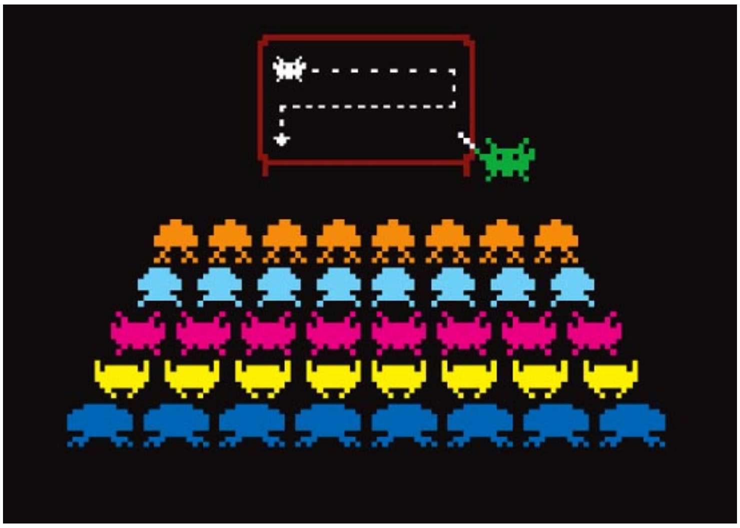
Lecture Notes on *Function Spaces*



"space invaders"

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Contents

Motivation

Examples of function spaces:

$$C(\mathbb{R}^n;\mathbb{R}),\ C^1([-1,1]),\ C^\alpha(\Omega;\mathbb{C}),\ L^p(X,\nu),\ W^{k,p}(\Omega),\ W^{s,p},\ B^s_{p,q},$$

where Ω domain in $\mathbb{R}^n$, $\alpha \in \mathbb{R}_+$, $1 \le p,q \le \infty$, (X,ν) measure space, $k \in \mathbb{N}$, $s \in \mathbb{R}_+$.

Philosophy: "Objectify" function spaces

Example: We can consider $H^{1/2}$ as

- *interpolation space*: $L^2 \cap H^1 \hookrightarrow H^{1/2} \hookrightarrow L^2 + H^1$, $H^{1/2} = \mathcal{F}(L^2, H^1)$
- *embedded space*: $H^1 \subset H^{1/2} \subset L^2$
- space of functions $f \in L^2$ with "one half" of a derivative: $|\xi|^{1/2}\hat{f} \in L^2$
- *trace space*: $\gamma : H^1(\mathbb{R}^n_+) \to H^{1/2}(\mathbb{R}^{n-1})$ is bounded and surjective.

Books

Author: *Title*, Reference	Which topics in the lecture? Comment	Where in the Notes?
Adams/Fournier *Sobolev Spaces*, [1]	Sobolev Spaces, Orlicz Spaces Functions on Domains	Sections 6.5, 9.3
Bennett/Sharpley: *Interpolation of Operators*, [2]	very nice elaborate proofs	Sections 6.2, 9.4, 9.5
Bergh/Löfström: *Interpolation Spaces,* [3]	often short proofs, nice structure for us	Chapters 7, 8 Section 9.2
Lunardi: *Interpolation Theory*, [5]	very good introduction, very nice proofs	Chapters 2,3,4,5
Lunardi: *Analytic Semigroups and Optimal Regularity of Parabolic Problems,* [4]	book on a different topic, for us: Hölder spaces	Section 6.3
Tartar: *An Introduction to Sobolev Spaces and Interpolation Spaces,* [6]	find history, references and ideas	inbetween
Triebel: *Interpolation Theory, Function Spaces, Differential Operators,* [7]	find everything, difficult for proofs	Chapter 1, everywhere inbetween

1 Basic Notions in Interpolation Theory

Picture:

$$\begin{tikzcd}
& \mathcal{A} \arrow[d, dash, dotted] & \\
A_0 \arrow[ur, dash, dotted] \arrow[r, dash, dashed] \arrow[d, "T|_{A_0}"] & A \arrow[r, dash, dashed] \arrow[d, "T|_{A}"] & A_1 \arrow[ul, dash, dotted] \arrow[d, "T|_{A_1}"] \\
B_0 \arrow[dr, dash, dotted] \arrow[r, dash, dashed] & B \arrow[r, dash, dashed] \arrow[d, dash, dotted] & B_1 \arrow[dl, dash, dotted] \\
& \mathcal{B} &
\end{tikzcd}$$

Let $\mathcal{A}, \mathcal{B}$ be linear Hausdorff spaces and $A_0, A_1 \subset \mathcal{A}$, $B_0, B_1 \subset \mathcal{B}$ be Banach spaces. Let $T : \mathcal{A} \to \mathcal{B}$ be a linear operator such that $T|_{A_0} : A_0 \to B_0$ and $T|_{A_1} : A_1 \to B_1$ are bounded. Then:

- $\{A_0, A_1\}$, $\{B_0, B_1\}$ are called *interpolation couples*.
- Question: Can we find Banach spaces $A \subset \mathcal{A}$, $B \subset \mathcal{B}$, such that for all T as above, $T \in \mathcal{L}(A, B)$?
- If the answer is "yes", then A, B are said to have the *interpolation property* with respect to $\{A_0, A_1\}$, $\{B_0, B_1\}$.
- We will often consider the special case $\mathcal{A} = \mathcal{B}$, $A_0 = B_0$, $A_1 = B_1$. In short: A has the *interpolation property* with respect to $\{A_0, A_1\}$.

Example 1.1. Let (X, μ) be a complete measure space, where μ is σ-finite. We write $L^p = L^p(X, \mu)$ for the Lebesgue spaces of complex or real-valued functions $f : X \to \mathbb{C}$ or $f : X \to \mathbb{R}$. It holds that

1. L^{p_θ} has the interpolation property with respect to $\{L^{p_0}, L^{p_1}\}$ if $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $0 < \theta < 1$.

2. The space $C^1([-1,1])$ $(= C^1([-1,1];\mathbb{R}))$ does not have the interpolation property with respect to $\{C([-1,1]), C^2([-1,1])\}$.

Theorem 1.2. (Convexity Theorem of Riesz/Thorin)
Let $1 \leq p_0, p_1, q_0, q_1 \leq \infty$, $p_0 \neq p_1$, $q_0 \neq q_1$ and T a linear operator such that $T : L^{p_i}(X,\mu) \to L^{q_i}(Y,\nu)$, $i \in (0,1)$ is bounded linear. Then for every $0 < \theta < 1$,

$$T : L^{p_\theta}(X,\mu) \to L^{q_\theta}(Y,\nu) \quad \textit{is linear and bounded,}$$

for $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $\frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. Furthermore, the estimate

$$\|T\|_{\mathcal{L}(L^{p_\theta},L^{q_\theta})} \leq C\|T\|^{1-\theta}_{\mathcal{L}(L^{p_0},L^{q_0})}\|T\|^{\theta}_{\mathcal{L}(L^{p_1},L^{q_1})}$$

holds true.

Remark 1.3. About the above theorem:

- If the L^p are complex-valued, $C = 1$. If they are real-valued, $C = 2$.
- Example 1 immediately follows from Theorem 1.2.
- Riesz 1926, Thorin 1939/48: Interpolation result which existed before interpolation theory. The direct proof contains ideas for general constructions of interpolation spaces (As and Bs in the picture). For us, this means that the proof will be given in a later Chapter :-).
- Why "convexity": The theorem shows that the function f given by $f(\frac{1}{p}, \frac{1}{q}) = \|T\|_{\mathcal{L}(L^p,L^q)}$ is logarithmically convex, i.e.

$$f\left((1-\theta)(\frac{1}{p_0}, \frac{1}{q_0}) + \theta(\frac{1}{p_1}, \frac{1}{q_1})\right) \leq f(\frac{1}{p_0}, \frac{1}{q_0})^{1-\theta} f(\frac{1}{p_1}, \frac{1}{q_1})^{\theta}$$

  or, in other words, $g = \log f$ is convex.

Reminder: $f \in C^k([-1,1]) \Leftrightarrow f : [-1,1] \to \mathbb{R}$ is k-times continuously differentiable and $\|f\|_{C^k([-1,1])} := \sum_{l=0}^{k} \frac{1}{l!} \sup_{x\in[-1,1]} |f^{(k)}(x)| < \infty$ (we can replace sup by max).

Theorem 1.4. (Mitjagin/Semenov '76)
For every $\varepsilon \in (0,1]$ let $V_\varepsilon : C([-1,1]) \to C([-1,1])$ be given by

$$(V_\varepsilon f)(x) := \int_{-1}^{1} \frac{x}{x^2 + y^2 + \varepsilon^2} (f(y) - f(0))\,\mathrm{d}y. \tag{1.1}$$

Then for all $\varepsilon \in (0,1]$, it holds that

1. *$V_\varepsilon \in C^\infty([-1,1])$,*
2. *$\|V_\varepsilon\|_{\mathcal{L}(C([-1,1]))} < 2\pi$,*

3. $\|V_\varepsilon\|_{\mathcal{L}(C^2([-1,1]))} < 5\pi + 2$,

4. given $f_\varepsilon(y) = \sqrt{y^2+\varepsilon^2} - \varepsilon$, *we get* $\|f_\varepsilon\|_{C^1([-1,1])} \le 2$, *but* $(V_\varepsilon f_\epsilon)'(0) > 2\ln(\frac{1}{5\varepsilon})$.

Corollary 1.5. *From Theorem 1.4 we get Example 2.*

Proof. The proof of the corollary will be given as an exercise. Proof of Theorem 1.4: In the following, we write C^k for $C^k([-1,1])$ and C^0 for $C([-1,1])$

1. Differentiation in the integral in (1.1).
2. Calculate:

$$\begin{aligned}
|V_\varepsilon f(x)| &\le \int_{-1}^{1} \frac{|x|}{x^2+y^2+\varepsilon^2} 2\|f\|_{C^0}\,\mathrm{d}y \\
&\underset{\text{symmetry}}{\le} 4\|f\|_{C^0}|x| \int_0^1 \frac{1}{x^2+y^2}\,\mathrm{d}y \\
&\le 4\|f\|_{C^0}|x|[\frac{1}{|x|}\arctan(\frac{y}{|x|})]_0^1 \\
&\le 2\pi\|f\|_{C^0}.
\end{aligned}$$

3. Identity map: $h(y) = y$, then

$$(V_\varepsilon h)(x) = \int_{-1}^{1} \frac{xy}{x^2+y^2+\varepsilon^2}\,\mathrm{d}y = 0 \tag{1.2}$$

for all $x \in [-1,1]$. Taylor Theorem: If $f \in C^2$, then

$$f(y) = f(0) + f'(0)y + r_2(f,y) \tag{1.3}$$

for some $r_2(f,\cdot) \in C^0$ and

$$|r_2(f,y)| = \frac{|f''(\vartheta y)|}{2} y^2 \le \|f\|_{C^2} y^2. \tag{1.4}$$

It follows from (1.2) and (1.3) that

$$\begin{aligned}
(V_\varepsilon f)(s) &= \int_{-1}^{1} \frac{x}{x^2+y^2+\varepsilon^2}[f(y) - f(0) - f'(0)y]\,\mathrm{d}y \\
&= \int_{-1}^{1} \frac{x}{x^2+y^2+\varepsilon^2} r_2(f,y)\,\mathrm{d}y.
\end{aligned}$$

Note that

$$|\frac{d}{dx}(\frac{x}{x^2+y^2+\varepsilon^2})| = |\frac{y^2+\varepsilon^2-x^2}{(x^2+y^2+\varepsilon^2)^2}| < \frac{1}{x^2+y^2+\varepsilon^2} < \frac{1}{y^2+\varepsilon^2}$$

and

$$|\frac{d^2}{dx^2}(\frac{x}{x^2+y^2+\varepsilon^2})| = \frac{2|x||x^2-3y^2-3\varepsilon^2|}{(x^2+y^2+\varepsilon^2)^3} < \frac{6|x|}{(x^2+y^2+\varepsilon^2)^2}.$$

In conclusion, from (1.4), we get

$$\begin{aligned}
|(V_\varepsilon f)'(x))| &\leq \int_{-1}^{1} \left| \frac{d}{dx}(\frac{x}{x^2+y^2+\varepsilon^2}) \right| |r_2(f,y)| \,\mathrm{d}y \\
&\leq \|f\|_{C^2} \int_{-1}^{1} \frac{y^2}{y^2+\varepsilon^2} \,\mathrm{d}y \leq 2\|f\|_{C^2}
\end{aligned}$$

and

$$\begin{aligned}
|(V_\varepsilon f)''(x))| &\leq \int_{-1}^{1} \frac{6|x|}{(x^2+y^2+\varepsilon^2)^2} \|f\|_{C^2} y^2 \,\mathrm{d}y \\
&\leq 12\|f\|_{C^2} \int_{0}^{1} \frac{|x|}{x^2+y^2} \,\mathrm{d}y \leq 6\pi\|f\|_{C^2},
\end{aligned}$$

so $\|V_\varepsilon f\|_{C^2} \leq (2\pi + 2 + 3\pi)\|f\|_{C^2}$ for all $f \in C^2$.

4. We see: The f_ε approximate $|\cdot|$. Note that

$$f_\varepsilon''(0) = \frac{\varepsilon^2}{(y^2+\varepsilon^2)^{3/2}}|_{y=0} = \frac{1}{\varepsilon} \stackrel{\varepsilon\to 0}{\longrightarrow} \infty$$

(i.e. $V_\varepsilon : C^2 \to C^2$ is ok!). Moreover, $|f_\varepsilon(y)| = \frac{y^2}{\sqrt{y^2+\varepsilon^2}+\varepsilon} < |y| \leq 1$ and $|f_\varepsilon'(y)| = \frac{|y|}{\sqrt{y^2+\varepsilon^2}} < 1$. The interesting part is:

$$\begin{aligned}
(V_\varepsilon f_\varepsilon)'(x) &= \int_{-1}^{1} \frac{y^2+\varepsilon^2-x^2}{(x^2+y^2+\varepsilon^2)^2}(\sqrt{y^2+\varepsilon^2}-\varepsilon) \,\mathrm{d}y \\
&\stackrel{x=0}{=} \int_{-1}^{1} \frac{\sqrt{y^2+\varepsilon^2}-\varepsilon}{y^2+\varepsilon^2} \,\mathrm{d}y \\
&\stackrel{u=\frac{y}{\varepsilon}}{=} 2\int_{0}^{1/\varepsilon} \frac{\sqrt{u^2+1}-1}{u^2+1} \,\mathrm{d}u \\
&= 2\int_{0}^{1/\varepsilon} \frac{1}{\sqrt{u^2+1}} \,\mathrm{d}u - 2\int_{0}^{1/\varepsilon} \frac{1}{u^2+1} \,\mathrm{d}u \\
&\geq 2\int_{0}^{1/\varepsilon} \frac{1}{u+1} \,\mathrm{d}u - 2\arctan(\frac{1}{\varepsilon}) \\
&\geq 2\ln(1+\frac{1}{\varepsilon}) - \pi > 2\ln(\frac{1+\frac{1}{\varepsilon}}{e^{\pi/2}}) > 2\ln(\frac{1}{5\varepsilon}).
\end{aligned}$$

□

Notation: We write $A \hookrightarrow B$ iff $\mathrm{id} : A \to B$ is bounded.

Lemma 1.6. *Let $\{A_0, A_1\}$ be an interpolation couple. Then*

$$A_0 + A_1 = \{a \in \mathcal{A} : \exists a_0 \in A_0, \exists a_1 \in A_1, a = a_0 + a_1\}$$

with the norm

$$\|a\|_{A_0+A_1} = \inf_{a=a_0+a_1, a_i \in A_i} (\|a_0\|_{A_0} + \|a_1\|_{A_1})$$

and

$$A_0 \cap A_1 = \{a \in \mathcal{A} : a \in A_0, a \in A_1\}$$

with the norm

$$\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1})$$

are Banach spaces. It holds that

$$A_0 \cap A_1 \hookrightarrow A_i \hookrightarrow A_0 + A_1.$$

Proof. Exercise. □

Definition 1.7. (Basic definitions in category theory)

1. A *category* consists of

   a) a class of *objects* $A, B, C, \ldots$ and

   b) a class of pairwise disjoint non-empty sets $[A, B]$. Each ordered pair (A, B) uniquely corresponds to a set $[A, B]$. The elements in $[A, B]$ are called *morphisms.*

2. For every ordered triplet (A, B, C) of objects, we have the composition of morphisms via
$$V : [B, C] \times [A, B] \to [A, C].$$
Notation: $f \in [A, B]$, $g \in [B, C]$, then $gf = V(g, f)$. Moreover, we have associativity
$$f \in [A, B], g \in [B, C], h \in [C, D], \text{ then } (hg)f = h(gf)$$
and for all objects A, there exists an identity $\mathrm{id}_A \in [A, A]$, such that for all $f \in [B, A]$, $g \in [A, B]$,
$$\mathrm{id}_A f = f \quad \text{and } g\mathrm{id}_A = g.$$

3. Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ be two categories. A map $\mathcal{F} : \mathfrak{C}_1 \to \mathfrak{C}_2$ is called a (covariant) functor, if

   a) for all objects A in $\mathfrak{C}_2$, $\mathcal{F}(A)$ is an object in $\mathfrak{C}_1$,

b) for all morphisms $f \in [A, B]$ in $\mathfrak{C}_2$, $\mathcal{F}(f) \in [\mathcal{F}(A), \mathcal{F}(B)]$ is a morphism in $\mathfrak{C}_1$,

c) $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ for all objects A in $\mathfrak{C}_2$,

d) $\mathcal{F}(gf) = \mathcal{F}(g)\mathcal{F}(f)$ for all morphisms $f \in [A, B], g \in [B, C]$.

Example 1.8. The notions categories are strong in some areas of mathematics, like geometry. For more examples, check e.g. Wikipedia :). For us, the following are relevant,

- Category $\mathfrak{C}_1$: (complex) Banach spaces $A, B, C, \dots$ are objects, bounded linear operators $T \in [A, B] = \mathcal{L}(A, B)$ are morphisms.
- Category $\mathfrak{C}_2$: interpolation couples $\{A_0, A_1\}, \{B_0, B_1\}, \dots$ are objects and
$$[\{A_0, A_1\}, \{B_0, B_1\}] = \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$$
are sets of morphisms, where $T : A_0 + A_1 \to B_0 + B_1$, $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ if $T|_{A_0} : A_0 \to B_0$, $T|_{A_1} : A_1 \to B_1$ are bounded.

Remark 1.9. The space $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ is a Banach space with the norm

$$\|T\|_{\mathcal{L}(\{A_0,A_1\},\{B_0,B_1\})} = \max(\|T|_{A_0}\|_{\mathcal{L}(A_0,B_0)}, \|T|_{A_1}\|_{\mathcal{L}(A_1,B_1)})$$

and

$$\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\}) \hookrightarrow \mathcal{L}(A_0 + A_1, B_0 + B_1). \tag{1.5}$$

Proof. Let

$$\begin{aligned} \mathcal{D} &= \{(U, V) : U \in \mathcal{L}(A_0, B_0), V \in \mathcal{L}(A_1, B_1), U = V \text{ on } A_0 \cap A_1\} \\ &\subset \mathcal{L}(A_0, B_0) \times \mathcal{L}(A_1, B_1), \end{aligned}$$

where

$$\|(U, V)\|_{\mathcal{D}} = \|(U, V)\|_{\mathcal{L}(A_0,B_0)\times\mathcal{L}(A_1,B_1)} = \max(\|U\|_{\mathcal{L}(A_0,B_0)}, \|V\|_{\mathcal{L}(A_1,B_1)}).$$

We show that

1. $\mathcal{D}$ is a closed subspace of $\mathcal{L}(A_0, B_0) \times \mathcal{L}(A_1, B_1)$, i.e. it is a Banach space,
2. $\mathcal{D}$ is isometrically isomorphic to $\mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$.

Regarding 1.: Let $(U_n, V_n) \to (U, V) \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ for some $(U_n, V_n)_n \subset \mathcal{D}$, so

$$\max(\|U - U_n\|_{\mathcal{L}(A_0,B_0)}, \|V - V_n\|_{\mathcal{L}(A_1,B_1)}) \to 0.$$

We have to show that $U = V$ on $A_0 \cap A_1$. For every $a \in A_0 \cap A_1$,

$$Ua - Va = (U - U_n)a + (U_n - V_n)a + (V_n - V)a$$

where $U_n - V_n \equiv 0$, $(U - U_n)a \to 0$ in B_0 and $(V_n - V)a \to 0$ in B_1. By embedding, $(U - U_n)a + (V_n - V)a \to 0$ in $B_0 + B_1$, so $U = V$ on $A_0 \cap A_1$.
Regarding 2.: We consider $j : \mathcal{L}(\{A_0, A_1\}, \{A_1, B_1\}) \to \mathcal{D}$, given by $j(T) = (T|_{A_0}, T|_{A_1})$. It follows that j is linear and isometric, therefore injective. It remains to show that j is surjective. Let $(\tilde{U}, \tilde{V}) \in \mathcal{D}$. We define $\tilde{T} : A_0 + A_1 \to B_0 + B_1$ by

$$\tilde{T}a = \tilde{T}(a_0 + a_1) = \tilde{U}a_0 + \tilde{U}a_1.$$

$\tilde{T}$ is well-defined: Let $a_0 + a_1 = a = a_0' + a_1'$, then $a_0' - a_0 = a_1 - a_1' \in A_0 \cap A_1$, so

$$\tilde{T}(a_0 + a_1) = \tilde{U}a_0 + \tilde{V}a_1 = \tilde{U}(a_0 - a_0') + \tilde{U}a_0' + \tilde{V}a_1' - \tilde{V}(a_1' - a_1) = \tilde{T}(a_0' + a_1')$$

since $\tilde{U} = \tilde{V}$ on $A_0 \cap A_1$. It is clear that $\tilde{T} \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$ and $j(\tilde{T}) = (\tilde{U}, \tilde{V})$. It remains to show (1.5). For all $a = a_0 + a_1$, $a_0 \in A_0$, $a_1 \in A_1$, we have

$$\begin{aligned} \|Ta\|_{B_0+B_1} &= \inf_{Ta=b_0+b_1, b_i \in B_i} (\|b_0\|_{B_0} + \|b_1\|_{B_1}) \\ &\leq \|Ta_0\|_{B_0} + \|Ta_1\|_{B_1} \\ &\leq \max\left(\|T|_{A_0}\|_{\mathcal{L}(A_0,B_0)}, \|T|_{A_1}\|_{\mathcal{L}(A_1,B_1)}\right)(\|a_0\|_{A_0} + \|a_1\|_{A_1}). \end{aligned}$$

We take the infimum over $a = a_0 + a_1$ to get that

$$\|Ta\|_{B_0+B_1} \leq \|T\|_{\mathcal{L}(\{A_0,A_1\},\{B_0,B_1\})} \|a\|_{A_0+A_1}.$$

□

We are now in a position to give a formulation of interpolation of Banach spaces in the language of categories.

Definition 1.10. A functor $\mathcal{F} : \mathfrak{C}_2 \to \mathfrak{C}_1$ is called *interpolation functor*, if for every $\{A_0, A_1\}$, $\{B_0, B_1\}$ in $\mathfrak{C}_2$,

$$A_0 \cap A_1 \hookrightarrow \mathcal{F}(\{A_0, A_1\}) \hookrightarrow A_0 + A_1$$

and

$$T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\}) \Rightarrow \mathcal{F}(T) = T|_{\mathcal{F}(\{A_0,A_1\})}.$$

If $\mathcal{F}$ is an interpolation functor, then the space $\mathcal{F}(\{A_0, A_1\})$ is called *interpolation space* with respect to $\{A_0, A_1\}$.

Note that this definition "fits" the picture at the beginning of the chapter, as if $\mathcal{F}(\{A_0, A_1\})$ is an interpolation space in the sense of the definition, it also has the interpolation property with respect to $\{A_0, A_1\}$. It is interesting that also something like the opposite holds true. If $\{A_0, A_1\}$ is an interpolation couple and there is a Banach spaces A such that $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$ and such that $\operatorname{Im}(T|_A) \subset A$ for all $T \in \mathcal{L}(\{A_0, A_1\}, \{A_0, A_1)\})$, then there exists an interpolation functor $\mathcal{F}_0$ such that $A = \mathcal{F}(\{A_0, A_1\})$. The proof of this fact is an exercise, given Theorem 1.14 at the end of this chapter.

Definition 1.11. An interpolation functor $\mathcal{F}$ is called *of type* θ, $0 < \theta < 1$, if there exists a constant $C > 0$, such that for all $\{A_0, A_1\}, \{B_0, B_1\}$,$T$ in $\mathfrak{C}_2$,

$$\|T\|_{\mathcal{L}(\mathcal{F}(\{A_0,A_1\},\mathcal{F}\{B_0,B_1\})} \leq C\|T\|^{1-\theta}_{\mathcal{L}(A_0,B_0)}\|T\|^{\theta}_{\mathcal{L}(A_1,B_1)}.$$

If we can choose $C = 1$, $\mathcal{F}$ is called *exact*. Note: It is always true that $C \geq 1$.

Even if we do not know whether a functor is of type θ or exact, the following theorem shows that we always have some estimate of this type, uniformly in T, but depending on the interpolation space.

Theorem 1.12. *Let $\mathcal{F}$ be an interpolation functor, $\{A_0, A_1\}$, $\{B_0, B_1\}$ interpolation couples and $A = \mathcal{F}(\{A_0, A_1\})$, $B = \mathcal{F}(\{B_0, B_1\})$ interpolation spaces. Then there exists a constant $C(A, B) > 0$ such that*

$$\|T\|_{\mathcal{L}(A,B)} \leq C(A, B) \max\left(\|T\|_{\mathcal{L}(A_0,B_0)}, \|T\|_{\mathcal{L}(A_1,B_1)}\right)$$

for all $T \in \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$.

