mu(\alpha_1+\alpha_2)+1)\gamma_{n+1}\\ &\quad+\binom{n+1}{3}+\sum_{i=0}^{n}\beta_i^2+2\sum_{i=1}^{n}\gamma_i,\end{aligned} \tag{64}$$

with initial values

$$\beta_0 = \frac{M_1}{M_0} = \frac{\mu\alpha_1\alpha_2\,{}_2F_1(1+\alpha_1, 1+\alpha_2; 2; \mu)}{{}_2F_1(\alpha_1, \alpha_2; 1; \mu)}, \quad \gamma_1 = \frac{\psi(\beta_0)}{1-\mu}. \tag{65}$$

6.2. Concluding remarks

(i) The polynomials have been computed for generalized Meixner polynomial of class one up to $n=10$ from the β_n, γ_n generated by the Laguerre–Freud equations given above and also from the Hankel representation of polynomials which requires the computation of the moments M_j up to $j=19$. These moments were computed from the moment recurrence relation for generalized Meixner polynomial:

$$(1-\mu)M_{k+2} = \alpha_1\alpha_2\mu M_k + (\alpha_1+\alpha_2)\mu M_{k+1} - \sum_{j=1}^{k}(-1)^j \binom{k}{j} M_{k+2-j}. \tag{66}$$

The polynomial coefficients in both approaches are written in terms of M_0 and M_1 using the initial values of the Laguerre–Freud recurrence given by Eqs. (65). Polynomials obtained in these two ways coincide of course and the Laguerre–Freud approach is obviously more efficient.

(ii) Using Eqs. (63) and (64), we have also computed numerically with Maple V.4, coefficients β_n and γ_n up to $n = 100\,000$, for several values of coefficients α_1, α_2 and μ.

The result of the plot for all cases indicates that the sequences γ_n/n^2, β_n/n are convergent. Assuming that they converge, their limits, $a(\mu)$ and $b(\mu)$, are obtained using Maple V.4 and Eqs. (63), (64) with the approximations: $\gamma_n \cong a(\mu)n^2$ and $\beta_n \cong b(\mu)n$, for n large. We obtain

$$a(\mu) = \frac{\mu}{(1-\mu)^2}, \qquad b(\mu) = \frac{1+\mu}{1-\mu}. \tag{67}$$

By the same way using numerical and symbolic computation with Maple V.4 and analysis of Eqs. (63) and (64) [5], the following asymptotic behaviour is observed for coefficients β_n and γ_n.

6.3. *Conjecture* [5]

The coefficients of the three-terms recurrence relation of the generalized Meixner polynomial of class one are given by

$$\beta_n = \frac{1+\mu}{1-\mu}n + \frac{\mu(\alpha_1+\alpha_2-1)}{1-\mu} + (\alpha_1-1)(\alpha_2-1)U(n),$$
$$\gamma_n = \frac{\mu(n+\alpha_1-1)(n+\alpha_2-1)}{(1-\mu)^2} - (\alpha_1-1)(\alpha_2-1)V(n), \tag{68}$$

where $U(n)$ and $V(n)$ are two positive sequences converging to zero.

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