In order to prove this theorem, we need the following lemma, whose proof is left as an exercise.

Lemma 1.13. *Let U, V, X, Y be Banach spaces such that $U \hookrightarrow X$, $V \hookrightarrow Y$ and $S : U \to V$ such that $S \in \mathcal{L}(X, Y)$. Then we also get $S \in \mathcal{L}(U, V)$.*

Proof. of Theorem 1.12. We use Lemma 1.13 with $U = \mathcal{L}(\{A_0, A_1\}, \{B_0, B_1\})$, $V = \mathcal{L}(A, B)$, $X = \mathcal{L}(A_0 + A_1, B_0 + B_1)$, $Y = \mathcal{L}(A, B_0 + B_1)$ and $S : \mathcal{L}(A_0 + A_1, B_0 + B_1) \to \mathcal{L}(A, B_0 + B_1)$ given by $S(T) = T|_A$. In view of (1.5), we only need to verify that $V \hookrightarrow Y$ and that S is bounded. The first can be derived directly from the embedding $B \hookrightarrow B_0 + B_1$. the boundedness of S follows from the embedding $A \hookrightarrow A_0 + A_1$. More precisely, for all $a \in A$,

$$\begin{aligned}\|Ta\|_{B_0+B_1} &\leq \|T\|_{\mathcal{L}(A_0+A_1,B_0+B_1)}\|a\|_{A_0+A_1} \\ &\leq C(A)\|T\|_{\mathcal{L}(A_0+A_1,B_0+B_1)}\|a\|_A,\end{aligned}$$

so $\|ST\|_{\mathcal{L}(A,B_0+B_1)} \leq C(A)\|T\|_{\mathcal{L}(A_0+A_1,B_0+B_1)}$. Clearly, the constants involved do not depend on T. □

As a closing to this abstract chapter, we look at the following theorem, justifiying the use of interpolation functors for interpolation theory.

Theorem 1.14. (Aronszajn/Gagliardo '65)
Let $\mathfrak{C}_1, \mathfrak{C}_2$ be as in Example 1.8, let $\{A_0, A_1\}$ be an interpolation couple and let A be such that

$$A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$$

and $T(A) \subset A$ for all $T \in \mathcal{L}(\{A_0, A_1\})$. Then there exists an interpolation functor $\mathcal{F}_0 : \mathfrak{C}_2 \to \mathfrak{C}_1$ such that $\mathcal{F}_0(\{A_0, A_1\}) = A$.

Proof. We give a proof in seven small steps.

1. Preliminary observation: By Lemma 1.13, $T(A) \subset A$ implies that $T \in \mathcal{L}(A)$.

2. Construction of $\mathcal{F}_0$: Let $\{X_0, X_1\}$ be an object in $\mathfrak{C}_2$. We define

$$\begin{aligned} X = \mathcal{F}_0(\{X_0, X_1\}) \quad &= \quad \{x \in X_0 + X_1 : x = \sum_{j=1}^{\infty} T_j a_j \text{ abs. conv. },\\ &\qquad a_j \in A, T_j \in \mathcal{L}(\{A_0, A_1\}, \{X_0, X_1\}), j \in \mathbb{N}, \text{where}\\ &\qquad \|x\|_X = \inf_{x=\sum_{j=1}^{\infty} T_j a_j} \sum_{j=1}^{\infty} \|T_j\|_{\mathcal{L}(\{A_0,A_1\},\{X_0,X_1\})} \|a_j\|_A < \infty\}. \end{aligned}$$

   It follows that $\|\cdot\|_X$ is a norm for X.

3. Show $X_0 \cap X_1 \hookrightarrow X$: Let $\varphi : A_0 + A_1 \to \mathbb{C}$ a bounded linear functional such that $\varphi(a^*) = 1$ for some $a^* \in A$. Define $\|\varphi\| = c^*$. For every $x \in X_0 \cap X_1$, we define

$$T_x : A_0 + A_1 \to X_0 + X_1, \quad T_x a := \varphi(a)x.$$

   It follows that for $i \in \{0, 1\}$ and $a \in A_i$,

$$\|T_x a\|_{X_i} = |\varphi(a)| \|x\|_{X_i} \leq c^* \|a\|_{A_0+A_1} \|x\|_{X_i} \leq c\|a\|_{A_i} \|x\|_{X_i},$$

   so $T_x \in \mathcal{L}(\{A_0, A_1\}, \{X_0, X_1\})$ and

$$\|T_x\|_{\mathcal{L}(\{A_0,A_1\},\{X_0,X_1\})} \leq c \max_i(\|T_x\|_{\mathcal{L}(A_i,X_i)}) \leq c \max_i \|x\|_{X_i} = c\|x\|_{X_0 \cap X_1}.$$

   Now for every $x \in X_0 \cap X_1$, $x = T_x a^* \in X$ and

$$\|x\|_X \leq \|T_x\|_{\mathcal{L}(\{A_0,A_1\},\{X_0,X_1\})} \|a^*\|_A \leq c\|a^*\|_A \|x\|_{X_0 \cap X_1}.$$

4. Show that $X \hookrightarrow X_0 + X_1$: Let $x = \sum_{j=1}^{\infty} T_j a_j \in X$. Then by Remark 1.9,

$$\begin{aligned} \|x\|_{X_0+X_1} \quad &\leq \quad \sum_{j=1}^{\infty} \|T_j a_j\|_{X_0+X_1}\\ &\leq \quad \sum_{j=1}^{\infty} \|T_j\|_{\mathcal{L}(A_0+A_1,X_0+X_1)} \|a_j\|_{A_0+A_1}\\ &\leq \quad c\sum_{j=1}^{\infty} \|T_j\|_{\mathcal{L}(\{A_0,A_1\},\{X_0,X_1\})} \|a_j\|_A. \end{aligned}$$

   Taking the inf, we get $\|x\|_{X_0+X_1} \leq c\|x\|_X$.

5. X is complete: If for some $(x_n)_n \subset X$, $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$ then by Step 4, $\sum_{n=1}^{\infty} \|x_n\|_{X_0+X_1} < \infty$, so that the limit $\sum_{n=1}^{\infty} x_n = x$ exists in X_0+X_1. Moreover, by definition, for all $n \in \mathbb{N}$, there exist $T_j^n \in \mathcal{L}(\{A_0, A_1\}, \{X_0, X_1\})$, $a_j^n \in A$, such that $x_n = \sum_{j=1}^{\infty} T_j^n a_j^n$ and
$$\sum_{j=1}^{\infty} \|T_j^n\|_{\mathcal{L}(\{A_0,A_1\},\{X_0,X_1\})} \|a_j^n\|_A < \|x_n\|_X + 2^{-n},$$
so $x = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} T_j^n a_j^n$. Taking the inf, we get $\|x\|_X < \sum_{n=1}^{\infty} \|x_n\|_X + 1$. It follows that $\sum_{n=1}^{m} x_n \overset{m\to\infty}{\to} x$ in X.

6. $\mathcal{F}_0$ on morphisms in $\mathfrak{C}_2$: Let $\{X_0, X_1\}$, $\{Y_0, Y_1\}$ be objects in $\mathfrak{C}_2$, and let $S \in \mathcal{L}(\{X_0, X_1\}, \{Y_0, Y_1\})$ be a morphism. We set $X = \mathcal{F}_0(\{X_0, X_1\})$, $Y = \mathcal{F}_0(\{Y_0, Y_1\})$ and $\mathcal{F}_0(S) := S|_X$. We need to show that $\mathcal{F}_0(S) \in \mathcal{L}(X, Y)$. Let $x = \sum_{j=1}^{\infty} T_j a_j$, where $T_j \in \mathcal{L}(\{A_0, A_1\}, \{X_0, X_1\})$, $a_j \in A$. Then $Sx = \sum_{j=1}^{\infty} ST_j a_j$, where $ST_j \in \mathcal{L}(\{A_0, A_1\}, \{Y_0, Y_1\})$. It follows that for every suitable choice of T_j, a_j,
$$\begin{aligned} \|Sx\|_Y &\le \sum_{j=1}^{\infty} \|ST_j\|_{\mathcal{L}(\{A_0,A_1\},\{Y_0,Y_1\})} \|a_j\|_A \\ &\le \|S\|_{\mathcal{L}(\{X_0,X_1\},\{Y_0,Y_1\})} \sum_{j=1}^{\infty} \|T_j\|_{\mathcal{L}(\{A_0,A_1\},\{X_0,X_1\})} \|a_j\|_A. \end{aligned}$$
Taking the inf, we get $\|Sx\|_Y \le \|S\|_{\mathcal{L}(\{X_0,X_1\},\{Y_0,Y_1\})} \|x\|_X$.

7. We show that $\mathcal{F}_0(\{A_0, A_1\}) = A$: It is clear that $A \subset \mathcal{F}_0(\{A_0, A_1\})$. For the opposite inclusion, let $a = \sum_{j=1}^{\infty} T_j a_j$ be in $\mathcal{F}_0(\{A_0, A_1\})$ as above. By Theorem 1.12, there exists a constant $c(A)$, such that for all $j \in \mathbb{N}$,
$$\|a\|_A \le \sum_{j=1}^{\infty} \|T_j a_j\|_A \le c(A) \sum_{j=1}^{\infty} \|T_j\|_{\mathcal{L}(\{A_0,A_1\})} \|a_j\|_A.$$
Taking the infimum, we get $\|a\|_A \le c(A) \|a\|_{\mathcal{F}_0(\{A_0,A_1\})}$.

□

2 The K-Method

Definition 2.1. Let $\{A_0, A_1\}$ an interpolation couple. Then *(Peetre's) K-functional*

$$K : \mathbb{R}_+ \times A_0 + A_1 \to \mathbb{R}$$

is defined as

$$K(t, a; A_0, A_1) = \inf_{a=a_0+a_1, a_i \in A_i} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}).$$

Notation: $K(t, a)$ instead of $K(t, a; A_0, A_1)$.

Lemma 2.2. *Fix $a \in A_0 + A_1$. Then $K(\cdot, a)$ is positive, monotonely increasing, concave and continuous. It holds that*

$$\min(1, t)\|a\|_{A_0+A_1} \leq K(t, a) \leq \max(1, t)\|a\|_{A_0+A_1}. \tag{2.1}$$

Proof. We see immediately: Positivity, monotonicity and (2.1). It remains to show that K is concave in t, i.e. for all $0 < \lambda < 1$,

$$K((1 - \lambda)t_1 + \lambda t_2, a) \geq (1 - \lambda)K(t_1, a) + \lambda K(t_2, a).$$

Let $0 < t_1 < t < t_2 < \infty$ and $\lambda = \frac{t-t_1}{t_2-t_1}$, so that $1 - \lambda = \frac{t_2-t}{t_2-t_1}$ and $t = (1 - \lambda)t_1 + \lambda t_2$. For every $a_0 \in A_0, a_1 \in A_1$ such that $a_0 + a_1 = a$, we have

$$\frac{t_2 - t}{t_2 - t_1}(\|a_0\|_{A_0} + t_1\|a_1\|_{A_1}) + \frac{t - t_1}{t_2 - t_1}(\|a_0\|_{A_0} + t_2\|a_1\|_{A_1}) = \|a_0\|_{A_0} + t\|a_1\|_{A_1}.$$

We first take the inf on the left hand side to get

$$\frac{t_2 - t}{t_2 - t_1}K(t_1, a) + \frac{t - t_1}{t_2 - t_1}K(t_2, a) \leq \|a_0\|_{A_0} + t\|a_1\|_{A_1}.$$

Now taking the inf on the right hand side gives

$$\frac{t_2 - t}{t_2 - t_1}K(t_1, a) + \frac{t - t_1}{t_2 - t_1}K(t_2, a) \leq K(t, a).$$

Since $K(\cdot, a)$ is concave and monotonely increasing, it is continuous. $\square$

Definition 2.3. Let $\{X_0, X_1\}$ an interpolation couple, $0 < \theta < 1$ and $1 \leq q \leq \infty$. Then we define

$$(X_0, X_1)_{\theta,q} := \{x \in X_0 + X_1 : \|x\|_{(X_0,X_1)_{\theta,q}} < \infty\},$$

where

$$\|x\|_{(X_0,X_1)_{\theta,q}} = \begin{cases} \left(\int_0^\infty [t^{-\theta}K(t, x)]^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, & q < \infty, \\ \sup_{0<t<\infty} t^{-\theta}K(t, x), & q = \infty. \end{cases}$$

Theorem 2.4. *Let $\{X_0, X_1\}$ an interpolation couple, $0 < \theta < 1$ and $1 \le q \le \infty$. Then*

1. *the space $(X_0, X_1)_{\theta,q}$ is an interpolation space with respect to $\{X_0, X_1\}$, i.e. there exists an interpolation functor $K_{\theta,q} : \mathfrak{C}_2 \to \mathfrak{C}_1$ such that $K_{\theta,q}(\{X_0, X_1\}) = (X_0, X_1)_{\theta,q}$. The functor $K_{\theta,q}$ is exact and of type θ.*

2. *For all $x \in (X_0, X_1)_{\theta,q}$,*

$$K(t, x) \le c_{\theta,q} t^{\theta} \|x\|_{(X_0,X_1)_{\theta,q}}. \tag{2.2}$$

Proof. In the following, let $X = (X_0, X_1)_{\theta,q}$. We first show (2.2). The case $q = \infty$ follows immediately. Otherwise, we first write $s^{-\theta q} = \theta q \int_s^\infty t^{-\theta q} \frac{\mathrm{d}t}{t}$. It follows that

$$\begin{aligned} s^{-\theta} K(s, x) &= (\theta q)^{\frac{1}{q}} K(s, x) \left(\int_s^\infty t^{-\theta q} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\le (\theta q)^{\frac{1}{q}} \left(\int_s^\infty t^{-\theta q} K(t, x)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\le (\theta q)^{\frac{1}{q}} \|x\|_X. \end{aligned}$$

Next, we show 1. in four steps.

1. $\|\cdot\|_X$ is a norm:

   - $\|x\|_X = 0 \;\Rightarrow\; K(t, x) \equiv 0 \;\overset{(2.1)}{\Rightarrow}\; x = 0$,
   - $K(\cdot, \lambda x) = |\lambda| K(\cdot, x) \;\Rightarrow\; \|\lambda x\|_X = |\lambda| \|x\|_X$,
   - $K(t, x_0 + x_1) \le K(t, x_1) + K(t, x_2) \;\overset{\text{Minkowski}}{\Rightarrow}\; \|x_0 + x_1\|_X \le \|x_0\|_X + \|x_1\|_X$.

2. X is complete: Let $(x_n)_n$ be a Cauchy sequence in X. By (2.2) and since $K(1, \cdot) = \|\cdot\|_{X_0+X_1}$, $(x_n)_n$ has a limit x in $X_0 + X_1$. Assume now $q < \infty$, analogous proof works otherwise. For every $\varepsilon > 0$, let $n_0(\varepsilon)$ be such that $\|x_m - x_n\| < \frac{\varepsilon}{2}$ for every $m > n > n_0(\varepsilon)$. Moreover, let $L > l > 0$. Then by monotonicity of K and (2.1), there is $m_0(\varepsilon, \theta, l, L)$, such that for all $m \ge m_0(\varepsilon, \theta, l, L)$, $m > n$,

$$\begin{aligned} \left(\int_l^L [t^{-\theta} K(t, x - x_n)]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} &\le \frac{\varepsilon}{2} + \left(\int_l^L [t^{-\theta} K(t, x - x_m)]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\le \frac{\varepsilon}{2} + L \|x - x_m\|_{X_0+X_1} \left(\int_l^L t^{-\theta q} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\le \frac{\varepsilon}{2} + L \left(\frac{1}{\theta q}\right)^{\frac{1}{q}} l^{-\theta} \|x - x_m\|_{X_0+X_1} < \varepsilon. \end{aligned}$$

   Passing to the limit $L \to \infty$, $l \to 0$ yields the claim.

3. Show that $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$: Let $x \in X_0 \cap X_1$, then $K(t,x) \leq \min(1,t)\|x\|_{X_0 \cap X_1}$. W.l.o.g., let $q < \infty$. Then

$$\begin{aligned} \|x\|_X^q &= \int_0^1 [t^{-\theta} K(t,x)]^q \frac{\mathrm{d}t}{t} + \int_1^\infty [t^{-\theta} K(t,x)]^q \frac{\mathrm{d}t}{t} \\ &\leq \|x\|_{X_0 \cap X_1} \left(\int_0^1 t^{(1-\theta)q} \frac{\mathrm{d}t}{t} + \int_1^\infty t^{-\theta q} \frac{\mathrm{d}t}{t} \right) \\ &\leq C_{\theta,q} \|x\|_{X_0 \cap X_1}. \end{aligned}$$

Clearly, $\|x\|_{X_0+X_1} = K(1,x) \overset{(2.2)}{\leq} c_{\theta,q}\|x\|_X$.

4. $K_{\theta,q}$ is an exact interpolation functor of type θ: Let $T \in \mathcal{L}(\{X_0, X_1\}, \{Y_0, Y_1\})$, $T \neq 0$, $x \in X_0 + X_1$, then

$$\begin{aligned} K(t, Ta; Y_0, Y_1) &= \inf_{Tx = y_0 + y_1, y_i \in Y_i} (\|y_0\|_{Y_0} + t\|y_1\|_{Y_1}) \\ &\leq \inf_{x = x_0 + x_1, x_i \in X_i} (\|Tx_0\|_{Y_0} + t\|Tx_1\|_{Y_1}) \\ &\leq \inf_{x = x_0 + x_1, x_i \in X_i} (\|T\|_{\mathcal{L}(X_0,Y_0)}\|x_0\|_{X_0} + t\|T\|_{\mathcal{L}(X_1,Y_1)}\|x_1\|_{X_1}) \\ &= \|T\|_{\mathcal{L}(X_0,Y_0)} \inf_{x = x_0 + x_1, x_i \in X_i} (\|x_0\|_{X_0} + t\frac{\|T\|_{\mathcal{L}(X_1,Y_1)}}{\|T\|_{\mathcal{L}(X_0,Y_0)}}\|x_1\|_{X_1}). \end{aligned}$$

We now set $\tau = t\frac{\|T\|_{\mathcal{L}(X_1,Y_1)}}{\|T\|_{\mathcal{L}(X_0,Y_0)}}$ to conclude that

$$K(t, Ta; Y_0, Y_1) \leq \|T\|_{\mathcal{L}(X_0,Y_0)} K(\tau, a; X_0, X_1).$$

It follows that for all $x \in X$,

$$\begin{aligned} \|Tx\|_Y &= \left(\int_0^\infty [t^{-\theta} K(t, Tx; Y_0, Y_1)]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\leq \|T\|_{\mathcal{L}(X_0,Y_0)} \left(\int_0^\infty [t^{-\theta} K(\tau, x; X_0, X_1)]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}} \\ &\leq \|T\|_{\mathcal{L}(X_0,Y_0)} \left(\frac{\|T\|_{\mathcal{L}(X_0,Y_0)}}{\|T\|_{\mathcal{L}(X_1,Y_1)}}\right)^{-\theta} \left(\int_0^\infty [\tau^{-\theta} K(\tau, x; X_0, X_1)]^q \frac{\mathrm{d}\tau}{\tau} \right)^{\frac{1}{q}} \\ &= \|x\|_X \|T\|_{\mathcal{L}(X_0,Y_0)}^{1-\theta} \|T\|_{\mathcal{L}(X_1,Y_1)}^{\theta}. \end{aligned}$$

If we define $K_{\theta,q}(T) = T|_{K_{\theta,q}(\{X_0,X_1\})}$, it follows that $K_{\theta,q}$ is an interpolation functor which is exact and of type θ.

□

Theorem 2.5. (Properties of $(X_0, X_1)_{\theta,q}$)
Let $\{X_0, X_1\}$ be an interpolation couple, $0 < \theta < 1$ and $1 \leq q \leq \infty$. We obtain the following properties of $(X_0, X_1)_{\theta,q}$.

1. $(X_0, X_1)_{\theta,q} = (X_1, X_0)_{1-\theta,q}$,

2. *for* $1 \leq q \leq r \leq \infty$,

$$(X_0, X_1)_{\theta,1} \hookrightarrow (X_0, X_1)_{\theta,q} \hookrightarrow (X_0, X_1)_{\theta,r} \hookrightarrow (X_0, X_1)_{\theta,\infty},$$

3. *if* $X_0 \hookrightarrow X_1$*, then for all* $0 < \theta < \eta < 1$ *and* $1 \leq q, r \leq \infty$*, we get* $(X_0, X_1)_{\theta,q} \hookrightarrow (X_0, X_1)_{\eta,r}$.

4. *if* $X_0 = X_1$*, then we get* $(X_0, X_1)_{\theta,q} = X_1 = X_0$ *in the sense that the norms are equivalent,*

5. *there is a constant* $C_{\theta,q} > 0$ *, such that for all* $x \in X_0 \cap X_1$,

$$\|x\|_{(X_0,X_1)_{\theta,q}} \leq C_{\theta,q} \|x\|_{X_0}^{1-\theta} \|x\|_{X_1}^{\theta},$$

6. *if we have a second interpolation couple* $\{Y_0, Y_1\}$ *such that* $X_i \hookrightarrow Y_i$*, then*

$$(X_0, X_1)_{\theta,q} \hookrightarrow (Y_0, Y_1)_{\theta,q},$$

7. *if* $q < \infty$ *and* $\theta \leq 0$ *or* $\theta \geq 1$ *or if* $q = \infty$ *and* $\theta < 0$ *or* $\theta > 1$*, then* $(X_0, X_1)_{\theta,q}$" $=$ "$\{0\}$.

Proof. Exercise. □

Remark. The following three chapters 3,4 and 5 follow closely Chapters 1 and 2 in [5].

3 The Trace Method

Let $0 < a < b \leq \infty$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let X be a Banach space. We consider in the following:

- Reminder: Bochner integral and $L^p((a,b),X)$.
- Reminder: Sobolev spaces $W^{m,p}((a,b);\mathbb{K})$.
- **New:** $W^{1,p}((a,b);X)$ as the Banach space of functions $f \in L^p((a,b);X)$, such that there exists a weak derivative $f' := g \in L^p((a,b);X)$, i.e.

  $$\int_a^b f(t)\varphi'(t)\,\mathrm{d}t = -\int_a^b g(t)\varphi(t)\,\mathrm{d}t \qquad \forall \varphi \in C_c^\infty(a,b).$$

  Norm: $\|f\|_{W^{1,p}((a,b);X)} = \|f\|_{L^p((a,b);X)} + \|f'\|_{L^p((a,b);X)}$. In particular, we have the fundamental theorem of calculus, i.e. if $f \in W^{1,p}(a,b;X)$ and $s,t \in (a,b)$, then

  $$f(t) = f(s) + \int_s^t f'(\tau)\,\mathrm{d}\tau. \tag{3.1}$$

3.1 Weighted L^p spaces

Definition 3.1. Let $1 \leq p \leq \infty$. We write $L_*^p(0,\infty)$ for the space of real or complex valued functions f such that

$$\begin{aligned}
\|f\|_{L_*^p(0,\infty)} &= \left(\int_0^\infty |f(t)|^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} < \infty \quad \text{if } p < \infty,\\
\|f\|_{L_*^\infty(0,\infty)} &= \operatorname*{ess\,sup}_t |f(t)| < \infty.
\end{aligned}$$

Lemma 3.2. (Hardy-Young inequality)
Let $f : (0,\infty) \to \mathbb{R}_+$ be measurable, $\alpha > 0$, $1 \leq p < \infty$, then

$$\int_0^\infty t^{-\alpha p}\left(\int_0^t f(s)\,\frac{\mathrm{d}s}{s}\right)^p \frac{\mathrm{d}t}{t} \leq \frac{1}{\alpha^p}\int_0^\infty s^{-\alpha p} f(s)^p\,\frac{\mathrm{d}s}{s}.$$

Proof. First, we substitute $\sigma = \frac{s}{t}$ to get that

$$\begin{aligned}
\Big(\int_0^\infty t^{-\alpha p-1}(\int_0^t f(s)\,\frac{\mathrm{d}s}{s})^p\,\mathrm{d}t\Big)^{1/p} &= \Big(\int_0^\infty t^{-\alpha p-1}(\int_0^1 f(t\sigma)\,\frac{\mathrm{d}\sigma}{\sigma})^p\,\mathrm{d}t\Big)^{1/p} \\
&= \Big\|\int_0^1 t^{-\alpha-1/p} f(t\sigma)\,\frac{\mathrm{d}\sigma}{\sigma}\Big\|_{L^p(0,\infty)} \\
&\leq \int_0^1 \Big(\int_0^\infty t^{-\alpha p-1}\frac{f(t\sigma)^p}{\sigma^p}\,\mathrm{d}t\Big)^{1/p}\,\mathrm{d}\sigma,
\end{aligned}$$

Next, we substitute $\tau = t\sigma$,

$$\begin{aligned}
&= \int_0^1 \Big(\int_0^\infty \sigma^{\alpha p-p}\tau^{-\alpha p-1} f(\tau)^p\,\mathrm{d}\tau\Big)^{1/p}\,\mathrm{d}\sigma \\
&= \Big(\int_0^1 \sigma^{\alpha-1}\,\mathrm{d}\sigma\Big)\Big(\int_0^\infty \tau^{-\alpha p} f(\tau)^p\,\frac{\mathrm{d}\tau}{\tau}\Big)^{1/p} \\
&= \frac{1}{\alpha}\Big(\int_0^\infty \tau^{-\alpha p} f(\tau)^p\,\frac{\mathrm{d}\tau}{\tau}\Big)^{1/p}.
\end{aligned}$$

□

Definition 3.3. For any Banach space X and $1 \leq p \leq \infty$, we define $L^p_*(X) = L^p_*(0,\infty;X)$ as the space of all Bochner measurable functions $f : (0,\infty) \to X$ such that $t \mapsto \|f(t)\|_X$ is in $L^p_*(0,\infty)$. The norm is given by $\|f\|_{L^p_*(X)} := \|t \mapsto \|f(t)\|_X\|_{L^p_*(0,\infty)}$.

3.2 The spaces $V(p,\theta,X_0,X_1)$

In the following, let $0 < \theta < 1$, $1 \leq p \leq \infty$ and $\{X_0, X_1\}$ an interpolation couple.

Definition 3.4. We define

$$\begin{aligned}
V(p,\theta,X_0,X_1) \;:=\; \{v \in W^{1,p}((a,b);X_0+X_1)\,\forall 0<a<b<\infty : t \mapsto t^\theta v(t) \in L^p_*(X_1) \\
\text{and}\ \ t \mapsto t^\theta v'(t) \in L^p_*(X_0)\}
\end{aligned}$$

and

$$\|v\|_{V(p,\theta,X_0,X_1)} := \|t^\theta v\|_{L^p_*(X_1)} + \|t^\theta v'\|_{L^p_*(X_0)}.$$

Lemma 3.5. *It holds that*

1. *$V(p,\theta,X_0,X_1)$ is a Banach space,*
2. *for all $v \in V(p,\theta,X_0,X_1)$ there exists a continuous extension of v to $t = 0$.*

Proof. The proof of the first assertion is given as an exercise. For the second assertion, we use (3.1), the Hölder inequality and the embedding $X_0 \hookrightarrow X_0 + X_1$ to get that for all $0 < s < t$, $1 < p < \infty$ and $p' = \frac{p}{p-1}$,

$$\begin{aligned} \|v(t) - v(s)\|_{X_0+X_1} &\leq \int_s^t \|\tau^{\theta-1/p} v'(\tau)\tau^{1/p-\theta}\|_{X_0+X_1}\,\mathrm{d}\tau \\ &\leq \left(\int_s^t \|\tau^{\theta-1/p} v'(\tau)\|^p_{X_0+X_1}\,\mathrm{d}\tau\right)^{1/p} \left(\int_s^t \tau^{(1/p-\theta)p'}\,\mathrm{d}\tau\right)^{1/p'} \\ &\leq C\|v'\|_{L^p_*(s,t;X_1)}(1+p'(1/p-\theta))^{-1/p'}(t^{1+p'(1/p-\theta)} - s^{1+p'(1/p-\theta)})^{1/p'} \\ &\leq C_{p,\theta}\|v\|_{V(\theta,p,X_0,X_1)}(t^{1+p'(1/p-\theta)} - s^{1+p'(1/p-\theta)})^{1/p'}. \end{aligned}$$

It follows that $v : \mathbb{R}_+ \to X_0 + X_1$ is continuous and if we put $s = 0$ and look at $t \to 0$ in the last line, we see that it can be extended continuously to $t = 0$. The cases $p = 1$ and $p = \infty$ are left as an exercise. □

3.3 Real interpolation by the trace method and equivalence

Theorem 3.6. *Let θ, p and $\{X_0, X_1\}$ be as above. Then*

$$X := (X_0, X_1)_{\theta,p} = \{x \in X_0 + X_1 : \exists v \in V(p, 1-\theta, X_1, X_0), v(0) = x\}$$

and

$$\|x\|_X \cong \inf_{v\in V(p,1-\theta,X_1,X_0),x=v(0)} \{\|v\|_{V(p,1-\theta,X_1,X_0)}\} =: \|x\|^{Tr}_X$$

In order to prove this theorem, we want to use the Hardy-Young inequality, through the following lemma.

Lemma 3.7. *Let u be a function such that $u_\theta := t \mapsto t^\theta u(t) \in L^p_*(0, a; X)$ for some Banach space X, $0 < a \leq \infty$, $0 < \theta < 1$ and $1 \leq p \leq \infty$. Then also the mean value*

$$v(t) := \frac{1}{t}\int_0^t u(s)\,\mathrm{d}s,\ t > 0$$

has this property and $\|v_\theta\|_{L^p_*(0,a;X)} \leq \frac{1}{1-\theta}\|u_\theta\|_{L^p_*(0,a;X)}$.

Proof. The proof is direct if we use the Hardy-Young inequality, Lemma 3.2, as for $1 \leq p < \infty$,

$$\begin{aligned} \int_0^a \|v_\theta(t)\|^p_X \frac{\mathrm{d}t}{t} &= \int_0^a t^{(\theta-1)p} \left\|\int_0^t u(s)\,\mathrm{d}s\right\|^p_X \frac{\mathrm{d}t}{t} \\ &\leq \int_0^a t^{(\theta-1)p} \left(\int_0^t \frac{\|su(s)\|_X}{s}\,\mathrm{d}s\right)^p \frac{\mathrm{d}t}{t} \\ &\leq (\frac{1}{1-\theta})^p \int_0^a s^{(\theta-1)p}\|su(s)\|^p_X \frac{\mathrm{d}s}{s} \\ &= (\frac{1}{1-\theta})^p \|u_\theta\|_{L^p_*(0,a;X)}. \end{aligned}$$

The case $p = \infty$ also follows immediately from the definition. □

We can now proceed to the proof of Theorem 3.6.

Proof. First we show that for a given $x \in X$, we can construct a function $v \in V(p, 1-\theta, X_1, X_0)$ such that x is the trace of v in $t = 0$ and such that $\|x\|_X^{Tr} \le C\|x\|_X$. This part of the proof is devided into four steps.

1. Let $x \in X$. Then for all $t > 0$ there exist $a_t \in X_0$ and $b_t \in X_1$ such that $x = a_t + b_t$ and
$$\|a_t\|_{X_0} + t\|b_t\|_{X_1} \le 2K(t, x). \tag{3.2}$$
It follows that
$$\|x - b_t\|_{X_0+X_1} \le \|a_t\|_{X_0+X_1} \le C\|a_t\|_{X_0} \le 2CK(t, x) \le 2Ct^\theta \|x\|_X$$
by (2.2). It therefore seems that $t \mapsto b_t$ would be a candidate for v, but it is not necessarily differentiable and it does not necessarily satisfy $t^{1-\theta} b_t \in L^p(\mathbb{R}_+; X_1)$ or $t^{1-\theta} b_t' \in L^p(\mathbb{R}_+; X_0)$. In the next step, we use b_t to construct a suitable candidate.

2. Let
$$u(t) := \sum_{n=1}^{\infty} b_{\frac{1}{n+1}} \chi_{(\frac{1}{n+1}; \frac{1}{n}]}(t) = \sum_{n=1}^{\infty} (x - a_{\frac{1}{n+1}}) \chi_{(\frac{1}{n+1}; \frac{1}{n}]}(t)$$
and let
$$v(t) = \frac{1}{t} \int_0^t u(s)\, \mathrm{d}s.$$
Then it still holds that $x = \lim_{t \to 0} u(t) = \lim_{t \to 0} v(t)$ in $X_0 + X_1$.

3. We show that $v_{1-\theta} \in L_*^p(\mathbb{R}_+; X_1)$. Of course, we want to use Lemma 3.7. By (3.2) and from the monotonicity of $t \mapsto K(t, x)$, we get
$$\begin{aligned} \|t^{1-\theta} u(t)\|_{X_1} &\le t^{-\theta} \sum_{n=1}^{\infty} t \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) 2(n+1) K\left(\frac{1}{n+1}, x\right) \\ &\le 4t^{-\theta} K(t, x). \end{aligned}$$
It follows that
$$\|u_{1-\theta}\|_{L_*^p(\mathbb{R}_+, X_1)} \le 4 \left(\int_0^\infty (t^{-\theta} K(t, x))^p \frac{\mathrm{d}t}{t} \right)^{1/p} = 4\|x\|_X, \tag{3.3}$$
so
$$\|v_{1-\theta}\|_{L_*^p(\mathbb{R}_+; X_1)} \le C\|x\|_X.$$

4. We show that $v'_{1-\theta} \in L^p_*(\mathbb{R}_+; X_0)$. We use the following remark, to ensure that we find $v \in W^{1,p}_{loc}(0, \infty, X_0 + X_1)$. If on some interval $I \subset \mathbb{R}$ we have $f, g \in L^1(I; \mathbb{R})$ and
$$f(x) - f(y) = \int_x^y g(s)\,\mathrm{d}s$$
for almost all $x, y \in I$, then $f \in W^{1,1}(I; \mathbb{R})$ and g is its weak derivative, as for $c, d \in I$ and all $\varphi \in C_c^\infty(I)$ with $\operatorname{supp}(\varphi) \in [c, d]$,
$$\int_c^d \varphi'(y)(f(y)-f(c))\,\mathrm{d}y = \int_c^d \int_c^y \varphi'(y) g(x)\,\mathrm{d}x\,\mathrm{d}y = \int_c^d \int_x^d \varphi'(y)\,\mathrm{d}y g(x)\,\mathrm{d}x = -\int_c^d \varphi(x) g(x)\,\mathrm{d}x.$$
We use this fact on $\|v(t)\|_{X_0+X_1}$ and that, by definition,
$$v(t) = x - \frac{1}{t}\int_0^t \sum_{n=1}^\infty a_{\frac{1}{n+1}} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(s)\,\mathrm{d}s$$
and
$$v'(t) = \frac{1}{t^2}\int_0^t g(s)\,\mathrm{d}s - \frac{1}{t} g(t)$$
holds true almost everywhere for $g(t) := \sum_{n=1}^\infty a_{\frac{1}{n+1}} \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) \in X_0$. Again, from the monotonicity of $t \mapsto K(t, x)$, it follows that
$$\|g(t)\|_{X_0} \leq \sum_{n=1}^\infty 2K\left(\frac{1}{n+1}, x\right) \chi_{(\frac{1}{n+1}, \frac{1}{n}]}(t) \leq 2K(t, x)$$
and therefore
$$\begin{aligned} \|t^{1-\theta} v'(t)\|_{X_0} &\leq \frac{t^{1-\theta}}{t^2}\int_0^t \|g(s)\|_{X_0}\,\mathrm{d}s + t^{-\theta}\|g(t)\|_{X_0} \\ &\leq 4t^{-\theta} K(t, x). \end{aligned}$$
As in (3.3), it follows that $\|v_{1-\theta}\|_{L^p_*(\mathbb{R}_+; X_0)} \leq C\|x\|_X$. In conclusion, we see that $v \in V(p, 1-\theta, X_1, X_0)$, that $x = \lim_{t\to 0} v(t)$ in $X_0 + X_1$ and that $\|x\|_X^{Tr} \leq \|v\|_{V(p,1-\theta,X_1,X_0)} \leq C\|x\|_X$.

We now look at the opposite inclusion and assume that $x \in X_0 + X_1$ is the trace of a function $v \in V(p, 1-\theta, X_1, X_0)$ at $t = 0$. We can write
$$x = x - v(t) + v(t) = -\int_0^t v'(s)\,\mathrm{d}s + v(t),\ t > 0,$$
cf. Lemma 3.5. Therefore, we see that
$$t^{-\theta} K(t, x) \leq t^{1-\theta}\|\frac{1}{t}\int_0^t v'(s)\,\mathrm{d}s\|_{X_0} + t^{1-\theta}\|v(t)\|_{X_1},\ t > 0.$$

By Lemma 3.7,

$$\|x\|_X = \|t^{-\theta}K(t,x)\|_{L^p_*(\mathbb{R}_+)} \le C\|v'_{1-\theta}\|_{L^p_*(\mathbb{R}_+;X_0)} + \|v_{1-\theta}\|_{L^p_*(\mathbb{R}_+,X_1)} \le C\|v\|_{V(p,1-\theta,X_1,X_0)}.$$

□

Remark 3.8. Let $x \in (X_0, X_1)_{\theta,p}$ be the trace of a function $v \in V(p, 1-\theta, X_1, X_0)$.

1. We can improve the regularity of v in the following way: For any smooth non-negative function $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}$ with compact support and $\int_0^\infty \varphi(s)\,\frac{\mathrm{d}s}{s} = 1$, we set
$$u(t) = \int_0^\infty \varphi(\frac{t}{\tau})v(\tau)\,\frac{\mathrm{d}\tau}{\tau} = \int_0^\infty \varphi(s)v(\frac{t}{s})\,\frac{\mathrm{d}s}{s}.$$
Then we get $u \in C^\infty(\mathbb{R}_+; X_0 \cap X_1)$, $u(0) = x$ and
$$\begin{aligned} t \mapsto t^{n-\theta}u^{(n)}(t) &\in L^p_*(\mathbb{R}_+; X_0),\ n \in \mathbb{N}, \\ t \mapsto t^{n+1-\theta}u^{(n)}(t) &\in L^p_*(\mathbb{R}_+; X_1),\ n \in \mathbb{N} \cup \{0\}, \end{aligned}$$
with norms estimated by $c(n)\|v\|_{V(p,1-\theta,X_1,X_0)}$. The proof is left as an exercise.
2. Let $\psi \in C_c^\infty([0,\infty))$ such that $\psi \equiv 1$ in $(0,1]$ or any right neighbourhood of 0. Then $u_\psi : t \mapsto \psi(t)u(t) \in V(p, 1-\theta, X_1, X_0)$ with trace x at $t = 0$, where u is chosen as in 1. Moreover, $\|u_\psi\|_{V(p,1-\theta,X_1,X_0)} \le C_\psi\|u\|_{V(p,1-\theta,X_1,X_0)}$ and it has compact support. This shows that we could also consider a subset of $V(p, 1-\theta, X_1, X_0)$ consisting of functions with compact support in order to define an equivalent trace space.

Corollary 3.9. *Let $1 < p < \infty$. Then $(X_0, X_1)_{1-1/p,p}$ is the set of the traces at $t = 0$ of the functions $u \in W^{1,p}(0,\infty; X_1) \cap L^p(0,\infty; X_0)$.*

Proof. Clearly, if $\theta = 1 - 1/p$, $u_{1-\theta} \in L^p_*(\mathbb{R}_+; X_i)$ iff $u \in L^p(\mathbb{R}_+; X_i)$. The corollary follows if we take into account Remark 3.8. □

The following example gives us an important motivation to consider the trace method.

Example 3.10. Let $\mathbb{R}^{n+1}_+$ denote the upper half-space $\{(t,x) \in \mathbb{R} \times \mathbb{R}^n : t > 0\}$. Then for $1 < p < \infty$, $(L^p(\mathbb{R}^n), W^{1,p}(\mathbb{R}^n))_{1-1/p,p}$ is the space of traces of functions $(t,x) \mapsto v(t,x) \in W^{1,p}(\mathbb{R}^{n+1}_+)$ at $t = 0$.

We close this chapter with a theoretical result on the real interpolation space $X = (X_0, X_1)_{\theta,p}$ which can be derived by the trace method.

Proposition 3.11. *Let $0 < \theta < 1$ and $\{X_0, X_1\}$ an interpolation couple. For $1 \le p < \infty$, $X_0 \cap X_1$ is dense in $X = (X_0, X_1)_{\theta,p}$.*

Proof. Let $x \in X$. By Remark 3.8, $x = v(0)$, where $v \in C^\infty(\mathbb{R}_+; X_0 \cap X_1) \cap V(p, 1 - \theta, X_1, X_0)$ and $t \mapsto t^{2-\theta} v' \in L^p_*(\mathbb{R}_+, X_1)$. We set

$$x_\varepsilon := v(\varepsilon) \in X_0 \cap X_1, \quad \forall \varepsilon > 0$$

and show that $x_\varepsilon \to x$ in X. We define

$$z_\varepsilon(t) := (v(\varepsilon) - v(t))\chi_{[0,\varepsilon]}(t)$$

to get $x_\varepsilon - x = z_\varepsilon(0)$, $z_\varepsilon \in W^{1,p}(a, b; X_0)$ for all $0 < a < b < \infty$ and $z'_\varepsilon(t) = -v'(t)\chi_{(0,\varepsilon)}(t)$. It follows that

$$\lim_{\varepsilon \to 0} \|t^{1-\theta} z'_\varepsilon(t)\|_{L^p_*(\mathbb{R}_+; X_0)} = 0.$$

We now show that $t \mapsto t^{1-\theta} z_\varepsilon(t) \in L^p_*(\mathbb{R}_+; X_1)$ by using

$$z_\varepsilon(t) = \int_t^\infty \chi_{(0,\varepsilon)}(s) v'(s) \,\mathrm{d}s$$

and a modified version of the Hardy-Young inequality: For $\alpha > 0, p \geq 1$ and positive φ, we have that

$$\int_0^\infty t^{\alpha p} (\int_t^\infty \varphi(s) \frac{\mathrm{d}s}{s})^p \frac{\mathrm{d}t}{t} \leq \frac{1}{\alpha^p} \int_0^\infty s^{\alpha p} \varphi(s)^p \frac{\mathrm{d}s}{s}, \tag{3.4}$$

which follows from Lemma 3.2 by substituting $\tau = \frac{1}{t}$ and $\sigma = \frac{1}{s}$. We get that

$$\begin{aligned} \int_0^\infty (t^{1-\theta} \|z_\varepsilon(t)\|_{X_1})^p \frac{\mathrm{d}t}{t} &\leq \int_0^\infty t^{(1-\theta)p} (\int_t^\infty \chi_{(0,\varepsilon)}(s) s \|v'(s)\|_{X_1} \frac{\mathrm{d}s}{s})^p \frac{\mathrm{d}t}{t} \\ &\leq (\frac{1}{1-\theta})^p \int_0^\infty \chi_{(0,\varepsilon)}(s) s^{(2-\theta)p} \|v'(s)\|^p_{X_1} \frac{\mathrm{d}s}{s}, \end{aligned}$$

so that

$$\lim_{\varepsilon \to 0} \|t^{1-\theta} z_\varepsilon(t)\|_{L^p_*(\mathbb{R}_+; X_1)} = 0.$$

In conclusion, $z_\varepsilon \to 0$ in $V(p, 1-\theta, X_1, X_0)$ as $\varepsilon \to 0$, so $\|x_\varepsilon - x\|^{Tr}_X \to 0$ as $\varepsilon \to 0$. By Theorem 3.6, $\lim_{\varepsilon \to 0} \|x_\varepsilon - x\|_X = 0$. $\square$

4 The Reiteration Theorem

In the following, let $\{X_0, X_1\}$ an interpolation couple. We define two classes of intermediate spaces.

Definition 4.1. Let $0 \le \theta \le 1$ and let X be a Banach space such that $X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1$.

1. We say that X belongs to the class J_θ between X_0 and X_1 if there exists a constant $c > 0$ such that
$$\|x\|_X \le c\|x\|_{X_0}^{1-\theta}\|x\|_{X_1}^{\theta}, \quad \forall x \in X_0 \cap X_1.$$
We write $X \in J_\theta(X_0, X_1)$.

2. We say that X belongs to the class K_θ between X_0 and X_1 if there is a constant $k > 0$ such that
$$K(t, x, X_0, X_1) \le kt^\theta\|x\|_X, \quad \forall x \in X, t > 0.$$
In this case we write $X \in K_\theta(X_0, X_1)$. If $\theta \in (0,1)$, this means that $X \hookrightarrow (X_0, X_1)_{\theta,\infty}$.

Proposition 4.2. *Let $\theta \in (0,1)$ and let X be an intermediate space for $\{X_0, X_1\}$. Then the following statements are equivalent:*

1. $X \in J_\theta(X_0, X_1)$,
2. $(X_0, X_1)_{\theta,1} \hookrightarrow X$.

Proof. The implication 2. $\Rightarrow$ 1. follows directly from Theorem 2.5 (5) with $q = 1$. We show that 1. $\Rightarrow$ 2.. For every $x \in (X_0, X_1)_{\theta,1}$, let $u \in V(1, 1-\theta, X_1, X_0) \cap C^\infty(\mathbb{R}_+; X_1 \cap X_0)$ such that $u(t) \to 0$ as $t \to \infty$ and $u(0) = x$. We set
$$v(t) = \frac{1}{t}\int_0^t u(s)\,\mathrm{d}s,$$
so that $t \mapsto t^{2-\theta}v'(t) \in L^1_*(\mathbb{R}_+; X_1)$ and $t \mapsto t^{1-\theta}v'(t) \in L^1_*(\mathbb{R}_+; X_0)$ follows, similarly as in Remark 3.8, by Lemma 3.7. It still holds that $v(0) = x$ and that $v(t) \to 0$ as $t \to \infty$, so
$$x = -\int_0^\infty v'(t)\,\mathrm{d}t.$$

Let c be such that $\|y\|_X \le c\|y\|_{X_1}^{\theta}\|y\|_{X_0}^{1-\theta}$ for every $y \in X_0 \cap X_1$. Then

$$\|v'(t)\|_X \le c\|v'(t)\|_{X_1}^{\theta}\|v'(t)\|_{X_0}^{1-\theta} = ct^{-1}\|t^{2-\theta}v'(t)\|_{X_1}^{\theta}\|t^{1-\theta}v'(t)\|_{X_0}^{1-\theta}.$$

Now by Hölder's inequality, as $1 = \frac{1}{1/\theta} + \frac{1}{1/(1-\theta)}$,

$$\begin{aligned} \|x\|_X &\le \|v'\|_{L^1(\mathbb{R}_+;X)} \le c\left\|t^{-\theta}\|t^{2-\theta}v'(t)\|_{X_1}^{\theta}t^{-(1-\theta)}\|t^{1-\theta}v'(t)\|_{X_0}^{1-\theta}\right\|_{L^1(\mathbb{R}_+)}. \\ &\le c\|t^{2-\theta}v'(t)\|_{L^1_*(\mathbb{R}_+,X_1)}^{\theta}\|t^{1-\theta}v'(t)\|_{L^1_*(\mathbb{R}_+,X_0)}^{1-\theta} \\ &\le c\|x\|_{(X_0,X_1)_{\theta,1}}. \end{aligned}$$

□

The above proposition shows that for $\theta \in (0,1)$, $X \in J_\theta(X_0,X_1) \cap K_\theta(X_0,X_1)$ iff $(X_0,X_1)_{\theta,1} \hookrightarrow X \hookrightarrow (X_0,X_1)_{\theta,\infty}$.

Example 4.3. Actually, two examples:

1. By Equation (2.2) and by Theorem 2.5, $(X_0,X_1)_{\theta,p} \in K_\theta(X_0,X_1) \cap J_\theta(X_0,X_1)$.
2. The space $C^1([-1,1])$ lies in

   $$K_{1/2}(C([-1,1]),C^2([-1,1])) \cap J_{1/2}(C([-1,1]),C^2([-1,1])),$$

   but, as we have seen, it is not an interpolation space. The proof is left as an exercise.

The following theorem shows that we can "iterate" the procedure of interpolating spaces, i.e. the real interpolation spaces of two suitable intermediate spaces is again a real interpolation space.

Theorem 4.4. (Reiteration Theorem)
Let $0 \le \theta_0 < \theta_1 \le 1$. We fix $\theta \in (0,1)$ and set $\omega = (1-\theta)\theta_0 + \theta\theta_1$. Then the following holds true.

1. *If for an interpolation couple $\{X_0,X_1\}$, there are intermediate spaces $E_i \in K_{\theta_i}(X_0,X_1)$, $i \in \{0,1\}$, then*

   $$(E_0,E_1)_{\theta,p} \hookrightarrow (X_0,X_1)_{\omega,p} \quad \text{for all } 1 \le p \le \infty.$$

2. *If on the other hand, $E_i \in J_{\theta_i}(X_0,X_1)$, then*

   $$(X_0,X_1)_{\omega,p} \hookrightarrow (E_0,E_1)_{\theta,p}.$$

In conclusion, if $E_i \in J_{\theta_i}(X_0,X_1) \cap K_{\theta_i}(X_0,X_1)$, then

$$(E_0,E_1)_{\theta,p} = (X_0,X_1)_{\omega,p} \quad \textit{for all } 1 \le p \le \infty,$$

with equivalence of the respective norms.

Proof. We first show 1.:
Let k_i be such that

$$K(t,e,X_0,X_1) \le k_i t^{\theta_i} \|e\|_{E_i}, \quad e \in E_i, t > 0.$$

Let $e_0 \in E_0$ and $e_1 \in E_1$ be such that $e = e_0 + e_1$. It follows that

$$K(t,e,X_0,X_1) \le K(t,e_0,X_0,X_1) + K(t,e_1,X_0,X_1) \le k_0 t^{\theta_0}\|e_0\|_{E_0} + k_1 t^{\theta_1}\|e_1\|_{E_1}.$$

Taking the infimum, it follows that

$$\begin{aligned} K(t,e,X_0,X_1) &\le \max\{k_0,k_1\} t^{\theta_0}(\|e_0\|_{E_0} + t^{\theta_1-\theta_0}\|e_1\|_{E_1}) \\ &\le \max\{k_0,k_1\} t^{\theta_0} K(t^{\theta_1-\theta_0}, e, E_0, E_1), \end{aligned}$$

so we get

$$t^{-\omega}K(t,e,X_0,X_1) \le \max\{k_0,k_1\} t^{-\theta(\theta_1-\theta_0)} K(t^{\theta_1-\theta_0}, e, E_0, E_1).$$

Setting $s = t^{\theta_1-\theta_0}$, we can conclude

$$\|e\|_{(X_0,X_1)_{\omega,p}} \le \max\{k_0,k_1\}\|s^{-\theta}K(s,e,E_0,E_1)\|_{L^p_*(0,\infty)} = \max\{k_0,k_1\}\|e\|_{(E_0,E_1)_{\theta,p}},$$

if $p < \infty$ and

$$\|e\|_{(X_0,X_1)_{\omega,\infty}} \le \max\{k_0,k_1\}\|e\|_{(E_0,E_1)_{\theta,p}}.$$

We now show 2.:
By Theorem 3.6 and Remark 3.8, every $x \in (X_0,X_1)_{\omega,p}$ is the trace at $t = 0$ of a function $v \in C^\infty(\mathbb{R}_+, X_0 \cap X_1)$ such that $v(\infty) = 0$, $v'_{1-\omega} \in L^p_*(\mathbb{R}_+, X_0)$, $v'_{2-\omega} \in L^p_*(\mathbb{R}_+, X_1)$ and

$$\|v'_{1-\omega}\|_{L^p_*(\mathbb{R}_+,X_0)} + \|v'_{2-\omega}\|_{L^p_*(\mathbb{R}_+,X_1)} \le k\|x\|^{Tr}_{(X_0,X_1)_{\omega,p}}. \tag{4.1}$$

We now consider the function

$$g(t) = v(t^{1/(\theta_1-\theta_0)})$$

and show that it belongs to $V(p, 1-\theta, E_1, E_0)$, so that $g(0) = v(0) = x \in (E_0,E_1)_{\theta,p}$ with the corresponding estimate.
First, we look at $\|v'(t)\|_{E_i}$, $t > 0$, $i \in \{0,1\}$. Let c_i such that

$$\|x\|_{E_i} \le c_i \|x\|_{X_0}^{1-\theta_i}\|x\|_{X_1}^{\theta_i}, \quad x \in X_0 \cap X_1.$$

We get

$$\|v'(t)\|_{E_i} \le \frac{c_i}{t^{1+\theta_i-\omega}}\|t^{1-\omega}v'(t)\|_{X_0}^{1-\theta_i}\|t^{2-\omega}v'(t)\|_{X_1}^{\theta_i}.$$

Now we calculate

$$1+\theta_0-\omega = 1+\theta_0-(1-\theta)\theta_0-\theta\theta_1 = 1-\theta(\theta_1-\theta_0)$$

and

$$1+\theta_1-\omega = 1+\theta_1-(1-\theta)\theta_0-\theta\theta_1 = 1+(1-\theta)(\theta_1-\theta_0)$$

to obtain that

$$\|v'_{1-\theta(\theta_1-\theta_0)}\|_{L^p_*(0,\infty,E_0)} \le c_0 k \|x\|^{Tr}_{(X_0,X_1)_{\omega,p}} \tag{4.2}$$

and that

$$\|v'_{1+(1-\theta)(\theta_1-\theta_0)}\|_{L^p_*(0,\infty,E_1)} \le c_1 k \|x\|^{Tr}_{(X_0,X_1)_{\omega,p}} \tag{4.3}$$

by Hölder's inequality and (4.1). We now use

$$v(t) = -\int_t^\infty v'(s)\,\mathrm{d}s \tag{4.4}$$

and the second Hardy-Young inequality (3.4) to get that

$$\begin{aligned}
\|v_{(1-\theta)(\theta_1-\theta_0)}\|_{L^p_*(0,\infty,E_1)} &\overset{(4.4)}{\le} \left(\int_0^\infty t^{(1-\theta)(\theta_1-\theta_0)p}\left(\int_t^\infty \|sv'(s)\|_{E_1}\frac{\mathrm{d}s}{s}\right)^p \frac{\mathrm{d}t}{t}\right)^{1/p} \\
&\overset{(3.4)}{\le} \frac{1}{(1-\theta)(\theta_1-\theta_0)}\left(\int_0^\infty s^{(1-\theta)(\theta_1-\theta_0)p+p}\|v'(s)\|^p_{E_1}\frac{\mathrm{d}s}{s}\right)^{1/p} \\
&\overset{(4.3)}{\le} \frac{1}{(1-\theta)(\theta_1-\theta_0)} c_1 k \|x\|^{Tr}_{(X_0,X_1)_{\omega,p}}.
\end{aligned}$$

With the substitution $s = t^{1/(\theta_1-\theta_0)}$, i.e. $t = s^{\theta_1-\theta_0}$ and $\mathrm{d}t = (\theta_1-\theta_0)s^{(\theta_1-\theta_0)-1}\,\mathrm{d}s$, it follows that

$$\begin{aligned}
\|g_{1-\theta}\|_{L^p_*(0,\infty,E_1)} &= \left(\int_0^\infty t^{(1-\theta)p}\|v(t^{1/(\theta_1-\theta_0)})\|^p_{E_1}\frac{\mathrm{d}t}{t}\right)^{1/p} \\
&= \left(\int_0^\infty s^{(\theta_1-\theta_0)(1-\theta)p}\|v(s)\|^p_{E_1}(\theta_1-\theta_0)\frac{\mathrm{d}s}{s}\right)^{1/p} \\
&= (\theta_1-\theta_0)^{1/p}\|v_{(1-\theta)(\theta_1-\theta_0)}\|_{L^p_*(0,\infty,E_1)} \\
&\le (1-\theta)^{-1}(\theta_1-\theta_0)^{1/p-1}c_1 k\|x\|^{Tr}_{(X_0,X_1)_{\omega,p}}.
\end{aligned}$$

Similarly, we look at

$$g'(t) = (\theta_1-\theta_0)^{-1}t^{(1/(\theta_1-\theta_0)-1)}v'(t^{1/(\theta_1-\theta_0)})$$

to get that

$$\begin{aligned}
\|g'_{1-\theta}\|_{L^p_*(0,\infty,E_0)} &= \frac{1}{\theta_1-\theta_0}\left(\int_0^\infty (t^{1-\theta+1/(\theta_1-\theta_0)-1}\|v'(t^{1/(\theta_1-\theta_0)})\|_{E_0})^p\frac{\mathrm{d}t}{t}\right)^{1/p} \\
&\overset{s=t^{1/(\theta_1-\theta_0)}}{=} (\theta_1-\theta_0)^{1/p-1}\|v'_{1-\theta(\theta_1-\theta_0)}\|_{L^p_*(0,\infty,E_0)} \\
&\overset{(4.2)}{\le} c_0 k(\theta_1-\theta_0)^{1/p-1}\|x\|^{Tr}_{(X_0,X_1)_{\omega,p}}.
\end{aligned}$$

In conclusion, we get that $g \in V(p, 1-\theta, E_1, E_0)$, so that $x = g(0) \in (E_0, E_1)_{\theta,p}$ by Theorem 3.6 and

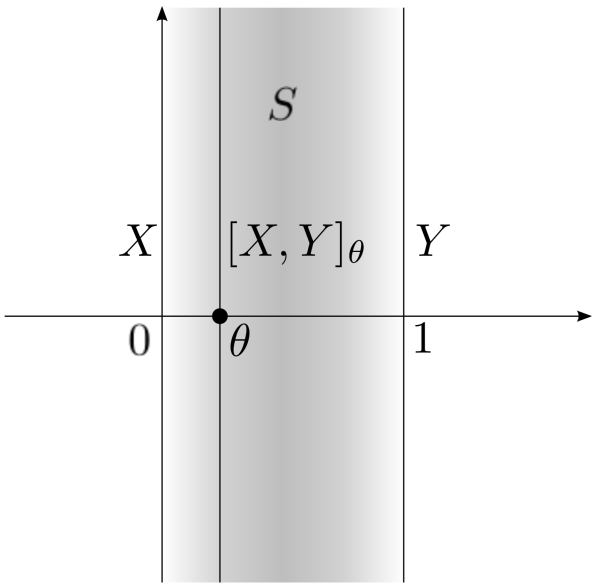
$$\begin{aligned}
\|x\|_{(E_0,E_1)_{\theta,p}} &\leq \|g\|_{V(p,1-\theta,E_1,E_0)} \leq \max\{c_i\}k(1-\theta)^{-1}(\theta_1-\theta_0)^{1/p-1}\|x\|^{Tr}_{(X_0,X_1)_{\omega,p}} \\
&= \max\{c_i\}k\frac{(\theta_1-\theta_0)^{1/p}}{(\theta_1-\omega)}\|x\|^{Tr}_{(X_0,X_1)_{\omega,p}}.
\end{aligned}$$

□

Corollary 4.5. *From Theorem 4.4 and Example 4.3 we immediately get that for any* $0 < \theta_0 < \theta_1 < 1$, $0 < \theta < 1$ *and* $1 \leq p, q_0, q_1 \leq \infty$,

$$((X_0, X_1)_{\theta_0,q_0}, (X_0, X_1)_{\theta_1,q_1})_{\theta,p} = (X_0, X_1)_{(1-\theta)\theta_0+\theta\theta_1,p}.$$

5 Complex Interpolation



Idea:

- $\{X, Y\}$ interpolation couple, construct interpolation space $[X, Y]_\theta$ for $\theta \in (0, 1)$
- only one parameter θ will do, we got two norms already...
- $f : S \to X + Y$ "nice" such that $f(it) \in X$ and $f(1 + it) \in Y$
- $[X, Y]_\theta = \{f(\theta)\} = \{f(\theta + it)\}$
- $f : S \to X + Y$ holomorphic $\rightsquigarrow$ maximum principle $\rightsquigarrow$ estimate interior by boundary

Roadmap:

1. properties of holomorphic functions $f : S \to X + Y$
2. definition of $[X, Y]_\theta$
3. $\mathcal{F}_\theta : \{X, Y\} \mapsto [X, Y]_\theta$ is exact interpolation functor of type θ
4. $[X, Y]_\theta \in J_\theta \cap K_\theta \rightsquigarrow$ Reiteration Theorem

5.1 X-valued holomorphic functions

Let X be a complex Banach space. For every set $\Omega \subset \mathbb{C}$ we say that $f : \Omega \to X$ is holomorphic in $S \subset \Omega$, if f is differentiable in every λ_0 in a neighbourhood of S, i.e. the limit

$$f'(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0}$$

exists in X.

Proposition 5.1. *Let $\Omega \subset \mathbb{C}$ be a bounded open set and $f : \overline{\Omega} \to X$ a function which is holomorphic on Ω and continuous on $\overline{\Omega}$. Then*

$$\|f(\xi)\|_X \leq \max\{\|f(z)\|_X : z \in \partial\Omega\} \quad \forall \xi \in \overline{\Omega}.$$

Proof. For every $\xi \in \overline{\Omega}$ we can choose $x'(\xi) \in X'$ in the dual space of X such that $\|f(\xi)\|_X = \langle f(\xi), x'(\xi)\rangle$ and $\|x'(\xi)\|_{X'} = 1$. We apply the maximum principle for $\mathbb{C}$-valued holomorphic functions to $z \mapsto \langle f(z), x'(\xi)\rangle$ to get

$$\begin{aligned} \|f(\xi)\|_X &= |\langle f(\xi), x'(\xi)\rangle| \leq \max\{|\langle f(z), x'(\xi)\rangle|, z \in \partial\Omega\} \\ &\leq \max\{\|f(z)\|_X, z \in \partial\Omega\}. \end{aligned}$$

□

Following the idea above, we want to consider holomorphic functions on the strip

$$S := \{x + iy \in \mathbb{C} : 0 \leq x \leq 1\}.$$

Also for this set, the maximum principle holds.

Proposition 5.2. *Let $f : S \to X$ be holomorphic on $\overset{\circ}{S}$ and continuous and bounded on S. Then*

$$\|f(\xi)\|_X \leq \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_X\}.$$

Proof. Let $\varepsilon \in (0,1)$ and $\xi_0 = x_0 + it_0 \in S$ such that

$$\|f(\xi_0)\| \geq (1-\varepsilon)\|f\|_{C(S,X)}.$$

We set $f_\delta(\xi) := e^{\delta(\xi-\xi_0)^2} f(\xi)$, $\xi = x + it$, so that $f_\delta(\xi_0) = f(\xi_0)$ and

$$\begin{aligned} \lim_{|t|\to\infty} e^{\delta(x+it-\xi_0)^2} f(\xi) &= \lim_{|t|\to\infty} e^{-\delta t^2 + 2\delta(it)(x-\xi_0) + \delta(x-\xi_0)^2} f(\xi) \\ &= \lim_{|t|\to\infty} e^{-\delta t(t-2t_0)} e^{\delta \mathrm{Re}(x-\xi_0)^2} f(\xi) \\ &= 0. \end{aligned}$$

We can now apply the maximum principle, Proposition 5.1, to f_δ on every domain $[0,1] \times [-M, M]$, $M > 0$. By the above calculation, for large M, only the vertical boundaries will be relevant.

$$\begin{aligned}\|f_\delta(\xi)\| &\leq \max\{\sup_{t\in\mathbb{R}} \|f_\delta(it)\|, \sup_{t\in\mathbb{R}} \|f_\delta(1+it)\|\} \\ &\leq \max\{e^{\delta \mathrm{Re}(1-\xi_0)^2}, e^{\delta \mathrm{Re}\, \xi_0^2}\} \sup_{t\in\mathbb{R}}\{e^{-\delta t(t-2t_0)}\} \max\{\sup_{t\in\mathbb{R}} \|f(it)\|, \sup_{t\in\mathbb{R}} \|f(1+it)\|\}.\end{aligned}$$

It is easy to show that $\sup_{t\in\mathbb{R}} e^{-\delta t(t-2t_0)} = e^{\delta t_0^2}$ is reached in $t = t_0$, so that in conclusion, for every $\varepsilon \in (0,1)$ there exists a sufficiently small δ such that

$$\begin{aligned}(1-\varepsilon)\|f\|_{C(S,X)} &\leq \|f(\xi_0)\| = \|f_\delta(\xi_0)\| \\ &\leq (1+\varepsilon) \max\{\sup_{t\in\mathbb{R}} \|f(it)\|, \sup_{t\in\mathbb{R}} \|f(1+it)\|\}.\end{aligned}$$

□

Theorem 5.3. (Three lines theorem)
Let $f : S \to X$ be holomorphic on $\overset{\circ}{S}$ and continuous and bounded on S. Then for all $0 < \theta < 1$, we have

$$\|f(\theta)\|_X \leq (\sup_{t\in\mathbb{R}} \|f(it)\|_X)^{1-\theta} (\sup_{t\in\mathbb{R}} \|f(1+it)\|_X)^\theta.$$

Proof. In the following, we use the abbreviations $M_0 := \sup_{t\in\mathbb{R}} \|f(it)\|_X$ and $M_1 := \sup_{t\in\mathbb{R}} \|f(1+it)\|_X$ and we consider the function $\varphi(z) = e^{\lambda z} f(z)$ where $\lambda = \log(M_0/M_1)$. By the maximum principle on S, Proposition 5.2,

$$\begin{aligned}\|f(\theta)\|_X &= |e^{-\lambda\theta}| \|\varphi(\theta)\|^\theta \|\varphi(\theta)\|^{1-\theta} \\ &\leq (\frac{M_1}{M_0})^\theta \max\{e^{\lambda it} M_0, e^{\lambda + \lambda it} M_1\}^\theta \max\{e^{\lambda it} M_0, e^{\lambda+\lambda it} M_1\}^{1-\theta} \\ &= (\frac{M_1}{M_0})^\theta M_0^\theta M_0^{1-\theta} \\ &= M_0^{1-\theta} M_1^\theta.\end{aligned}$$

□

5.2 The spaces $[X,Y]_\theta$ and basic properties

Definition 5.4. The space $G(X,Y)$ is defined as the space of all functions $f : S \to X+Y$ such that

1. f is holomorphic in $\overset{\circ}{S}$ and continuous and bounded on S with values in $X+Y$.

2. It holds that $t \mapsto f(it) \in C_b(\mathbb{R}; X)$, $t \mapsto f(1+it) \in C_b(\mathbb{R}; Y)$ and

$$\|f\|_{G(X,Y)} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y\} < \infty.$$

$G_0(X,Y)$ is a subspace of $G(X,Y)$ which imposes the additional properties

$$\lim_{|t|\to\infty} \|f(it)\|_X = 0, \qquad \lim_{|t|\to\infty} \|f(1+it)\|_Y = 0.$$

In the exercises, we show that both $G(X,Y)$ and $G_0(X,Y)$ are Banach spaces, continuously embedded in $C_b(S, X+Y)$.

We only cite the following technical lemma. (The reason is: the proof is difficult)

Lemma 5.5. *The linear hull of the set of functions $z \mapsto e^{\delta z^2 + \lambda z} a$, $\delta > 0$, $\lambda \in \mathbb{R}$, $a \in X \cap Y$, is dense in $G_0(X,Y)$.*

Definition 5.6. (Complex Interpolation Spaces)
For every $\theta \in [0,1]$, we set

$$\begin{aligned} [X,Y]_\theta &:= \{f(\theta) : f \in G(X,Y)\}, \\ \|x\|_{[X,Y]_\theta} &:= \inf_{f\in G(X,Y), f(\theta)=x} \|f\|_{G(X,Y)}. \end{aligned}$$

We see that $[X,Y]_\theta$ is a Banach space from the fact that it is isomorphic to the quotient space $G(X,Y)/\mathcal{N}_\theta$, where $\mathcal{N}_\theta$ is the closed subspace of functions $f \in G(X,Y)$, satisfying $f(\theta) = 0$.

Proposition 5.7. (Properties of $[X,Y]_\theta$)

1. *If $\theta \in (0,1)$, it holds that $[X,Y]_\theta = [Y,X]_{1-\theta}$.*
2. *The space $[X,Y]_\theta$ can be defined equivalently from the space $G_0(X,Y)$.*
3. *If $X = Y$, then $[X,X]_\theta = X$.*
4. *For every $t \in \mathbb{R}$ and $f \in G(X,Y)$, $f(\theta+it) \in [X,Y]_\theta$ for every $\theta \in (0,1)$ and*

$$\|f(\theta+it)\|_{[X,Y]_\theta} = \|f(\theta)\|_{[X,Y]_\theta}.$$

5. *For every $\theta \in (0,1)$, we get that $[X,Y]_\theta$ is an intermediate space of X, Y, i.e.*

$$X \cap Y \hookrightarrow [X,Y]_\theta \hookrightarrow X+Y.$$

6. *For every $\theta \in (0,1)$, $X \cap Y$ is dense in $[X,Y]_\theta$.*

Proof.

1. Follows directly from the definition (reflect fs)

2. For every $f \in G(X,Y)$ and $\delta > 0$, we can define $f_\delta(z) = e^{\delta(z-\theta)^2} f(z)$. We already know this function, that $f_\delta(\theta) = f(\theta)$ and that it lies in $G_0(X,Y)$. By definition,
$$\|f_\delta\|_{G(X,Y)} \leq \max\{e^{\delta\theta^2}, e^{\delta(1-\theta)^2}\}\|f\|_{G(X,Y)},$$
from $\delta \to 0$, we see
$$\inf_{f\in G(X,Y), f(\theta)=x} \|f\|_{G(X,Y)} = \inf_{f\in G_0(X,Y), f(\theta)=x} \|f\|_{G(X,Y)}.$$

3. This follows from the maximum principle. For $x \in [X,X]_\theta$, we have
$$\|x\|_X = \|f(\theta)\|_X \overset{m.p.}{\leq} \|f\|_{G(X,X)}.$$
Taking the inf gives $\|x\|_X \leq \|x\|_{[X,X]_\theta}$. On the other hand, the constant function $c_x : z \mapsto x$, $z \in S$ is in $G(X,X)$, so that for every $x \in X$, $\|x\|_X = \|c_x\|_{G(X,X)} \geq \|x\|_{[X,X]_\theta}$.

4. Let $g(z) = f(z+it)$. We see directly that $g \in G(X,Y)$ and that $\|g\|_{G(X,Y)} = \|f\|_{G(X,Y)}$. It follows that $f(\theta+it) = g(\theta) \in [X,Y]_\theta$ and that the norm doesn't change.

5. Let $x \in X \cap Y$. Again, the constant function $c_x(z) = x$ belongs to $G(X,Y)$ and
$$\|c_x\|_{G(X,Y)} \leq \max\{\|x\|_X, \|x\|_Y\},$$
so that $x = c_x(\theta) \in [X,Y]_\theta$ and $\|x\|_{[X,Y]_\theta} \leq \|x\|_{X\cap Y}$.
On the other hand, if $x = f(\theta) \in [X,Y]_\theta$, then
$$\begin{aligned} \|x\|_{X+Y} &\leq \|f(\theta)\|_{X+Y} \\ &\overset{m.p.}{\leq} \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X+Y}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X+Y}\} \\ &\leq C\max\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_Y\} \\ &= C\|f\|_{G(X,Y)}. \end{aligned}$$
Taking the infimum, we get $\|x\|_{X+Y} \leq C\|x\|_{[X,Y]_\theta}$.

6. This follows directly from Lemma 5.5. For every $x \in [X,Y]_\theta$, there is a function $f \in G_0(X,Y)$ such that $f(\theta) = x$, by 2. By Lemma 5.5, there is a sequence of functions given by
$$f_n(z) = \sum_{i=1}^{m_n} \mu_i e^{\delta_i z^2 + \lambda_i z} a_i \in X \cap Y$$
such that $f_n \to f$ in $G_0(X,Y)$. Setting $x_n = f_n(\theta)$, we have $\|x - x_n\|_{[X,Y]_\theta} = \|f(\theta) - f_n(\theta)\|_{[X,Y]_\theta} \leq \|f - f_n\|_{G_0(X,Y)}$.

□

Proposition 5.8. *We let $V(X,Y)$ the linear hull of the functions $z \mapsto \varphi(z)x, S \to X \cap Y$, where $\varphi \in G_0(\mathbb{C},\mathbb{C})$ and $x \in X \cap Y$. For every $x \in X \cap Y$, we get*

$$\|x\|_{[X,Y]_\theta} = \inf_{f \in V(X,Y), f(\theta)=x} \|f\|_{G(X,Y)}.$$

Proof. We can approximate the norm of x by choosing for every $\varepsilon > 0$ a function $f_0 \in G_0(X,Y)$ such that $x = f_0(\theta)$ and $\|f_0\|_{G(X,Y)} \leq \|x\|_{[X,Y]_\theta} + \varepsilon$. We define a function $r \in G(\mathbb{C},\mathbb{C})$ by

$$r(z) = \frac{z-\theta}{z+\theta}, \quad z \in S$$

and set

$$f_1(z) = \frac{f_0(z) - e^{(z-\theta)^2}x}{r(z)}, \quad z \in S.$$

It follows that $f_1 \in G_0(X,Y)$, as $f_0 \in G_0(X,Y)$, $z \mapsto e^{(z-\theta)^2}x \in G_0(X,Y)$ and $|r(z)| \leq 1$, $r(z) \neq 0$ if $z \neq 0$ and $r'(\theta) \neq 0$. By Lemma 5.5, it follows that there exists an approximating function

$$f_2(z) = \sum_{i=1}^{n} \mu_i e^{\delta_i z^2 + \lambda_i z} x_i$$

with $\delta_i > 0$, $\lambda_i \in \mathbb{R}$ and $x_i \in X \cap Y$ such that $\|f_1 - f_2\|_{G(X,Y)} \leq \varepsilon$. We now set

$$f(z) = e^{(z-\theta)^2}x + r(z)f_2(z), \quad z \in S.$$

It follows that $f \in V(X,Y)$ and that

$$\begin{aligned}
\|f\|_{G(X,Y)} &\leq \|f_0\|_{G(X,Y)} + \|f - f_0\|_{G(X,Y)} \\
&\leq \|x\|_{[X,Y]_\theta} + \varepsilon + \|e^{(\cdot-\theta)^2}x + rf_1 - f_0\|_{G(X,Y)} + \|r(f_2 - f_1)\|_{G(X,Y)} \\
&\leq \|x\|_{[X,Y]_\theta} + 2\varepsilon.
\end{aligned}$$

□

5.3 The complex interpolation functor

Theorem 5.9. *For every $\theta \in (0,1)$,*

$$\mathcal{F}_\theta : \mathfrak{C}_2' \to \mathfrak{C}_1', \{X,Y\} \mapsto [X,Y]_\theta$$

is an exact interpolation functor of type θ, where $\mathfrak{C}_i'$ denotes the categories containing complex Banach spaces and complex interpolation couples.

Proof. Let $\{X, Y\}, \{\overline{X}, \overline{Y}\}$ be complex interpolation couples and $T \in \mathcal{L}(\{X, \overline{X}\}, \{Y, \overline{Y}\})$. For every $x \in [X, Y]_\theta$, let $f \in G(X, Y)$ such that $f(\theta) = x$. We set

$$g(z) = \left(\frac{\|T\|_{\mathcal{L}(X,\overline{X})}}{\|T\|_{\mathcal{L}(Y,\overline{Y})}} \right)^{z-\theta} Tf(z), \quad z \in S.$$

It follows that $g \in G(\overline{X}, \overline{Y})$ and that

$$\begin{aligned} \|g(it)\|_{\overline{X}} &\leq \|T\|^{1-\theta}_{\mathcal{L}(X,\overline{X})} \|T\|^{\theta}_{\mathcal{L}(Y,\overline{Y})} \|f(it)\|_X, \\ \|g(1+it)\|_{\overline{Y}} &\leq \|T\|^{1-\theta}_{\mathcal{L}(X,\overline{X})} \|T\|^{\theta}_{\mathcal{L}(Y,\overline{Y})} \|f(1+it)\|_Y. \end{aligned}$$

Therefore, $\|g\|_{G(\overline{X},\overline{Y})} \leq \|T\|^{1-\theta}_{\mathcal{L}(X,\overline{X})} \|T\|^{\theta}_{\mathcal{L}(Y,\overline{Y})} \|f\|_{G(X,Y)}$ and so $Tx = g(\theta) \in [\overline{X}, \overline{Y}]_\theta$. We have the estimate

$$\|Tx\|_{[\overline{X},\overline{Y}]_\theta} \leq \|g\|_{G(\overline{X},\overline{Y})} \leq \|T\|^{1-\theta}_{\mathcal{L}(X,\overline{X})} \|T\|^{\theta}_{\mathcal{L}(Y,\overline{Y})} \|f\|_{G(X,Y)},$$

so that by taking the infimum over f, we get $\|T\|_{\mathcal{L}([X,Y]_\theta,[\overline{X},\overline{Y}]_\theta)} \leq \|T\|^{1-\theta}_{\mathcal{L}(X,\overline{X})} \|T\|^{\theta}_{\mathcal{L}(Y,\overline{Y})}$. □

Following the ideas for Theorem 5.9, we can prove the following result.

Theorem 5.10. *Let $\{X, Y\}$, $\{\overline{X}, \overline{Y}\}$ again be interpolation couples. For every $z \in S$ let $T_z \in \mathcal{L}(X \cap Y, \overline{X} + \overline{Y})$ be such that $z \mapsto T_z x$ is in $G(\overline{X}, \overline{Y})$ for every $x \in X \cap Y$. Moreover, assume that $T_{it} \in \mathcal{L}(X, \overline{X})$ and that $T_{1+it} \in \mathcal{L}(Y, \overline{Y})$. Assume further that the following suprema are finite:*

$$M_0 := \sup_{t\in\mathbb{R}} \|T_{it}\|_{\mathcal{L}(X,\overline{X})}, \qquad M_1 := \sup_{t\in\mathbb{R}} \|T_{1+it}\|_{\mathcal{L}(Y,\overline{Y})}.$$

In this case, for every $\theta \in (0, 1)$, we get

$$\|T_\theta x\|_{[\overline{X},\overline{Y}]_\theta} \leq M_0^{1-\theta} M_1^{\theta} \|x\|_{[X,Y]_\theta},$$

so that T_θ extends to an operator in $\mathcal{L}([X, Y]_\theta, [\overline{X}, \overline{Y}]_\theta)$.

Proof. The proof is an exercise, using the proof of Theorem 5.9 and Proposition 5.8. □

5.4 The space $[X, Y]_\theta$ is of class J_θ and of class K_θ.

Corollary 5.11. *For every $\theta \in (0, 1)$ we have*

$$\|y\|_{[X,Y]_\theta} \leq \|y\|_X^{1-\theta} \|y\|_Y^{\theta}, \quad y \in X \cap Y,$$

i.e. $[X, Y]_\theta \in J_\theta(X, Y)$.

Proof. The idea is the same as for real interpolation spaces. We consider the operators $T_y\lambda = \lambda y$, $T \in \mathcal{L}(\{\mathbb{C},\mathbb{C}\},\{X,Y\})$ and use Theorem 5.9, i.e. the exactness of the interpolation functor, to get

$$\|y\|_{[X,Y]_\theta} = \|T_y\|_{\mathcal{L}(\mathbb{C},[X,Y]_\theta)} \leq \|y\|_X^{1-\theta}\|y\|_Y^\theta.$$

□

To prove that $[X,Y]_\theta \in K_\theta(X,Y)$ needs more work. The main idea is to use a Poisson integral formula for Banach-space valued holomorphic functions on the strip S.

Theorem 5.12. *For $\theta \in (0,1)$, the spaces $[X,Y]_\theta$ are in the class $K_\theta(X,Y)$.*

Proof. We do not give a detailed proof, but name the basic steps and ingredients.

1. Preliminary observation: for $a \in [X,Y]_\theta$, in order to estimate

$$K(t,a,X,Y) = \inf_{a=x+y}\{\|x\|_X + t\|y\|_Y\},$$

we split a and therefore $f \in G(X,Y)$ with $a = f(\theta)$ into $f = f_X + f_Y$, where $f_X : S \to X$, $f_Y : S \to Y$, recovering estimates for f_X and f_Y in terms of the boundary values $f(it)$, $f(1+it)$.

2. The Poisson formula: The Dirichlet problem on the open unit ball D, given by

$$\begin{aligned} \Delta u &= 0, \quad \text{in } D, \\ u(\lambda) &= h(\lambda), \quad \text{on } \partial D, \end{aligned} \tag{5.1}$$

can be solved by the Poisson formula

$$u(\xi) = \frac{1}{2\pi}\int_{|\lambda|=1} h(\lambda)\frac{1-|\xi|^2}{|\xi-\lambda|^2}\,\mathrm{d}\lambda. \tag{5.2}$$

If h is continuous, u is unique. Therefore if f is holomorphic on D and bounded on ∂D, it satisfies (5.2) with $u = f$ and $h = f|_{\partial D}$.

3. The solution formula (5.2) transfers from D to S: By the Riemannian Mapping Theorem, this will work for all connected open subsets of $\mathbb{C}$. If there is a conformal map $\varphi : \Omega \to D$, then the Dirichlet problem (5.1) is equivalent to the Dirichlet problem

$$\begin{aligned} \Delta v &= 0, \quad \text{in } \Omega, \\ v(\omega) &= (h\circ\varphi)(\omega), \quad \text{on } \partial\Omega \end{aligned}$$

and $v = u \circ \varphi$. In particular,

$$\xi(z) = \frac{e^{\pi i z} - i}{e^{\pi i z} + i}, \quad z \in \overline{S},$$

is a conformal map from S to D. The Poisson formula for S is thus given by

$$\begin{aligned} v(z) &= \int_{-\infty}^{\infty} e^{\pi(y-t)}\sin(\pi x)\Bigg[\frac{h(it)}{\sin^2(\pi x)+(\cos(\pi x)-e^{\pi(y-t)})^2} \\ &\quad + \frac{h(1+it)}{\sin^2(\pi x)+(\cos(\pi x)+e^{\pi(y-t)})^2}\Bigg]\,\mathrm{d}t, \end{aligned} \tag{5.3}$$

where $z = x + iy \in \overset{\circ}{S}$ and it is satisfied by every function $f \in G(\mathbb{C}, \mathbb{C})$ with v and h replaced by f.

4. It follows from the Hahn-Banach theorem, that equation (5.3) also holds in $X+Y$ for $f \in G(X,Y)$, as for every $z' \in (X+Y)'$, the function $f_{z'} : z \mapsto \langle f(z), z'\rangle$ is holomorphic and satisfies (5.3). In particular, we can write $f = f_X + f_Y$, where

$$f_X(z) = \int_{-\infty}^{\infty} e^{\pi(y-t)}\sin(\pi x)\frac{f(it)}{\sin^2(\pi x)+(\cos(\pi x)-e^{\pi(y-t)})^2}\,\mathrm{d}t, \tag{5.4}$$

$$f_Y(z) = \int_{-\infty}^{\infty} e^{\pi(y-t)}\sin(\pi x)\frac{f(1+it)}{\sin^2(\pi x)+(\cos(\pi x)+e^{\pi(y-t)})^2}\,\mathrm{d}t. \tag{5.5}$$

5. The two kernels in (5.4) and (5.5) are positive and we have that

$$\begin{aligned} \int_{-\infty}^{\infty} e^{\pi(y-t)}\sin(\pi x)&\Bigg[\frac{1}{\sin^2(\pi x)+(\cos(\pi x)-e^{\pi(y-t)})^2} \\ &+\frac{1}{\sin^2(\pi x)+(\cos(\pi x)+e^{\pi(y-t)})^2}\Bigg]\,\mathrm{d}t = 1, \end{aligned}$$

from considering $f \equiv 1$, so that

$$\begin{aligned} 0 &< \int_{-\infty}^{\infty} e^{\pi(y-t)}\sin(\pi x)\frac{1}{\sin^2(\pi x)+(\cos(\pi x)-e^{\pi(y-t)})^2}\,\mathrm{d}t < 1, \\ 0 &< \int_{-\infty}^{\infty} e^{\pi(y-t)}\sin(\pi x)\frac{1}{\sin^2(\pi x)+(\cos(\pi x)+e^{\pi(y-t)})^2}\,\mathrm{d}t < 1. \end{aligned}$$

It follows that

$$\begin{aligned} \|f_X(z)\|_X &\le \sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_X, \quad z \in \overline{S} \quad \text{and} \\ \|f_Y(z)\|_Y &\le \sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_Y, \quad z \in \overline{S}. \end{aligned}$$

6. From these estimates we get that for every $t > 0$, $f \in G(X,Y)$ and $f(\theta) = a \in [X,Y]_\theta$,

$$K(t,a,X,Y) \le \|f_X(\theta)\|_X + t\|f_Y(\theta)\|_Y \le \sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_X + t\sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_Y.$$

For every f, we can also apply this estimate to the function $g : z \mapsto t^{\theta - z} f(z)$, which is also in $G(X,Y)$ with $g(\theta) = a$, to get that

$$K(t,a) \leq t^{\theta} \sup_{\tau \in \mathbb{R}} \|f(i\tau)\|_X + t^{\theta-1} t \sup_{\tau \in \mathbb{R}} \|f(1+i\tau)\|_Y \leq 2t^{\theta} \|f\|_{G(X,Y)}.$$

Taking the infimum over f yields the claim.

□

Corollary 5.13. *It follows that for every* $0 < \theta, \theta_0, \theta_1 < 1$, $1 \leq p \leq \infty$, *we have*

$$([X,Y]_{\theta_0}, [X,Y]_{\theta_1})_{\theta,p} = (X,Y)_{\theta_0(1-\theta)+\theta\theta_1,p}$$

and that

$$(X,Y)_{\theta,1} \hookrightarrow [X,Y]_{\theta} \hookrightarrow (X,Y)_{\theta,\infty}.$$

Remark 5.14. More reiteration and relations of real and complex interpolation spaces, without proof.

1. In general, it is not true that $[X,Y]_\theta = (X,Y)_{\theta,p}$ for some p. We will see later that if X and Y are Hilbert spaces,

$$[X,Y]_\theta = (X,Y)_{\theta,2} \text{ for } 0 < \theta < 1.$$

2. If $X \hookrightarrow Y$ or $Y \hookrightarrow X$, OR if X and Y are reflexive and $X \cap Y$ is dense in X and in Y, then

$$[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_{\theta} = [X,Y]_{(1-\theta)\theta_0+\theta\theta_1}$$

3. If X, Y are reflexive, $0 < \theta, \theta_0, \theta_1 < 1$, and $1 < p < \infty$,

$$[(X,Y)_{\theta_0,p}, (X,Y)_{\theta_1,p}]_{\theta} = (X,Y)_{(1-\theta)\theta_0+\theta\theta_1,p}.$$

6 Examples

6.1 Complex interpolation of L^p -spaces

Let (Ω, μ) be a measure space with a σ-finite measure μ. For $1 \leq q \leq \infty$, we use the notation $L^q = L^q(\Omega, \mu)$.

Theorem 6.1. *Let $\theta \in (0,1)$ and $1 \leq p_0, p_1 \leq \infty$. Then we have*

$$[L^{p_0}, L^{p_1}]_\theta = L^p, \quad \text{where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

with coinciding norms.

Proof. Note: see Theorem 1.2.
Strategy: We show that

1. for every $a \in L^{p_0} \cap L^{p_1}$, $\|a\|_{[L^{p_0}, L^{p_1}]_\theta} \leq \|a\|_{L^p}$ and
2. for every $a \in L^{p_0} \cap L^{p_1}$, $\|a\|_{L^p} \leq \|a\|_{[L^{p_0}, L^{p_1}]_\theta}$,

so that the identity is an isometry between $L^{p_0} \cap L^{p_1}$ with the $[L^{p_0}, L^{p_1}]_\theta$-norm and with the L^p-norm. The claim follows from $L^{p_0} \cap L^{p_1}$ being dense both in $[L^{p_0}, L^{p_1}]_\theta$ and in L^p and from the fact that $[L^{p_0}, L^{p_1}]_\theta$ and L^p embed into $L^{p_0} + L^{p_1}$. Let now $a \in L^{p_0} \cap L^{p_1}$. W.l.o.g. (exercise), we say that $\|a\|_{L^p} = 1$.

1. For every $z \in S$, we set

$$\begin{aligned} f(z)(x) &= |a(x)|^{p(\frac{1-z}{p_0} + \frac{z}{p_1})} \frac{a(x)}{|a(x)|}, \quad \text{for } x \in \Omega, \text{if } a(x) \neq 0, \\ f(z)(x) &= 0, \quad \text{for } x \in \Omega, \text{if } a(x) = 0. \end{aligned}$$

   It follows that $f \in G(L^{p_0}, L^{p_1})$, i.e. f is holomorphic on $\mathring{S}$ and continuous on S with values in $L^{p_0} + L^{p_1}$, $t \mapsto f(it)$ is bounded and continuous with values in L^{p_1} , $t \mapsto f(1+it)$ is bounded and continuous with values in L^{p_1} , since we have

$$\begin{aligned} |f(it)(x)| = |a|^{p/p_0}, &\qquad \|f(it)\|_{L^{p_0}} \leq \|a\|_{L^p}^{p/p_0}, \\ |f(1+it)(x)| = |a|^{p/p_1}, &\qquad \|f(1+it)\|_{L^{p_1}} \leq \|a\|_{L^p}^{p/p_1}. \end{aligned}$$

   It follows that $\|f\|_{G(L^{p_0}, L^{p_1})} = 1$ and so since $f(\theta) = a$, we have

$$\|a\|_{[L^{p_0}, L^{p_1}]_\theta} \leq 1 = \|a\|_{L^p}.$$

2. We know that

$$1 = \|a\|_{L^p} = \sup\left\{ |\int_\Omega a(x)b(x)\,\mathrm{d}x| : b \in L^{p_0'} \cap L^{p_1'} \cap (\text{simple functions}), \|b\|_{L^{p'}} = 1 \right\}$$

(easy: $\frac{1}{p'} = \frac{1-\theta}{p_0'} + \frac{\theta}{p_1'}$, where $\frac{1}{q'} = 1 - \frac{1}{q}$). For all $b \in L^{p_0'} \cap L^{p_1'}$ with $\|b\|_{L^{p'}} = 1$, we again define

$$\begin{aligned} g(z)(x) &= |b(x)|^{p'(\frac{1-z}{p_0'} + \frac{z}{p_1'})} \frac{b(x)}{|b(x)|}, \quad \text{for } x \in \Omega, \text{if } b(x) \neq 0, \\ g(z)(x) &= 0, \quad \text{for } x \in \Omega, \text{if } b(x) = 0 \end{aligned}$$

and we define for every $f \in G(L^{p_0}, L^{p_1})$ with $f(\theta) = a$,

$$F(z) := \int_\Omega f(z)(x) g(z)(x)\,\mathrm{d}x, \quad z \in S.$$

It follows that F is holomorphic, by the following argument. From the definition, g is holomorphic on S with values in $L^{p_0'} \cap L^{p_1'}$, as in particular, $b \in L^{p_0' p'(\frac{1-z}{p_0'} + \frac{z}{p_1'})} \cap L^{p_1' p'(\frac{1-z}{p_0'} + \frac{z}{p_1'})}$. It follows that $g(z) \in L^{p_0'} \cap L^{p_1'}$ is in the dual of $L^{p_0} + L^{p_1}$ for all $z \in S$. It follows that

$$\begin{aligned} \lim_{h\to 0} \frac{F(z+h) - F(z)}{h} &= \lim_{h \to 0} \frac{\langle f(z+h), g(z+h) - g(z)\rangle}{h} \\ &\quad + \lim_{h\to 0} \frac{\langle f(z+h) - f(z), g(z)\rangle}{h} \\ &= \langle f(z), g'(z)\rangle + \langle f'(z), g(z)\rangle, \end{aligned}$$

where $\langle , \rangle$ is the dual pairing for $L^{p_0} + L^{p_1}$. Moreover, F is continuous and bounded on S, so that by the maximum principle 5.2, for every $z \in S$,

$$F(z) \leq \max\{\sup_{t\in\mathbb{R}} |F(it)|, \sup_{t\in\mathbb{R}} |F(1+it)|\},$$

where

$$|F(it)| \leq \|f(it)\|_{L^{p_0}} \|g(it)\|_{L^{p_0'}} = \|f(it)\|_{L^{p_0}} \|b\|_{L^{p'}}^{p'/p_0'} = \|f(it)\|_{L^{p_0}},$$

$$|F(1+it)| \leq \|f(1+it)\|_{L^{p_1}} \|g(1+it)\|_{L^{p_1'}} = \|f(1+it)\|_{L^{p_1}} \|b\|_{L^{p'}}^{p'/p_1'} = \|f(1+it)\|_{L^{p_1}},$$

so that

$$|F(z)| \leq \|f\|_{G(L^{p_0}, L^{p_1})}, \quad z \in S.$$

It follows that

$$|\int_\Omega a(x)b(x)\,\mathrm{d}x| = |F(\theta)| \leq \|f\|_{G(L^{p_0}, L^{p_1})}.$$

Taking the *sup* over b and then the *inf* over f gives

$$\|a\|_{L^p} \leq \|a\|_{[L^{p_0}, L^{p_1}]_\theta}.$$

□

6.2 Real interpolation of L^p-spaces

Again, let (Ω, μ) be a measure space with a σ-finite measure μ. For $1 \leq q \leq \infty$, we use the notation $L^q = L^q(\Omega, \mu)$. Now, we may consider real- as well as complex-valued functions. As a roadmap for this section, we will

- define Lorentz spaces $L^{p,q} = L^{p,q}(\Omega, \mu)$
- show that $(L^{p_0}, L^{p_1})_{\theta,q} = L^{p,q}$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{P_1}$
- prove the Marcinkiewicz Theorem

6.2.1 Lorentz spaces

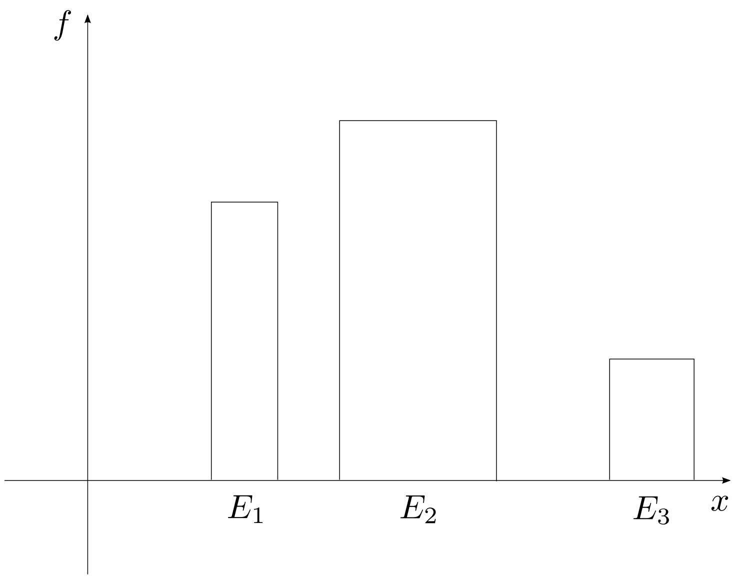
Definition 6.2. (non-increasing rearrangement)
Let $f \in L^1 + L^\infty$. We define the *distribution function* $\mu_f : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ of f by

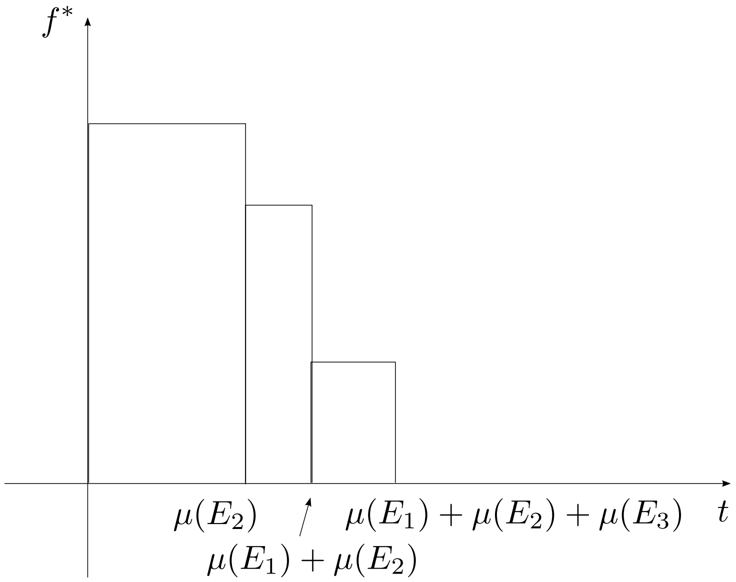
$$\mu_f(\sigma) = \mu(\{x \in \Omega : |f(x)| > \sigma\}).$$

The *non-increasing rearrangement* of f onto $[0, \infty)$ is given by $f^* : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{\infty\}$,

$$f^*(t) = \inf\{\sigma : \mu_f(\sigma) \leq t\},$$

where $\inf \emptyset = \infty$.





Proposition 6.3. (Properties of μ_f and f^*)
The functions μ_f and f^* are non-increasing, positive and right-continuous. Moreover, the functions $|f|$ and f^* are equimeasurable, *i.e. for every $0 < \sigma_0 < \sigma_1$,*

$$|\{t > 0 : f^*(t) \in [\sigma_0, \sigma_1]\}| = \mu\{x \in \Omega : |f(x)| \in [\sigma_0, \sigma_1]\}. \tag{6.1}$$

Proof. Clear: non-increasing, positive, equimeasurable.
Right-continuity of μ_f: We look at the sets $E(\sigma) = \{x \in \Omega : |f(x)| > \sigma\}$. For every $\sigma_0 \in \mathbb{R}_+$, we have

$$E(\sigma_0) = \bigcup_{\sigma > \sigma_0} E(\sigma) = \bigcup_{n \in \mathbb{N}} E(\sigma_0 + \frac{1}{n}).$$

It follows by the monotone convergence theorem that

$$\mu_f(\sigma_0 + \frac{1}{n}) = \mu(E(\sigma_0 + \frac{1}{n})) \nearrow \mu(E(\sigma_0)) = \mu_f(\sigma_0).$$

Right-continuity of f^* : Note that f^* itself is the distribution function of μ_f with respect to the Lebesgue measure m on $\mathbb{R}_+$, i.e. $f^*(t) = \sup\{\sigma : \mu_f(\sigma) > t\} = m_{\mu_f}(\sigma)$, so right-continuity follows as for μ_f. $\square$

Proposition 6.4. *The functions f and f^* have the same L^p-norms, i.e. for $1 \leq p < \infty$,*

$$\begin{aligned} \int_\Omega |f|^p \, \mathrm{d}\mu &= \int_0^\infty (f^*(t))^p \, \mathrm{d}t \qquad \textit{and} \\ \operatorname{ess\,sup}_\Omega f &= \operatorname{ess\,sup}_{\mathbb{R}_+ \cup \{0\}} f^* = f^*(0). \end{aligned}$$

Proof. Exercise. □

Definition 6.5. (Lorentz spaces)
For $1 \le p, q \le \infty$, we define the *Lorentz spaces* by

$$L^{p,q}(\Omega,\mu) := \left\{ f \in L^1 + L^\infty : \|f\|_{L^{p,q}} = \left(\int_0^\infty (t^{1/p} f^*(t))^q \, \frac{\mathrm{d}t}{t} \right)^{1/q} < \infty \right\}$$

for $q < \infty$ and by

$$L^{p,\infty}(\Omega,\mu) := \left\{ f \in L^1 + L^\infty : \|f\|_{L^{p,\infty}} = \sup_{t>0} t^{1/p} f^*(t) < \infty \right\}$$

otherwise. The $L^{p,\infty}$-spaces are also called *Marcinkiewicz spaces*. Note that $L^{p,p} = L^p$ by Proposition 6.4.

6.2.2 Lorentz spaces and the K-functional

Theorem 6.6. *For $0 < \theta < 1$ and $1 \le q \le \infty$, we have that*

$$(L^1, L^\infty)_{\theta,q} = L^{\frac{1}{1-\theta},q}.$$

Proof. We prove the theorem in two steps:

1. We show that
$$K(t, f, L^1, L^\infty) = \int_0^t f^*(s) \, \mathrm{d}s \tag{6.2}$$
for all $t > 0$. Note that historically, the Lorentz spaces were used before this connection to real interpolation spaces was discovered. In the following, let $t > 0$ and
$$\begin{aligned} E_t &= \{x \in \Omega : |f(x)| > f^*(t)\}, \\ D_t &= \{x \in \Omega : |f(x)| = f^*(t)\}, \end{aligned}$$
so that $\mu(E_t) \le t \le \mu(E_t \cup D_t)$. From (6.1) and Proposition 6.4, we get that for every $y \in D_t$ - which we need if $\mu(D_t) \ne 0$ - ,
$$\int_0^t f^*(s) \, \mathrm{d}s = \int_{E_t} |f(x)| \, \mathrm{d}\mu + (t - \mu(E_t))|f(y)|. \tag{6.3}$$
Rigorously, this can be seen from the following argument. We may write
$$\begin{aligned} (\chi_{E_t} f)^*(s) &= \inf\{\sigma : \mu(\{x \in \Omega : |\chi_{E_t}(x)| \cdot |f(x)| > \sigma\}) \le s\} \\ &= \inf\{\sigma : \mu(\{x \in \Omega : |f(x)| > \sigma\} \cap E_t) \le s\} \\ &= \inf\{\sigma : \mu(\{x \in \Omega : |f(x)| > \max(\sigma, f^*(t))\}) \le s\}. \end{aligned} \tag{6.4}$$

In the case $s \geq \mu(E_t)$, if $f^*(t) > 0$, we get $(\chi_{E_t} f)^*(s) = 0$, as

$$\mu(\{x \in \Omega : |f(x)| > f^*(t)\}) = \mu(E_t) \leq s,$$

otherwise if $f^*(t) = 0$, then $(\chi_{E_t} f)^*(s) = f^*(s)$. In the case $s < \mu(E_t)$, we see from the last line in (6.4) that $(\chi_{E_t} f)^*(s) = f^*(s)$. If $f^*(t) = 0$, then (6.3) follows because $|f(y)| = 0$. If $f^*(t) \neq 0$, then

$$\int_0^t f^* = \int_0^{\mu(E_t)} f^* + \int_{\mu(E_t)}^t f^* = \int_{E_t} |f| + \int_{\mu(E_t)}^t f^*$$

by Proposition 6.4. Moreover, we have $f^*(s) = f^*(t) = |f(y)|$ for every $s \in [\mu(E_t), t]$ as clearly $f^*(t) \leq f^*(s) \leq f^*(\mu(E_t))$ and by definition,

$$f^*(\mu(E_t)) = \inf\{\sigma > 0 : \mu\{x : |f(x)| > \sigma\} \leq \mu\{x : |f(x)| > |f^*(t)|\}\} \leq f^*(t).$$

This gives (6.3).
It follows that for a decomposition $f = f_1 + f_\infty$ with $f_1 \in L^1$ and $f_\infty \in L^\infty$, we have

$$\begin{aligned}\int_0^t f^*(s)\,\mathrm{d}s \quad &\leq \quad \int_{E_t} |f_1|\,\mathrm{d}\mu + (t - \mu(E_t))|f_1(y)| \\ &\quad + \mu(E_t) \sup_{x \in E_t} |f_\infty(x)| + (t - \mu(E_t))|f_\infty(y)| \\ &\leq \quad \|f_1\|_{L^1} + t\|f_\infty\|_{L^\infty}.\end{aligned}$$

For the other inequality, we may choose the following decomposition of f,

$$f_1(x) = \begin{cases} f(x) - \frac{f(x)}{|f(x)|} f^*(t) & \text{for } x \in E_t, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$f_\infty = f - f_1.$$

First, we show that

$$\|f_1\|_{L^1} = \int_0^{\mu(E_t)} f^*(s) - f^*(t)\,\mathrm{d}s.$$

This follows from Proposition 6.4 if $f^*(t) = 0$. Otherwise,

$$\|f_1\|_{L^1} = \int_\Omega |\chi_{E_t} f_1| = \int_0^{\mu(E_t)} f_1^*(s)\,\mathrm{d}s$$

by (6.4). On E_t, we have $f_1 = (|f| - f^*(t))e^{i \arg f}$, so that

$$|f_1| = (|f| - f^*(t))\chi_{E_t}.$$

Since $|f_1|^* = f_1^*$, we get

$$\begin{aligned} f_1^* &= \inf\{\sigma : \mu(\{x : \chi_{E_t}(|f| - f^*(t))\}) \le s\} \\ &= \inf\{\sigma : \mu(\{x \in E_t : |f| > \sigma + f^*(t)\}) \le s\} \\ &\overset{|f|>f^*(t) \text{ only on } E_t}{=} \inf\{\sigma : \mu(\{x : |f| > \sigma + f^*(t)\}) \le s\} \\ &= \inf\{\sigma > f^*(t) : \mu(\{x : |f| > \sigma\}) \le s\} - f^*(t). \end{aligned} \tag{6.5}$$

Now if $\sigma \le f^*(t)$, then

$$\mu(\{x : |f| > \sigma\}) \ge \mu(\{x : |f| > f^*(t)\}) = \mu(E_t) > s,$$

so that from (6.5) we see that

$$f_1^*(s) = \inf\{\sigma : \mu(\{x : |f| > \sigma\}) \le s\} - f^*(t) = f^*(s) - f^*(t).$$

It follows that

$$\|f_1\|_{L^1} = \int_0^{\mu(E_t)} f^*(s) - f^*(t)\,\mathrm{d}s \le \int_0^t f^*(s)\,\mathrm{d}s - tf^*(t)$$

and that

$$|f_\infty(x)| \le |f^*(t)\chi_{E_t}(x)\| \le f^*(t), \qquad x \in \Omega,$$

so that

$$K(t, f, L^1, L^\infty) \le \|f_1\|_{L^1} + t\|f_\infty\|_{L^\infty} = \int_0^t f^*(s)\,\mathrm{d}s.$$

2. From (6.2), the claim follows fairly directly. Note that $\int_0^t f^*(s)\,\mathrm{d}s \ge tf^*(t)$, as f^* is non-increasing, so that for $q < \infty$, we have

$$\begin{aligned} \|f\|_{(L^1,L^\infty)\theta,q} &= \left[\int_0^\infty t^{-\theta q} K^q(t,f)\,\frac{\mathrm{d}t}{t}\right]^{1/q} \\ &\ge \left[\int_0^\infty t^{(1-\theta)q} (f^*(t))^q\,\frac{\mathrm{d}t}{t}\right]^{1/q} \\ &= \|f\|_{L^{\frac{1}{1-\theta},q}}, \end{aligned}$$

and for $q = \infty$,

$$\sup_{t>0} |t^{-\theta} K(t,f)| \ge \sup_{t>0} |t^{1-\theta} f^*(t)| = \|f\|_{L^{\frac{1}{1-\theta},\infty}}.$$

On the other hand, by the Hardy-Young inequality, Lemma 3.2, we have

$$\begin{aligned} \|t^{-\theta} K(t,f)\|_{L_*^q}^q &= \int_0^\infty t^{-\theta q} \left(\int_0^t s f^*(s)\,\frac{\mathrm{d}s}{s}\right)^q \frac{\mathrm{d}t}{t} \\ &\le \frac{1}{\theta^q} \int_0^\infty s^{(1-\theta)q} (f^*(s))^q\,\frac{\mathrm{d}s}{s} \\ &= \frac{1}{\theta^q} \|f\|_{L^{\frac{1}{1-\theta},q}}^q \end{aligned}$$

and also

$$\sup_{t>0} |t^{-\theta} K(t,f)| \leq t^{-\theta} \left(\int_0^t \frac{\mathrm{d}s}{s^{1-\theta}} \right) \sup_{s>0} (s^{1-\theta} f^*(s)) = \frac{1}{\theta} \|f\|_{L^{\frac{1}{1-\theta},\infty}}.$$

□

Theorem 6.7. *For $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$ and $1 \leq q \leq \infty$ we have*

$$(L^{p_0}, L^{p_1})_{\theta,q} = L^{p,q}, \quad \textit{where } \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

In particular, if $p_0 < q < p_1$ and $\theta = \frac{p_1}{q}\left(\frac{p_0-q}{p_0-p_1}\right)$, then

$$(L^{p_0}, L^{p_1})_{\theta,q} = L^q.$$

Moreover, for $1 < p_0 < p_1 < \infty$, $0 < \theta < 1$ and $1 \leq q_0 \leq q_1 \leq \infty$, we have

$$(L^{p_0,q_0}, L^{p_1,q_1})_{\theta,q} = L^{p,q}$$

for p as above.

Proof. By Theorem 6.6, we know that $L^p = (L^1, L^\infty)_{1-1/p,p}$, so $L^p \in J_{1-1/p} \cap K_{1-1/p}(L^1, L^\infty)$. We can therefore apply the Reiteration Theorem 4.4 to get

$$(L^{p_0}, L^{p_1})_{\theta,q} = (L^1, L^\infty)_{\omega,q},$$

where $\omega = (1-\theta)(1-1/p_0) + \theta(1-1/p_1)$. By Theorem 6.6, $(L^1, L^\infty)_{\omega,q} = L^{p,q}$, where $\frac{1}{p} = 1 - \omega = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The same argument yields the third statement. □

6.2.3 The Marcinkiewicz Theorem

Definition 6.8. Let (Ω, μ) and (Λ, ν) be two σ-finite measure spaces. A linear operator $T : L^1(\Omega,\mu) + L^\infty(\Omega,\mu) \to L^1(\Lambda,\nu) + L^\infty(\Lambda,\nu)$ is called *of weak type* (p,q), if there is an $M > 0$ such that

$$\sup_{\sigma>0} \sigma(\nu\{y \in \Lambda : |Tf(y)| > \sigma\})^{1/q} = \sup_{t>0} t^{1/q} (Tf)^*(t) \leq M \|f\|_{L^p(\Omega,\mu)}.$$

for all $f \in L^p(\Omega,\mu)$, i.e. the restriction of T to $L^p(\Omega,\mu)$ is bounded from $L^p(\Omega,\mu)$ to $L^{q,\infty}(\Lambda,\nu)$.
T is said to be *of strong type* (p,q) if its restriction to $L^p(\Omega,\mu)$ is bounded from $L^p(\Omega,\mu)$ to $L^q(\Lambda,\nu)$.

Clearly, every operator of strong type (p,q) is also of weak type (p,q).

Theorem 6.9. *Let $T : L^1(\Omega,\mu)+L^\infty(\Omega,\mu) \to L^1(\Lambda,\nu)+L^\infty(\Lambda,\nu)$ be of weak type (p_0,q_0) and (p_1,q_1) with constants M_0, M_1, respectively and $1 < p_0, p_1 < \infty$, $1 < q_0, q_1 < \infty$, $q_0 \neq q_1$ and $p_0 \leq q_0$, $p_1 \leq q_1$. For $0 < \theta < 1$ let*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then T is of strong type (p,q) and there is a constant C, independent of θ, such that

$$\|Tf\|_{L^q(\Lambda,\nu)} \leq C M_0^{1-\theta} M_1^\theta \|f\|_{L^p(\Omega,\mu)}$$

for all $f \in L^p(\Omega,\mu)$.

Proof. For $i = 0,1$, T is bounded from $L^{p_i}(\Omega,\mu)$ to $L^{q_i,\infty}(\Lambda,\nu)$ with norm M_i. Since real interpolation is exact of type θ, Theorem 2.4, it follows that

$$\|T\|_{\mathcal{L}((L^{p_0},L^{p_1})_{\theta,p},(L^{q_0,\infty},L^{q_1,\infty})_{\theta,p})} \leq M_0^{1-\theta} M_1^\theta.$$

By Theorem 6.7, $(L^{p_0}, L^{p_1})_{\theta,p} = L^{p,p} = L^p(\Omega,\mu)$ and $(L^{q_0,\infty}, L^{q_1,\infty})_{\theta,p} = L^{q,p}(\Lambda,\nu)$. Since $p_i \leq q_i$, also $p \leq q$, so that $L^{q,p} \hookrightarrow L^{q,q} = L^q(\Lambda,\nu)$ by Theorem 2.5. This yields the claim. □

6.3 Hölder spaces

Reminder: $f \in C^1(\Omega) \Leftrightarrow f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable and $\|f\|_{C^1(\Omega)} := \|f\|_\infty + \sum_{i=1}^n \|D_i f\|_\infty < \infty$.

Definition 6.10. For $\theta \in (0,1)$, we define the *Hölder space* $C^\theta(\Omega)$ for every open set $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, as the space of bounded and uniformly θ-Hölder continuous functions f with the norm

$$\|f\|_{C^\theta(\Omega)} = \|f\|_\infty + [f]_{C^\theta} = \|f\|_\infty + \sup_{x,y\in\Omega, x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\theta}.$$

Theorem 6.11. *For $\theta \in (0,1)$, we have that*

$$(C(\mathbb{R}^n), C^1(\mathbb{R}^n))_{\theta,\infty} = C^\theta(\mathbb{R}^n)$$

with equivalence of norms.

Proof. For convenience, we may write C, C^1 etc. in the following. First we show that $(C, C^1)_{\theta,\infty} \hookrightarrow C^\theta$: Clearly, for every $f \in C^1$, $\|f\|_\infty \leq \|f\|_{C^1}$. For every decomposition $f = f_0 + f_1$, $f_0 \in C$, $f_1 \in C^1$, we have $\|f\|_\infty \leq \|f_0\|_\infty + \|f_1\|_\infty$, so

$$\|f\|_\infty \leq K(1, f, C, C^1) \leq \|f\|_{(C,C^1)_{\theta,\infty}}.$$

Moreover, if $x \neq y$, we have that

$$|f(x) - f(y)| \leq |f_0(x) - f_0(y)| + |f_1(x) - f_1(y)| \leq 2\|f_0\|_\infty + \|f_1\|_{C^1}|x-y|.$$

Taking the infimum over these decompositions gives

$$|f(x) - f(y)| \leq 2K(|x-y|, f, C, C^1) \leq 2|x-y|^\theta \|f\|_{(C,C^1)_{\theta,\infty}}$$

by (2.2), so that f must be θ-Hölder continuous and $\|f\|_{C^\theta} = \|f\|_\infty + [f]_\theta \leq 3\|f\|_{(C,C^1)_{\theta,\infty}}$.

Now we show that $C^\theta \hookrightarrow (C, C^1)_{\theta,\infty}$: Let $f \in C^\theta$. We want to decompose f into a C- and a C^1-part with good estimates. Let $\varphi \in C^\infty$ such that $\varphi \geq 0$, $\operatorname{supp} \varphi \subset B_1$ and $\|\varphi\|_{L^1} = 1$. For every $t > 0$, we consider

$$\begin{aligned} f_{1,t}(x) &= \frac{1}{t^n} \int_{\mathbb{R}^n} f(y)\varphi(\frac{x-y}{t})\,\mathrm{d}y, \\ f_{0,t}(x) &= f(x) - f_{1,t}(x). \end{aligned} \tag{6.6}$$

It follows that $f_{0,t}(x) = \frac{1}{t^n}\int_{\mathbb{R}^n}(f(x) - f(x-z))\varphi(\frac{z}{t})\,\mathrm{d}z$, so

$$\|f_{0,t}\|_\infty \leq [f]_{C^\theta} \frac{1}{t^n} \int_{\mathbb{R}^n} |y|^\theta \varphi(\frac{y}{t})\,\mathrm{d}y = t^\theta [f]_{C^\theta} \int_{\mathbb{R}^n} |w|^\theta \varphi(w)\,\mathrm{d}w.$$

It also follows that $\|f_{1,t}\|_\infty \leq \|f\|_\infty$. It remains to look at the derivatives, for $i = 1, \ldots, n$,

$$D_i f_{1,t}(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} f(y)(D_i\varphi)(\frac{x-y}{t})\,\mathrm{d}y.$$

Since φ has compact support, we have $\int_{\mathbb{R}^n} D_i\varphi(\frac{x-y}{t})\,\mathrm{d}y = 0$. We get that

$$D_i f_{1,t}(x) = \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} (f(x-z) - f(x))(D_i\varphi)(\frac{z}{t})\,\mathrm{d}z, \tag{6.7}$$

so that

$$\begin{aligned} \|D_i f_{1,t}\|_\infty &\leq \frac{1}{t^{n+1}} \int_{\mathbb{R}^n} \frac{|f(x-z) - f(x)|}{|z|^\theta} |z|^\theta (D_i\varphi)(\frac{z}{t})\,\mathrm{d}z \\ &= t^{\theta-1}[f]_{C^\theta} \int_{\mathbb{R}^n} |w|^\theta (D_i\varphi)(w)\,\mathrm{d}w. \end{aligned}$$

For $0 < t \leq 1$, it follows that

$$t^{-\theta} K(t, f) \leq t^{-\theta}(\|f_{0,t}\|_\infty + t\|f_{1,t}\|_{C^1}) \leq C\|f\|_{C^\theta}.$$

For $t > 1$, we set $f_{0,t} = f$ and $f_{1,t} = 0$ to immediately get that

$$t^{-\theta} K(t, f) \leq t^{-\theta}\|f\|_\infty \leq \|f\|_\infty.$$

It follows that $f \in (C, C^1)_{\theta,\infty}$ and that $\|f\|_{(C,C^1)_{\theta,\infty}} \leq C\|f\|_{C^\theta}$, with C depending only on the choice of φ. $\square$

Remark 6.12. The following results are mostly taken from [4], Chapter 2: 1. follows immediately, 2., 3., 4, could also be an exercise:

1. By Reiteration, Theorem 4.4:
$$(C^\alpha, C^\beta)_{\theta,\infty} = ((C, C^1)_{\alpha,\infty}, (C, C^1)_{\beta,\infty})_{\theta,\infty} = (C, C^1)_{\omega,\infty} = C^\omega,$$
where $\alpha, \beta \in (0, 1)$ and $\omega = (1-\theta)\alpha + \theta\beta$.
2. It holds that
$$"(C, C^1)_{1,\infty}" = \mathrm{Lip}(\mathbb{R}^n),$$
where $\mathrm{Lip}(\mathbb{R}^n)$ is the space of Lipschitz continuous and bounded functions,
$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} = \|f\|_\infty + \sup_{x\neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$
3. C^1 is not dense in C^θ.
4. For any interval $I \subset \mathbb{R}$, X a Banach space, $0 < \theta < 1$,
$$(C(I; X), C^1(I, X))_{\theta,\infty} = C^\theta(I; X).$$
5. Let $m \in \mathbb{N}$ and $\theta \in (0, 1)$. If θm is not an integer,
$$(C, C^m)_{\theta,\infty} = C^{\theta m}.$$
6. In the situation of 3., if $\theta m = 1$, then
$$(C, C^m)_{\theta,\infty} = \mathcal{C}^1,$$
where for every $0 < \alpha < 2$, $\mathcal{C}^\alpha$ is given by
$$\begin{aligned} \mathcal{C}^\alpha &= \{f \in C : [f]_{\mathcal{C}^\alpha} = \sup_{x\neq y} \frac{|f(x) - 2f(\frac{x+y}{2}) + f(y)|}{|x-y|^\alpha} < \infty\}, \\ \|f\|_{\mathcal{C}^\alpha} &= \|f\|_\infty + [f]_{\mathcal{C}^\alpha}. \end{aligned}$$
For $\alpha \neq 1$, these spaces are equivalent to the C^α's.

6.4 Slobodeckii spaces

Dilemma I: How to define $W^{\alpha,p}(\mathbb{R}^n)$ for $\alpha \notin \mathbb{N}$, $1 \leq p < \infty$ and $n \in \mathbb{N}$?

1. $W^{\alpha,p} = (L^p, W^{1,p})_{\alpha,p}$ for $0 < \alpha < 1$.

2. $W^{\alpha,p}$ is the set of functions $f \in L^p$, such that

$$[f]_{\alpha,p} = \left(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x - y|^{\alpha p + n}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/p} < \infty,$$

endowed with the norm $\|f\|_{W^{\alpha,p}} = \|f\|_{L^p} + [f]_{W^{\alpha,p}}$.

Theorem 6.13. *The two definitions above coincide.*

Dilemma II: How to define $W^{\alpha,p}(\Omega)$ for open sets $\Omega \subset \mathbb{R}^n$?

1. as in 1. above.
2. $W^{\alpha,p}(\Omega)$ is the space of functions which are restrictions of functions in $W^{\alpha,p}(\mathbb{R}^n)$ to Ω.

Theorem 6.14. *The two definitions above coincide if Ω is a domain with uniformly C^1-boundary.*

Definition 6.15. On an open set $\Omega \subset \mathbb{R}^n$, for $\alpha \in (0, 1)$, the *(Sobolev-)Slobodeckii spaces* $W_p^\alpha(\Omega)$ are defined by

$$W_p^\alpha(\Omega) = (L^p(\Omega), W^{1,p}(\Omega))_{\alpha,p}.$$

Proof. of Theorem 6.13. It works similarly as the proof for Hölder spaces, Theorem 6.11. First we show that $W_p^\alpha \hookrightarrow W^{\alpha,p}$, defined as in 2.:

Note that for every $f_1 \in W^{1,p}$ and $h \in \mathbb{R}^n \backslash \{0\}$, we have that (exercise)

$$\left(\int_{\mathbb{R}^n} \left(\frac{|f_1(x + h) - f_1(x)|}{|h|}\right)^p \mathrm{d}x\right)^{1/p} \leq \||Df_1|\|_{L^p}.$$

For all $f \in (L^p, W^{1,p})_{\alpha,p}$ and $h \in \mathbb{R}^n$, we set $f_{0,h} \in L^p$ and $f_{1,h} \in W^{1,p}$ such that $f = f_{0,h} + f_{1,h}$ and

$$\|f_{0,h}\|_{L^p} + |h|\|f_{1,h}\|_{W^{1,p}} \leq 2K(|h|, f).$$

Moreover,

$$\frac{|f(x + h) - f(x)|^p}{|h|^{\alpha p + n}} \leq 2^{p-1} \left(\frac{|f_{0,h}(x + h) - f_{0,h}(x)|^p}{|h|^{\alpha p + n}} + \frac{|f_{1,h}(x + h) - f_{1,h}(x)|^p}{|h|^{\alpha p + n}}\right)$$

and so

$$\begin{aligned}
\int_{\mathbb{R}^n} \frac{|f(x + h) - f(x)|^p}{|h|^{\alpha p + n}} \,\mathrm{d}x \quad &\leq \quad 2^{p-1} \int_{\mathbb{R}^n} \left(\frac{|f_{0,h}(x + h) - f_{0,h}(x)|^p}{|h|^{\alpha p + n}} + \frac{|f_{1,h}(x + h) - f_{1,h}(x)|^p}{|h|^{\alpha p + n}}\right) \mathrm{d}x \\
&\leq \quad 2^p \frac{\|f_{0,h}\|_{L^p}^p}{|h|^{\alpha p + n}} + 2^{p-1} \frac{|h|^p \||Df_{1,h}|\|_{L^p}^p}{|h|^{\alpha p + n}} \\
&\leq \quad C_p |h|^{-\alpha p - n} (\|f_{0,h}\|_{L^p} + |h|\|f_{1,h}\|_{W^{1,p}})^p \\
&\leq \quad C_p |h|^{-\alpha p - n} K(|h|, f)^p.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[f]^p_{\alpha,p} &= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x+h)-f(x)|^p}{|h|^{\alpha p+n}}\,\mathrm{d}x\,\mathrm{d}h \\
&\leq C_p \int_{\mathbb{R}^n} |h|^{-\alpha p-n} K(|h|,f)^p\,\mathrm{d}h \\
&= C_{p,n} \int_0^\infty \frac{K(r,f)^p}{r^{\alpha p+1}}\,\mathrm{d}r \\
&= C_{p,n} \|f\|^p_{(L^p,W^{1,p})_{\alpha,p}}.
\end{aligned}$$

From the embedding $(L^p,W^{1,p})_{\alpha,p} \hookrightarrow L^p + W^{1,p} = L^p$ we also get that $\|f\|_{L^p} \leq C\|f\|_{(L^p,W^{1,p})_{\alpha,p}}$, so that in conclusion,

$$\|f\|_{W^{\alpha,p}} \leq C\|f\|_{(L^p,W^{1,p})_{\alpha,p}}.$$

We now check the other embedding: For every $f \in W^{\alpha,p}$, we define $f_{0,t}$ and $f_{1,t}$ as in (6.6). It follows that

$$\begin{aligned}
\|f_{0,t}\|^p_{L^p} &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x)-f(y)|\frac{1}{t^n}\varphi(\frac{x-y}{t})\,\mathrm{d}y\right)^p \mathrm{d}x \\
&\leq \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)-f(y)|^p \frac{1}{t^n}\varphi(\frac{x-y}{t})\,\mathrm{d}y\,\mathrm{d}x
\end{aligned}$$

from the Hölder inequality applied to

$$|f(x)-f(y)|(t^{-n}\varphi(\frac{x-y}{t}))^{1/p}\cdot(t^{-n}\varphi(\frac{x-y}{t}))^{1-1/p}.$$

We get that

$$\begin{aligned}
\int_0^\infty t^{-\alpha p}\|f_{0,t}\|^p_{L^p}\frac{\mathrm{d}t}{t} &\leq \int_0^\infty \left(\int_{\mathbb{R}^n\times\mathbb{R}^n} t^{-\alpha p}|f(x)-f(y)|^p\frac{1}{t^n}\varphi(\frac{x-y}{t})\,\mathrm{d}y\,\mathrm{d}x\right)\frac{\mathrm{d}t}{t} \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)-f(y)|^p \left(\int_0^\infty t^{-\alpha p}\frac{1}{t^n}\varphi(\frac{x-y}{t})\frac{\mathrm{d}t}{t}\right)\mathrm{d}y\,\mathrm{d}x \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^n} |f(x)-f(y)|^p \left(\int_{|x-y|}^\infty t^{-\alpha p}\frac{1}{t^n}\varphi(\frac{x-y}{t})\frac{\mathrm{d}t}{t}\right)\mathrm{d}y\,\mathrm{d}x \\
&\leq \frac{\|\varphi\|_\infty}{\alpha p+n}\int_{\mathbb{R}^n\times\mathbb{R}^n}\frac{|f(x)-f(y)|^p}{|x-y|^{\alpha p+n}}\,\mathrm{d}y\,\mathrm{d}x \\
&= C[f]^p_{\alpha,p}.
\end{aligned}$$

By using (6.7), we get that

$$\int_0^\infty t^{(1-\alpha)p}\|D_i f_{1,t}\|^p_{L^p}\frac{\mathrm{d}t}{t} \leq \frac{C_i^{p-1}\|D_i\varphi\|_\infty}{\alpha p+n}[f]^p_{\alpha,p},$$

where $C_i = \int_{\mathbb{R}^n} |D_i\varphi(y)|\,\mathrm{d}y$. We also see directly that $\|f_{1,t}\|_{L^p} \le \|f\|_{L^p}\|\varphi\|_{L^1} = \|f\|_{L^p}$, so

$$\int_0^1 t^{(1-\alpha)p}\|f_{1,t}\|_{L^p}^p \frac{\mathrm{d}t}{t} \le \frac{1}{(1-\alpha)p}\|f\|_{L^p}^p.$$

It follows that

$$t^{-\alpha}K(t,f,L^p,W^{1,p}) \le t^{-\alpha}\|f_{0,t}\|_{L^p} + t^{1-\alpha}\|f_{1,t}\|_{W^{1,p}} \in L_*^p(0,1)$$

and that its norm is estimated by $C\|f\|_{W^{\alpha,p}}$. Again, this suffices to get that $f \in (L^p, W^{1,p})_{\alpha,p}$. □

6.5 Functions on domains

This section follows parts of Chapters 3,4 and 5 in [1], in particular, pages 147-152.

Definition 6.16. A domain $\Omega \subset \mathbb{R}^n$ satisfies the *uniform C^m-regularity condition* if there exists a locally finite open cover $\{U_j\}$ of $\partial\Omega$ and a corresponding sequence $\{\Phi_j\}$ of m-Diffeomorphisms taking U_j onto the unit ball $B_1(0)$ with inverses $\Psi_j = \Phi_j^{-1}$ such that

1. for some finite R, every intersection of $R+1$ of the sets U_j is empty,
2. for some $\delta > 0$,
$$\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \delta\} \subset \cup_j \Psi_j(B_{1/2}(0)),$$
3. for each j, $\Phi_j(U_j \cap \Omega) = \{y \in B_1(0) : y_n > 0\}$,
4. there is a finite constant $M > 0$, such that for every j, $\|\Phi_j\|_{C^m(U_j)} \le M$ and $\|\Psi_j\|_{C^m(B_1(0))} \le M$.

Proof. Of Theorem 6.14. We use that for every uniform C^1-domain Ω, there is an extension operator E, such that

$$E \in \mathcal{L}(L^p(\Omega), L^p(\mathbb{R}^n)), \quad E \in \mathcal{L}(W^{1,p}(\Omega), W^{1,p}(\mathbb{R}^n)), \tag{6.8}$$

and $E \in \mathcal{L}(W^{\alpha,p}(\Omega), W^{\alpha,p}(\mathbb{R}^n))$. Here, $W^{\alpha,p}(\mathbb{R}^n)$ is given as in Dilemma I.2 and $W^{\alpha,p}(\Omega)$ is given as the space of functions which are restrictions of functions in $W^{\alpha,p}(\mathbb{R}^n)$ to Ω. This operator can be constructed similarly as before, first for the half-space, then by localization. However, in order to get that $E \in \mathcal{L}(W^{\alpha,p}(\Omega), W^{\alpha,p}(\mathbb{R}^n))$, we take functions from $W^{1,p}(\mathbb{R}^n_+)$ and extend by exact reflection, not as in the proof of Theorem 6.19 for continuously differentiable functions.

From (6.8), it follows that $E \in \mathcal{L}(W_p^\alpha(\Omega), W_p^\alpha(\mathbb{R}^n))$ as real interpolation is exact of type θ. For every $f \in W^{\alpha,p}(\Omega)$, we have that $Ef \in W^{\alpha,p}(\mathbb{R}^n) = W_p^\alpha(\mathbb{R}^n)$. Moreover, the restriction operator $R : f \in L^p(\mathbb{R}^n) \mapsto f|_\Omega$ clearly belongs to $\mathcal{L}(L^p(\mathbb{R}^n), L^p(\Omega)) \cap$

$\mathcal{L}(W^{1,p}(\mathbb{R}^n), W^{1,p}(\Omega))$ and thus to $\mathcal{L}(W_p^\alpha(\mathbb{R}^n), W_p^\alpha(\Omega))$. It follows that $R(E(f))$ belongs to $W_p^\alpha(\Omega)$ and we have the estimate

$$\|f\|_{W_p^\alpha(\Omega)} \leq C_E \|E(f)\|_{W_p^\alpha(\mathbb{R}^n)} \leq C_{E,\alpha,p} \|E(f)\|_{W^{\alpha,p}(\mathbb{R}^n)} \leq C\|f\|_{W^{\alpha,p}(\Omega)}.$$

On the other hand, for $f \in W_p^\alpha(\Omega)$, we know that $Ef \in W^{\alpha,p}(\mathbb{R}^n)$ and so $R(Ef) = f \in W^{\alpha,p}(\Omega)$ and

$$\|f\|_{W^{\alpha,p}(\Omega)} \leq \|E(f)\|_{W_p^\alpha(\mathbb{R}^n)} \leq \|f\|_{W_p^\alpha(\Omega)}.$$

$\square$

Remark. In the same way, we get that for $\Omega \subset \mathbb{R}^n$ with a uniformly C^1-boundary,

$$(C(\Omega), C^1(\Omega))_{\theta,\infty} = C^\theta(\Omega).$$

7 Function Spaces from Fourier Analysis

Reminder I:

- $\mathcal{S}(\mathbb{R}^n)$ Schwartz space, $\mathcal{S}'(\mathbb{R}^n)$ the dual space of $\mathcal{S}(\mathbb{R}^n)$
- Fourier transform:
$$(\mathcal{F}\varphi)(\xi) = \hat{\varphi}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(x\cdot\xi)}\varphi(x)\,\mathrm{d}x,$$
and $\mathcal{F}, \mathcal{F}^{-1} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $\mathcal{F} \in \mathcal{L}(L^2(\mathbb{R}^n))$ unitary, $\mathcal{F} : L^p \to L^{p'}$ if $1 \le p \le 2$.
- $D^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(x^\alpha\varphi(x))$ and $\xi^\alpha(\mathcal{F}\varphi)(\xi) = (-i)^{|\alpha|}\mathcal{F}(D^\alpha\varphi(x))$.

Definition 7.1. (Bessel potential spaces)
Let $1 < p < \infty$ and $s \in \mathbb{R}$, we define
$$H^{s,p}(\mathbb{R}^n) := \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^{s,p}(\mathbb{R}^n)} := \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L^p(\mathbb{R}^n)} < \infty \right\}.$$

Remark 7.2. Hopefully, we will be able to show that $H^{\theta m,p} = [L^p, W^{m,p}]_\theta$.

Reminder II: Sobolev spaces: $1 \le p \le \infty$, $k \in \mathbb{N}$,
$$W^{k,p}(\mathbb{R}^n) := \left\{ f \in L^p(\mathbb{R}^n) : \|f\|_{W^{k,p}(\mathbb{R}^n)} := \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^p(\mathbb{R}^n)} < \infty \right\}.$$

Question: If $1 < p < \infty$, $k \in \mathbb{N}$, then $H^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$?

1. $k = 0$: $L^p(\mathbb{R}^n) = H^{0,p}(\mathbb{R}^n) = W^{0,p}(\mathbb{R}^n)$
2. $p = 2$: true, using Plancherel
3. $p \ne 2$: also true, using Mikhlin multiplier theorem

We immediately see 1., we look at 2. and 3. in the following.

Note that for any multi-index $|\alpha| \le k$ and $\xi \in \mathbb{R}^n$,
$$|\xi^\alpha| \le |\xi|^{|\alpha|} \le (1+|\xi|^2)^{\frac{k}{2}} \tag{7.1}$$

and that for every $f \in H^{k,2}$, we automatically have $f \in L^2$. It follows that

$$\|f\|_{W^{k,2}} \leq \sum_{|\alpha|\leq k} \|\mathcal{F}D^\alpha f\|_{L^2} \leq \sum_{|\alpha|\leq k} \|\xi^\alpha \mathcal{F}f\|_{L^2} \leq C\|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}f\|_{L^2} = C\|f\|_{H^{k,2}}. \tag{7.2}$$

Conversely, let $\rho : \mathbb{R} \to [0,1]$, $\rho \in C^\infty(\mathbb{R})$ and $\rho(s) = -\rho(-s)$ such that

$$\rho(t) = \begin{cases} 0, & t \leq \frac{1}{2} \\ 1, & t \geq 0 \end{cases}, \text{ so that } \frac{(1+|\xi|^2)^{\frac{k}{2}}}{1+\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k} \underset{|\xi|\to 0,\infty}{\longrightarrow} 1 .$$

It follows that

$$(1+|\xi|^2)^{\frac{k}{2}} \leq C_{n,k}\left(1+\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k\right). \tag{7.3}$$

We get that

$$\begin{aligned}
\|f\|_{H^{k,2}} &= \|(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}f(\xi)\|_{L^2} \\
&= \|\frac{(1+|\xi|^2)^{\frac{k}{2}}}{1+\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k}\left(1+\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k\right)\mathcal{F}f(\xi)\|_{L^2} \\
&\leq C_{n,k}\left(\|\mathcal{F}f\|_{L^2} + \sum_{i=1}^n \|\rho^k(\xi_i)\xi_i^k \mathcal{F}f(\xi)\|_{L^2}\right) \\
&\leq C_{n,k}\left(\|\mathcal{F}f\|_{L^2} + \sum_{i=1}^n \|(-i)^k \mathcal{F}(\frac{\partial^k}{\partial_i x^k} f)(x)\|_{L^2}\right) \\
&\leq C\|f\|_{W^{k,2}}.
\end{aligned}$$

Now we look at the case $p \neq 2$ and introduce a preliminary concept, without proofs.

Definition 7.3. Let $1 \leq p \leq \infty$ and $m \in L^\infty(\mathbb{R}^n;\mathbb{C})$. Then m is called a *Fourier multiplier,* $m \in \mathfrak{M}_p$, if there is a constant $c > 0$, such that for all $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\|\mathcal{F}^{-1}m\mathcal{F}f\|_{L^p} \leq c\|f\|_{L^p},$$

so that the operator

$$T_m : \mathcal{S}(\mathbb{R}^n) \to C_0(\mathbb{R}^n),\ f \mapsto \mathcal{F}^{-1}m\mathcal{F}f$$

can be extended to a bounded operator on L^p.

Theorem 7.4. (Mikhlin-Hörmander)
Let $1 < p < \infty$. Then $m \in C^{\lfloor \frac{n}{2}\rfloor+1}(\mathbb{R}^n\backslash\{0\})$ is a Fourier multiplier in $\mathfrak{M}_p$ if there is a constant $c_m > 0$ such that

$$|\xi|^{|\beta|}|D^\beta m(\xi)| \leq c_m$$

for all $\xi \in \mathbb{R}^n\backslash\{0\}$ and multi-indices $|\beta| \leq \lfloor \frac{n}{2}\rfloor + 1$.

Proof. For example, [3], Theorem 6.1.6 for a proof and a more general version of the theorem. □

Theorem 7.5. *Let $1 < p < \infty$, $k \in \mathbb{N}_0$, $s \in \mathbb{R}$. Then $H^{s,p}(\mathbb{R}^n)$ is a Banach space and we have*

$$W^{k,p}(\mathbb{R}^n) = H^{k,p}(\mathbb{R}^n).$$

Proof. The idea is as in the case $p = 2$, only we use Theorem 7.4, too. We know that

$$\|f\|_{W^{k,p}} = \sum_{|\alpha|\le k} \|D^\alpha(\mathcal{F}^{-1}\mathcal{F}f)\|_{L^p} = \sum_{|\alpha|\le k} \|\mathcal{F}^{-1}(\xi^\alpha \mathcal{F}f(\xi))\|_{L^p}.$$

It follows that $\|f\|_{W^{k,p}} \le C\|f\|_{H^{k,p}}$, if we can show that

$$\|\mathcal{F}^{-1}(\xi^\alpha \mathcal{F}f(\xi))\|_{L^p} \le C_{n,k}\|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{k}{2}}\mathcal{F}f\|_{L^p}$$

for all α. We define

$$m_{\alpha,k}(\xi) = \frac{\xi^\alpha}{(1+|\xi|^2)^{\frac{k}{2}}}$$

for $|\alpha| \le k$ and show that it is a Fourier multiplier. For $|\alpha| = 0$, we have $|m_{\alpha,k}(\xi)| \le 1$ by (7.1). If $|\alpha| = 1$, we get

$$\begin{aligned} |\xi|\left|\frac{\partial}{\partial\xi_j}\frac{\xi^\alpha}{(1+|\xi|^2)^{k/2}}\right| &= |\xi|\left|\frac{\alpha_j\xi_1^{\alpha_1}\cdot\dots\cdot\xi_j^{\alpha_j-1}\cdot\dots\cdot\xi_n^{\alpha_n}(1+|\xi|^2)^{k/2} - \xi_j k(1+|\xi|^2)^{k/2-1}}{(1+|\xi|^2)^k}\right| \\ &= \frac{|\xi|}{(1+|\xi|^2)^{k/2}}\left|\alpha_j - \frac{\xi_j k}{(1+|\xi|^2)}\right| \le 2k. \end{aligned}$$

By more similar calculations and induction, the claim can be proved. It follows that

$$\begin{aligned} \|\mathcal{F}^{-1}\xi^\alpha\mathcal{F}f\|_{L^p} &= \|\mathcal{F}^{-1}m_{\alpha,k}(1+|\xi|^2)^{k/2}\mathcal{F}f\|_{L^p} \\ &= \|\mathcal{F}^{-1}m_{\alpha,k}\mathcal{F}\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F}f\|_{L^p} \\ &\le C_{n,k}\|\mathcal{F}^{-1}(1+|\xi|^2)^{k/2}\mathcal{F}f\|_{L^p} \end{aligned}$$

for every $f \in L^p$.
For the converse estimate, we define

$$\tilde{m}_{\rho,k}(\xi) = \frac{(1+|\xi|^2)^{k/2}}{1+\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k}$$

and show that it is a Fourier multiplier. For $|\alpha| = 0$, we have $|\tilde{m}_{\rho,k}(\xi)| \le C_{n,k}$ by (7.3). For $|\alpha| = 1$, we use the notation $1+\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k =: \tilde{\rho}$, $(1+|\xi|^2) =: M_\xi$ and calculate:

$$\begin{aligned} |\xi|\left|\frac{\partial}{\partial\xi_j}\tilde{m}_{\rho,k}(\xi)\right| &= |\xi|\left|\frac{\xi_j k M_\xi^{k/2-1}\tilde{\rho} - M_\xi^{k/2}[k\rho^{k-1}(\xi_j)\rho'(\xi_j)\xi_j^k + \rho^k(\xi_j)k\xi_j^{k-1}]}{\tilde{\rho}^2}\right| \\ &\le C_{n,k}|\xi|\frac{M_\xi^{k/2-1}}{\tilde{\rho}^2}k\left|M_\xi^{(k+1)/2} - M_\xi\left(C_k + M_\xi^{(k-1)/2}\right)\right| \\ &\le C_{n,k}M_\xi^{1/2}\frac{M_\xi^{k/2-1}}{\tilde{\rho}^2}M_\xi^{k/2+1/2} \le C_{n,k}\tilde{m}_{\rho,k}^2(\xi) \le C_{n,k}. \end{aligned}$$

Again, we could calculate similarly for $|\alpha| > 1$ and use induction to get the claim. We now use that $\xi \mapsto \rho^k(\xi_j)(-i)^k$ is a Fourier multiplier for every $1 \le j \le n$, to get that

$$\begin{aligned}
\|f\|_{H^{k,p}} &\le \|\mathcal{F}^{-1}\tilde{m}_{\rho,k}\tilde{\rho_k}\mathcal{F}f\|_{L^p} \le C_{\tilde{m}}\|\mathcal{F}^{-1}\tilde{\rho_k}\mathcal{F}f\|_{L^p} \\
&\le C(\|f\|_{L^p} + \|\mathcal{F}^{-1}(\sum_{i=1}^n \rho^k(\xi_i)\xi_i^k)\mathcal{F}f\|_{L^p}) \\
&= C(\|f\|_{L^p} + \|\mathcal{F}^{-1}(\sum_{i=1}^n \rho^k(\xi_i)(-i)^k\mathcal{F}(\partial_i^k f))\|_{L^p}) \\
&\le C(\|f\|_{L^p} + \sum_{i=1}^n \|\partial_i^k f\|_{L^p}) \le C\|f\|_{W^{k,p}}.
\end{aligned}$$

□

We will come back to Bessel potential spaces, but now want to define and consider Besov spaces.

Definition 7.6. A sequence of functions $(\varphi_j)_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ is called *dyadic partition of unity*, if and only if

1. $\operatorname{supp}\varphi_j \subset \overline{A}_j$, $j \in \mathbb{N}$, where
$$\begin{aligned}
A_0 &= B_2(0), \\
A_j &= \{x \in \mathbb{R}^n : 2^{j-1} < |x| < 2^{j+1}\}, \quad j \neq 0,
\end{aligned}$$
2. $\sum_{j=0}^\infty \varphi_j(x) = 1$,
3. for all multi-indices $\gamma = (\gamma_1, ..., \gamma_n) \in \mathbb{N}_0^n$, there is a constant c_γ, such that $2^{j|\gamma|}|D^\gamma\varphi_j(x)| \le c_\gamma$.

Example: If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset B_2(0)$ and $\varphi(x) = 1$ for $|x| \le 1$, then setting $\varphi_0 = \varphi$ and $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$ for $j \neq 0$ yields a dyadic partition of unity.

Definition 7.7. (Besov spaces)
Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $(\varphi_j)_{j\in\mathbb{N}}$ a dyadic partion of unity. Then the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are the spaces of all functions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} := \begin{cases} \left(\sum_{j=0}^\infty 2^{jsq}\|\mathcal{F}^{-1}\varphi_i\mathcal{F}f\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q}, & q < \infty \\ \sup_{j\in\mathbb{N}_0} 2^{js}\|\mathcal{F}^{-1}\varphi\mathcal{F}f\|_{L^p(\mathbb{R}^n)}, & q = \infty \end{cases} < \infty.$$

Remark 7.8. We will show that $(L^p, W^{m,p})_{\theta,q} = B^{\theta m}_{p,q}$, so that in particular, $B^s_{p,p} = W^s_p$.

Proposition 7.9. $B^s_{p,q}(\mathbb{R}^n)$ *is a Banach space and choosing two different dyadic partitions yields equivalent norms.*

Proof. Exercise.

In order to get interpolation results for Besov spaces, we start from the following type of spaces. □

Definition 7.10. Let X be a Banach space, $\sigma \in \mathbb{R}$ and $1 \leq p \leq \infty$. Then $l_p^\sigma(X)$ is the space of sequences $(x_j)_{j\in\mathbb{N}} \subset X$ such that

$$\|x\|_{l_p^\sigma(X)} := \begin{cases} \left(\sum_{j=0}^\infty 2^{j\sigma p}\|x_j\|_X^p\right)^{1/p}, & p < \infty \\ \sup_{j\in\mathbb{N}_0} 2^{j\sigma}\|x_j\|_X, & p = \infty \end{cases} < \infty.$$

Remark 7.11. It is a Banach space. If $X = \mathbb{C}$ and $\sigma = 0$, then $l_p^\sigma(X) = l^p$. Moreover, if $1 \leq r \leq p \leq \infty$, then $l_r^\sigma \hookrightarrow l_p^\sigma$.

Proof. Exercise. □

Theorem 7.12. *Let X be a Banach space, $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$, $1 \leq p_0, p_1, p \leq \infty$ and $0 < \theta < 1$. Then we get*

$$(l_{p_0}^{s_0}(X), l_{p_1}^{s_1}(X))_{\theta,p} = l_p^s(X),$$

where $s = (1-\theta)s_0 + \theta s_1$.

Proof. Use Theorem 2.5.6 and show that

$$(l_\infty^{s_0}, l_\infty^{s_1})_{\theta,p} \hookrightarrow l_p^s \hookrightarrow (l_1^{s_0}, l_1^{s_1})_{\theta,p} \tag{7.4}$$

to get that

$$l_p^s \hookrightarrow (l_1^{s_0}, l_1^{s_1})_{\theta,p} \hookrightarrow (l_{p_0}^{s_0}, l_{p_1}^{s_1})_{\theta,p} \hookrightarrow (l_\infty^{s_0}, l_\infty^{s_1})_{\theta,p} \hookrightarrow l_p^s.$$

We now first show the first embedding in (7.4). We know that

$$K(t, x, l_\infty^{s_0}, l_\infty^{s_1}) = \inf_{x=x^0+x^1}(\sup_j 2^{js_0}\|x_j^0\|_X + t\sup_j 2^{js_1}\|x_j^1\|_X)$$

for all $x \in l_\infty^{s_0} + l_\infty^{s_1} = l_\infty^{\min(s_0,s_1)}$. We look at

$$\hat{x}_j^0 = \begin{cases} x_j, & 2^{js_0} \leq t2^{js_1}, \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } \hat{x}^1 = x - \hat{x}^0,$$

to see that

$$K(t, x, l_\infty^{s_0}, l_\infty^{s_1}) \leq 2\sup_j \min(2^{js_0}, t2^{js_1})\|x_j\|_X.$$

On the other hand, clearly,

$$\sup_j \min(2^{js_0}, t2^{js_1})\|x_j\| \quad \leq \quad \sup_j(2^{js_0}\|x_j^0\|_X + t2^{js_1}\|x_j^1\|_X)$$

for all decompositions $x = x^0 + x^1$, so that

$$K(t, x, l_\infty^{s_0}, l_\infty^{s_1}) \sim \sup_j \min(2^{js_0}, t2^{js_1}) \|x_j\|_X.$$

Wlog, we assume that $s_0 > s_1$, since we could consider $(l_{p_1}^{s_1}, l_{p_0}^{s_0})_{1-\theta,q}$ instead. We treat the case when $p < \infty$ and use the decomposition $(0, \infty) = \cup_{k=-\infty}^{\infty} [2^{(k-1)(s_0-s_1)}, 2^{k(s_0-s_1)})$. For $x \in (l_\infty^{s_0}, l_\infty^{s_1})_{\theta,p}$,

$$\begin{aligned}
\|x\|^p_{(l_\infty^{s_0}, l_\infty^{s_1})_{\theta,p}} &= \sum_{k=-\infty}^{\infty} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} t^{-\theta p} K(t, x, l_\infty^{s_0}(X), l_\infty^{s_1}(X))^p \,\frac{\mathrm{d}t}{t} \\
&\geq \sum_{k=-\infty}^{\infty} 2^{-\theta p k(s_0-s_1)} \sup_j \min(2^{js_0 p}, 2^{kp(s_0-s_1)} 2^{js_1 p}) \|x_j\|_X^p \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} \frac{\mathrm{d}t}{t} \\
&\geq C \sum_{k=-\infty}^{\infty} 2^{-\theta p k(s_0-s_1)} \min(2^{ks_0 p}, 2^{kp(s_0-s_1)} 2^{ks_1 p}) \|x_k\|_X^p (s_0 - s_1) \ln(2) \\
&= C \sum_{k=0}^{\infty} 2^{kps} \|x_k\|_X^p = C \|x\|^p_{l_p^s}
\end{aligned}$$

The case $p = \infty$ follows analogously.
Now we show the second embedding in (7.4). Again, we may assume that $s_0 > s_1$. Analogously to the l_∞-situation, we see that

$$K(t, x, l_1^{s_0}, l_1^{s_1}) \sim \sum_{j=0}^{\infty} \min(2^{js_0}, t2^{js_1}) \|x_j\|_X.$$

For $p < \infty$ and $x \in l_p^s$, we calculate

$$\begin{aligned}
\|x\|^p_{(l_1^{s_0}, l_1^{s_1})_{\theta,p}} &\leq \sum_{k=-\infty}^{\infty} \int_{2^{(k-1)(s_0-s_1)}}^{2^{k(s_0-s_1)}} 2^{-\theta p(k-1)(s_0-s_1)} \left[\sum_{j=0}^{\infty} \min(2^{js_0}, 2^{k(s_0-s_1)+js_1}) \|x_j\|_X \right]^p \frac{\mathrm{d}t}{t} \\
&\leq C 2^{\theta p(s_0-s_1)} \sum_{k=-\infty}^{\infty} 2^{kps} \left[\sum_{j=0}^{\infty} \min(2^{(j-k)s_0}, 2^{(j-k)s_1}) \|x_j\|_X \right]^p \\
&= C \sum_{k=-\infty}^{\infty} 2^{kps} \left[\sum_{j=-\infty}^{k} 2^{(j-k)s_0} \|x_j\|_X + \sum_{j=k+1}^{\infty} 2^{(j-k)s_1} \|x_j\|_X \right]^p.
\end{aligned}$$

For every t_0, t_1 such that $s_0 > t_0 > s > t_1 > s_1$, we have

$$\begin{aligned}\sum_{j=-\infty}^{k} 2^{(j-k)s_0}\|x_j\|_X &\leq 2^{-ks_0}\left(\sum_{j=-\infty}^{k} 2^{j(s_0-t_0)p'}\right)^{1/p'}\left(\sum_{j=-\infty}^{k} 2^{jt_0p}\|x_j\|_X^p\right)^{1/p}\\ &\leq C2^{-kt_0}\left(\sum_{j=-\infty}^{k} 2^{jt_0p}\|x_j\|_X^p\right)^{1/p},\\ \sum_{j=k+1}^{\infty} 2^{(j-k)s_1}\|x_j\|_X &\leq 2^{-ks_1}\left(\sum_{j=k+1}^{\infty} 2^{j(s_1-t_1)p'}\right)^{1/p'}\left(\sum_{j=k+1}^{\infty} 2^{jt_1p}\|x_j\|_X^p\right)^{1/p}\\ &\leq C2^{-kt_1}\left(\sum_{j=k+1}^{\infty} 2^{jt_1p}\|x_j\|_X^p\right)^{1/p},\end{aligned}$$

so that

$$\begin{aligned}\|x\|^p_{(l_1^{s_0},l_1^{s_1})_{\theta,p}} &\leq C\sum_{k=-\infty}^{\infty} 2^{kps}\left[2^{-kt_0p}\sum_{j=-\infty}^{k} 2^{jt_0p}\|x_j\|_X^p + 2^{-kt_1p}\sum_{j=k+1}^{\infty} 2^{jt_1p}\|x_j\|_X^p\right]\\ &\leq C\sum_{j=-\infty}^{\infty} 2^{jsp}\|x_j\|_X^p = C\|x\|^p_{l_p^s}.\end{aligned}$$

This proves the theorem. □

We now summarize several results on real and complex interpolation of Besov and Bessel potential spaces in the following theorem, and then set out to give a proof for the "real" part, 1.-4. and some arguments for the "complex" part, 5. and 6.

Theorem 7.13. *Let $0 < \theta < 1$, $s, s_0 \neq s_1 \in \mathbb{R}$, $1 \leq p, p_0, p_1, q, q_0, q_1 \leq \infty$ and*

$$s_\theta = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then we get

1. $(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} = B^{s_\theta}_{p,q}$,
2. $(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{\theta,p_\theta} = B^{s_\theta}_{p_\theta,q_\theta}$, $\quad p_\theta = q_\theta, q_0, q_1 \neq 1$
3. $(H^{s_0,p}, H^{s_1,p})_{\theta,q} = B^{s_\theta}_{p,q}$, $\quad 1 < p < \infty$,
4. $(H^{s,p_0}, H^{s,p_1})_{\theta,p_\theta} = H^{s,p_\theta}$, $\quad 1 < p_0, p_1 < \infty$,
5. $[B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1}]_\theta = B^{s_\theta}_{p_\theta,q_\theta}$,

6. $[H^{s_0,p_0}, H^{s_1,p_1}]_\theta = H^{s_\theta,p_\theta}, \quad 1 < p_0, p_1 < \infty.$

Let $(\psi_i)_i$ be a dyadic partition of unity. To show 1., by looking at the map

$$S_\psi : B^s_{p,q} \to l^s_q(L^p),\ f \mapsto (\mathcal{F}^{-1}\psi_j\mathcal{F}f)_j,$$

we get the idea that we can interpolate $B^s_{p,q}$ as the we interpolate $l^s_q(L^p)$ by Theorem 7.12. The following notion is helpful.

Definition 7.14. An object X in a category is called a *retract* of another object Y, if there are morphisms $S : X \to Y$ and $R : Y \to X$ in the category, such that $R \circ S = \mathrm{id}_X$. In this case, the map R is called *retraction*, and S is called *coretraction.*

Clearly, the following holds true.

Lemma 7.15. *If $\{X_0, X_1\}$ is a retract of $\{Y_0, Y_1\}$ in $\mathfrak{C}_2$ with retraction R and coretraction S, then for every interpolation functor $\mathcal{F}$, $\mathcal{F}(\{X_0, X_1\})$ is a retract of $\mathcal{F}(\{Y_0, Y_1\})$ with retraction $\mathcal{F}(R) = R$ and coretraction S.*

Proposition 7.16. *The space $B^s_{p,q}$ is a retract of $l^s_q(L^p)$ with the coretraction S and the retraction*

$$R_\varphi : l^s_q(L^p) \to B^s_{p,q}, \quad (x_j)_j \mapsto \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j\mathcal{F}x_j.$$

Proof. Clearly, $S_\psi \in \mathcal{L}(B^s_{p,q}, l^s_q(L^p))$. On the other hand,

$$\begin{aligned} \|R_\varphi(x_j)_j\|_{B^s_{p,q}} &= \left(\sum_{j=0}^{\infty} 2^{jsq}\|\mathcal{F}^{-1}\psi_j\mathcal{F}(\sum_{k=0}^{\infty}\mathcal{F}^{-1}\varphi_k\mathcal{F}x_j)\|^q_{L^p}\right)^{1/q} \\ &\le c\left(\sum_{j=0}^{\infty} 2^{jsq}\|x_j\|^q_{L^p}\right)^{1/q} \end{aligned}$$

if we use that $\psi_j\varphi_{j-1}$, $\psi_j\varphi_j$ and $\psi_j\varphi_{j+1}$ are Fourier multipliers on L^p. Clearly,

$$R_\varphi S_\psi f = \sum_{j=0}^{\infty}\mathcal{F}^{-1}\varphi_j\psi_j\mathcal{F}f = \sum_{j=0}^{\infty}\mathcal{F}^{-1}\psi_j\mathcal{F}f = f.$$

□

Theorem 7.13.1 now follows from Theorem 7.12 and Proposition 7.16.
Next, we want to show 2.

Proposition 7.17. *For $1 < q_0, q_1 \le \infty$ and an interpolation couple $\{X_0, X_1\}$, we have*

$$(l^{s_0}_{q_0}(X_0), l^{s_1}_{q_1}(X_1))_{\theta,q_\theta} = l^{s_\theta}_{q_\theta}((X_0, X_1)_{\theta,q_\theta}).$$

Proof. Unpopular but elementary exercise. □

We get that by this Proposition and Theorem 6.7,

$$(B^{s_0}_{p_0,q_0}, B^{s_1}_{p_1,q_1})_{\theta,p_\theta} = (l^{s_0}_{q_0}(L^{p_0}), l^{s_1}_{,q_1}(L^{p_1}))_{\theta,p_\theta} = l^{s_\theta}_{q_\theta}((L^{p_0}, L^{p_1})_{\theta,p_\theta}) = l^{s_\theta}_{q_\theta}(L^{p_\theta}) = B^{s_\theta}_{q_\theta,p_\theta},$$

which is 2.
In order to show 3., we use the following embeddings.

Proposition 7.18. *For $1 < p < \infty$ and $s \in \mathbb{R}$, we have*

$$B^s_{p,1} \hookrightarrow H^{s,p} \hookrightarrow B^s_{p,\infty}, \tag{7.5}$$

and in particular,

$$B^0_{p,1} \hookrightarrow L^p \hookrightarrow B^0_{p,\infty}. \tag{7.6}$$

Proof. We note that (7.5) follows from (7.6) if we use that the operator $J^\sigma = \mathcal{F}^{-1}(1 + |\xi|^2)^{\sigma/2}\mathcal{F}$ is an isomorphism from $H^{s,p}$ to $H^{s-\sigma,p}$ and from $B^s_{p,q}$ to $B^{s-\theta}_{p,q}$. For the first embedding in (7.6), let $f \in B^0_{p,1}$. We can write

$$f = \sum_{j=0}^{\infty} \mathcal{F}^{-1}\varphi_j \mathcal{F} f,$$

to see that

$$\|f\|_{L^p} \leq \sum_{j=0}^{\infty} \|\mathcal{F}^{-1}\varphi_j \mathcal{F} f\|_{L^p} = \|f\|_{B^0_{p,1}}.$$

For the second embedding, let $(\varphi_j)_j$ as in the example above Definition 7.7. For $f \in \mathcal{S}$, we get

$$\begin{aligned}
\mathcal{F}^{-1}\varphi_j \mathcal{F} f(x) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi_j(\xi) \int_{\mathbb{R}^n} e^{-iy\xi} f(y)\,\mathrm{d}y\,\mathrm{d}\xi \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} f(x-z) \int_{\mathbb{R}^n} e^{iz\xi}\varphi_j(\xi)\,\mathrm{d}\xi\,\mathrm{d}z \\
&= 2^{(j-1)n}(2\pi)^{-n} \int_{\mathbb{R}^n} f(x-z) \int_{\mathbb{R}^n} e^{iz2^{j-1}\eta}\varphi_j(2^{j-1}\eta)\,\mathrm{d}\eta\,\mathrm{d}z \\
&= 2^{(j-1)n}(2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x-z)\mathcal{F}^{-1}\varphi_1(2^{j-1}z)\,\mathrm{d}z.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|\mathcal{F}^{-1}\varphi_j \mathcal{F} f\|_{L^p} &\leq C2^{(j-1)n}\|f\|_{L^p} \int_{\mathbb{R}^n} |\mathcal{F}^{-1}\varphi_1(2^{j-1}z)|\,\mathrm{d}z \\
&= C\|f\|_{L^p}\|\mathcal{F}^{-1}\varphi_1\|_{L^1}.
\end{aligned}$$

In conclusion,

$$\|f\|_{B^0_{p,\infty}} = \sup_{j\in\mathbb{N}_0} \|\mathcal{F}^{-1}\varphi_j \mathcal{F} f\|_{L^p} \leq C\|f\|_{L^p},$$

which shows the second embedding in (7.6). □

By 1., the above Proposition and Theorem 2.5.6, we have

$$B^{s_\theta}_{p,q} = (B^{s_0}_{p,1}, B^{s_1}_{p,1})_{\theta,q} \hookrightarrow (H^{s_0,p}, H^{s_1,p})_{\theta,q} \hookrightarrow (B^{s_0}_{p,\infty}, B^{s_1}_{p,\infty})_{\theta,q} = B^{s_\theta}_{p,q}.$$

This proves 3. Note that in particular, $B^{\theta k}_{p,q} = (L^p, W^{k,p})_{\theta,q}$ by Theorem 7.5, so that $B^s_{p,p} = W^{s,p}$ if $s \notin \mathbb{Z}$.
Regarding 4., we know that $(L^{p_0}, L^{p_1})_{\theta,p_\theta} = L^{p_\theta}$ by Theorem 6.7. The claim follows if we again use the fact that $J^s : H^{s,p} \to L^p$ is an isomorphism.
For 5., the idea is to use Proposition 7.16 and the complex interpolation result

$$[l^{s_0}_{q_0}(X_0), l^{s_1}_{q_1}(X_1)]_\theta = l^{s_\theta}_{q_\theta}([X_0, X_1]_\theta).$$

For 6., we have to work much harder. The idea is to show that $H^{s,p}$ is a retract of the space $L^p(l^s_2)$, which involves defining the Fourier transform for Hilbert space-valued functions and a corresponding Mikhlin multiplier theorem and then use the Riesz-Thorin-type result

$$[L^{p_0}(X_0), L^{p_1}(X_1)]_\theta = L^{p_\theta}([X_0, X_1]_\theta).$$

Reminder: Sobolev embedding theorem:

$$W^{k,p} \hookrightarrow L^q, \quad k - \frac{n}{p} \geq -\frac{n}{q}, k - \frac{n}{p} > 0.$$

Remark, without proof: We have

$$B^s_{p,p} \hookrightarrow H^{s,p} \hookrightarrow B^s_{p,2}, \quad 1 < p \leq 2,$$

but

$$B^s_{p,2} \hookrightarrow H^{s,p} \hookrightarrow B^s_{p,p}, \quad 2 \leq p < \infty. \tag{7.7}$$

For Besov and Bessel potential spaces, we get the following embedding results.

Theorem 7.19. *Assume that*

$$s - \frac{n}{p} = s_1 - \frac{n}{p_1},$$

then

$$B^s_{p,q} \hookrightarrow B^{s_1}_{p_1,q_1} \quad 1 \leq p \leq p_1 \leq \infty, 1 \leq q \leq q_1 \leq \infty, s, s_1 \in \mathbb{R},$$

and

$$H^{s,p} \hookrightarrow H^{s_1,p_1}, \quad 1 < p \leq p_1 < \infty, s, s_1 \in \mathbb{R}.$$

In particular, $W^{s,p} \hookrightarrow W^{s_1,p_1}$.

Proof. We prove the first embedding directly and then use interpolation to get the second. It suffices to show the estimate

$$\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^{p_1}} \leq C2^{nk(\frac{1}{p}-\frac{1}{p_1})}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p} \tag{7.8}$$

to get that

$$\begin{aligned}\|f\|_{B^{s_1}_{p_1,q_1}} &= \left(\sum_{k=0}^{\infty} 2^{ks_1q_1}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|^{q_1}_{L^{p_1}}\right)^{1/q_1}\\ &\leq C\left(\sum_{k=0}^{\infty} 2^{ksq}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|^{q}_{L^{p}}\right)^{1/q}\\ &\leq C\|f\|_{B^{s}_{p,q}},\end{aligned}$$

as $q_1 \geq q$. We show (7.8).

Reminder: Young's inequality. Let $k \in L^\rho$, $f \in L^p$, $1 < p < \rho'$, $\frac{1}{p_1} = \frac{1}{p} - \frac{1}{\rho'}$, then

$$\|k * f\|_{L^{p_1}} \leq \|k\|_{L^\rho}\|f\|_{L^p}.$$

Proof: Riesz-Thorin....

We write: $\mathcal{F}^{-1}\varphi_k\mathcal{F}f = \mathcal{F}^{-1}(\varphi_k \cdot \hat{f}) = \check{\varphi}_k * f \in L^p$ and

$$\begin{aligned}\mathcal{F}^{-1}\varphi_k\mathcal{F}f &= \sum_{j=0}^{\infty}\mathcal{F}^{-1}\varphi_k\varphi_j\mathcal{F}f\\ &= \sum_{j=k-1}^{k+1}\mathcal{F}^{-1}\varphi_j\varphi_k\mathcal{F}f\\ &= \sum_{j=k-1}^{k+1}\check{\varphi}_j * (\check{\varphi}_k * f).\end{aligned}$$

By Young's inequality,

$$\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^{p_1}} \leq \sum_{j=k-1}^{k+1}\|\check{\varphi}_j\|_{L^\rho}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p}. \tag{7.9}$$

We now estimate $\|\check{\varphi}_j\|_{L^\rho}$. Note that by the example above Definition 7.6, we can write $\varphi_j(x) = \varphi(2^{-j}x) - \varphi(2^{-j+1}x)$. It follows that

$$\begin{aligned}\check{\varphi}_j(\xi) &= \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(2^{-j}x)\,\mathrm{d}x - \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(2^{-j+1}x)\,\mathrm{d}x\\ &= 2^{jn}\check{\varphi}(2^j\xi) - 2^{(j-1)n}\check{\varphi}(2^{j-1}\xi).\end{aligned}$$

We directly get

$$\begin{aligned}\|\check{\varphi}_j\|_{L^\rho} &\leq \left(\int_{\mathbb{R}^n}(2^{jn}\check{\varphi}(2^j\xi))^\rho\,\mathrm{d}\rho\right)^{1/\rho} + \left(\int_{\mathbb{R}^n}(2^{(j-1)n}\check{\varphi}(2^{(j-1)}\xi))^\rho\,\mathrm{d}\rho\right)^{1/\rho}\\ &= 2^{jn(1-1/\rho)}\|\check{\varphi}\|_{L^\rho} + 2^{(j-1)n(1-1/\rho)}\|\check{\varphi}\|_{L^\rho}\\ &\leq 2\cdot 2^{jn(1/p-1/p_1)}\|\check{\varphi}\|_{L^\rho}.\end{aligned}$$

This implies (7.8) via (7.9).

It remains to prove the second embedding. It suffices to consider the case $s_1 = 0$ and then apply the isomorphism J^{s_1}. For every $1 < p \le p_1$, by definition, we see that $B^{\tilde{s}}_{p,q} \hookrightarrow B^r_{p,q}$ if $\tilde{s} \ge r$. If we set

$$\tilde{s} = s - \frac{n}{p} - \frac{n}{p_1}$$

and moreover use the first embedding and (7.6), we get

$$B^s_{p,1} \hookrightarrow B^{\tilde{s}}_{p_1,1} \hookrightarrow B^0_{p_1,1} \hookrightarrow L^{p_1}.$$

Now choose $0 < \theta < 1$ and $s' < s$. We can then define s'', p_1' and p_1'' via

$$\begin{aligned} s &= (1-\theta)s' + \theta s'', \\ s' - \frac{n}{p} &= -\frac{n}{p_1'}, \\ s'' - \frac{n}{p} &= -\frac{n}{p_1''}, \end{aligned}$$

to get

$$\begin{aligned} B^{s'}_{p,1} &\hookrightarrow L^{p_1'}, \\ B^{s''}_{p,1} &\hookrightarrow L^{p_1''}. \end{aligned}$$

By 7.13.1, Theorem 2.5.6 and Theorem 6.7, this implies

$$B^s_{p,\infty} = (B^{s'}_{p,1}, B^{s''}_{p,1})_{\theta,\infty} \hookrightarrow (L^{p_1'}, L^{p_1''})_{\theta,\infty} \hookrightarrow L^{p_1,\infty}.$$

From (7.5), it follows that

$$H^{s,p} \hookrightarrow B^s_{p,\infty} \hookrightarrow L^{p_1,\infty}.$$

Now choose again θ and $1 < q_1' < p_1$. Define q' by $s - \frac{n}{q'} = -\frac{n}{q_1'}$, so that $1 < q' \le q_1'$. We can then define q'' and q_1'' via

$$\begin{aligned} \frac{1}{p} &= \frac{1-\theta}{q'} + \frac{\theta}{q''}, \\ \frac{1}{p_1} &= \frac{1-\theta}{q_1'} + \frac{\theta}{q_1''} \end{aligned}$$

and obtain $s - \frac{n}{q''} = -\frac{n}{q_1''}$. As above, we can get the inclusions

$$\begin{aligned} H^{s,q'} &\hookrightarrow L^{q_1',\infty}, \\ H^{s,q''} &\hookrightarrow L^{q_1'',\infty}, \end{aligned}$$

and therefore, by Theorem 7.13.4, Theorem 2.5.6 and Theorem 6.7,

$$H^{s,p} = (H^{s,q'}, H^{s,q''})_{\theta,p} \hookrightarrow (L^{q_1',\infty}, L^{q_1'',\infty})_{\theta,p} = L^{p_1,p}.$$

Since $p \le p_1$, we have $L^{p_1,p} \hookrightarrow L^{p_1,p_1} = L^{p_1}$, which proves the claim. □

8 A Trace Theorem

We consider the trace operator

$$T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^{n-1}),$$

given by

$$Tf(x') = f(0, x'), \quad x' = (x_2, \ldots, x_n).$$

We want to show the following.

Theorem 8.1. *Let $1 < p < \infty$, $1 \leq q \leq \infty$, $s > 1/p$. Then T can be extended to a bounded operator*

$$T : B^s_{p,q}(\mathbb{R}^n) \to B^{s-1/p}_{p,q}(\mathbb{R}^{n-1}), \tag{8.1}$$

$$T : H^{s,p}(\mathbb{R}^n) \to B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}). \tag{8.2}$$

Proof. First note that $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^{s,p}(\mathbb{R}^n)$ and $B^s_{p,q}(\mathbb{R}^n)$ as long as $1 \leq p, q < \infty$, $s \in \mathbb{R}_+$. We know that $\mathcal{S}$ is dense in L^p. Given $f \in H^{s,p}$, we can approximate in L^p by g_n and therefore we can approximate in $H^{s,p}$ by $J^{-s} g_n \in \mathcal{S}$. Density in $B^s_{p,q}$ then follows from Theorem 7.13.3. Thus, in the following, we consider functions in $\mathcal{S}(\mathbb{R}^n)$ and show the norm estimates.

The Theorem holds for all $s > 1/p$. We will only give a proof for $1 > s > 1/p$, which is the crucial part.

We split the proof into several steps. Step 2 is the crucial one. Its proof will be given further below, in a seperate theorem. Step 1 is a trace theorem we may obtain from the trace method. The remaining steps are concerned with using interpolation results suitably.

Step 1. As an exercise, we deduce from the trace method, that $W^{1-1/p}_p(\mathbb{R}^{n-1})$ is the space of functions which are traces of $W^{1,p}(\mathbb{R}^n)$, i.e. $T : H^{1,p}(\mathbb{R}^n) \to B^{1-1/p}_{p,p}(\mathbb{R}^{n-1})$ is bounded and onto.

Step 2. We show below that $T : B^{1/p}_{p,1}(\mathbb{R}^n) \to L^p(\mathbb{R}^{n-1})$ is bounded.

Step 3. From 1 and 2, we can now deduce (8.1). We set $0 < \theta < 1$ such that $s = (1-\theta) + \frac{\theta}{p}$. Then by Theorem 7.13.1,

$$B^s_{p,q}(\mathbb{R}^n) = (B^1_{p,1}(\mathbb{R}^n), B^{1/p}_{p,1}(\mathbb{R}^n))_{\theta,q} \xrightarrow{T} (B^{1-1/p}_{p,p}(\mathbb{R}^{n-1}), L^p(\mathbb{R}^{n-1}))_{\theta,q} = B^{(1-\theta)(1-1/p)}_{p,q}(\mathbb{R}^n).$$

It remains to calculate $(1-\theta)(1-1/p) = s - \frac{\theta}{p} - (1-\theta)\frac{1}{p} = s - \frac{1}{p}$.

Step 4. By the embedding (7.7), we immediately get (8.2) if $2 \le p < \infty$:

$$H^{s,p}(\mathbb{R}^n) \hookrightarrow B^s_{p,p}(\mathbb{R}^n) \xrightarrow{T} B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}).$$

Step 5. It remains to show (8.2) in the case $1 < p \le 2$. We know that

$$\begin{aligned} T &: H^{s_0,2}(\mathbb{R}^n) \to B^{s_0-1/2}_{2,2}(\mathbb{R}^{n-1}), \qquad s_0 > 1/2, \\ T &: H^{1,p_1}(\mathbb{R}^n) \to B^{s-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1}), \qquad 1 < p_1 < \infty. \end{aligned}$$

From the complex embeddings, Theorem 7.8.5 and 7.8.6, for $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1}$ and $s = (1-\theta)s_0 + \theta$, we get

$$\begin{aligned} T : H^{s,p}(\mathbb{R}^n) &= [H^{s_0,2}(\mathbb{R}^n), H^{1,p_1}(\mathbb{R}^n)]_\theta \\ &\xrightarrow{T} [B^{s_0-1/2}_{2,2}(\mathbb{R}^{n-1}), B^{1-1/p_1}_{p_1,p_1}(\mathbb{R}^{n-1})]_\theta \\ &= B^{(1-\theta)(s_0-1/2)+\theta(1-1/p_1)}_{\frac{1}{\frac{1-\theta}{2}+\frac{\theta}{p_1}}, \frac{1}{\frac{1-\theta}{2}+\frac{\theta}{p_1}}}(\mathbb{R}^{n-1}) \\ &= B^{s-1/p}_{p,p}(\mathbb{R}^{n-1}). \end{aligned}$$

Now if $s > 1/p$ and $1 < p \le 2$, we can choose p_1, $1 < p_1 < p$, so that

$$\begin{aligned} s_0 &= \frac{s-\theta}{1-\theta} > \frac{1/p-\theta}{1-\theta} = \frac{\frac{1-\theta}{2} + \frac{\theta}{p_1} - \theta}{1-\theta} \\ &> \frac{\frac{1-\theta}{2} + \frac{\theta}{p} - \theta}{1-\theta} \overset{p\le 2}{\ge} \frac{\frac{1}{2}-\theta}{1-\theta} \\ &= 1 - \frac{1/2}{1-\theta} > \frac{1}{2}. \end{aligned}$$

□

To really show the theorem, it now remains to show Step 2. We will give an equivalent characterization of $B^s_{p,q}(\mathbb{R}^n)$ for $s > 0$, which helps. We define the *modulus of continuity* by

$$\omega^m_p(t, f) = \sup_{|y|<t} \|\Delta^m_y f\|_{L^p},$$

where Δ^m_y is the m-th order difference operator

$$\Delta^m_y f(x) = \sum_{k=0}^m \binom{m}{k} (-1)^k f(x+ky).$$

Interesting for us is the first order:

$$\Delta^1_y f(x) = f(x) - f(x+y), \quad \omega^1_p(t,f) = \sup_{|y|<t} \|f(x) - f(x+y)\|_{L^p}.$$

The following holds.

Theorem 8.2. *Let $s > 0$, $m, N \in \mathbb{N}_0$, such that $m + N > s$ and $0 \le N < s$. Then for $1 \le p, q \le \infty$,*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|f\|_{L^p} + \sum_{j=1}^{n} \left(\int_0^\infty (t^{N-s} \omega_p^m(t, \frac{\partial^N f}{\partial x_j^N}))^q \, \frac{\mathrm{d}t}{t} \right)^{1/q}.$$

The proof follows the proof in [3], Theorem 6.2.5. Before we go into the details, we need to consider some more facts about Fourier multipliers, in particular, because we would like to cover the cases $p = 1, \infty$.

Theorem 8.3. *We let $1 \le p \le \infty$ and consider the space $\mathfrak{M}_p$ of all Fourier multipliers $\rho \in \mathcal{S}'(\mathbb{R}^n)$ with norm $\|\rho\|_{\mathfrak{M}_p} = \|T_\rho\|_{\mathcal{L}(L^p)}$, where $T_\rho f = \mathcal{F}^{-1} \rho \mathcal{F} f$. Then the following holds true.*

1. *We have $\mathfrak{M}_p = \mathfrak{M}_{p'}$ with equal norms.*
2. *$\rho \in \mathfrak{M}_\infty$ iff*
$$|\mathcal{F}^{-1} \rho \mathcal{F} f(0)| \le C \|f\|_\infty$$
for all $f \in \mathcal{S}(\mathbb{R}^n)$.
3. *If $\rho \in \mathfrak{M}_{p_0} \cap \mathfrak{M}_{p_1}$, $1 \le p_0, p_1 \le \infty$, then*
$$\|\rho\|_{\mathfrak{M}_{p_\theta}} \le \|\rho\|^{1-\theta}_{\mathfrak{M}_{p_0}} \|\rho\|^{\theta}_{\mathfrak{M}_{p_1}}.$$
4. *In particular, if $1 \le p < q \le 2$, then*
$$\mathfrak{M}_1 \hookrightarrow \mathfrak{M}_p \hookrightarrow \mathfrak{M}_q \hookrightarrow \mathfrak{M}_2.$$
5. *Let $a : \mathbb{R}^n \to \mathbb{R}^m$ be a surjective affine transformation. Then the map $\tilde{a}$ defined by*
$$(\tilde{a}\rho)(x) = \rho(a(x))$$
is isometric from $\mathfrak{M}_p(\mathbb{R}^m)$ to $\mathfrak{M}_p(\mathbb{R}^n)$ (the space of Fourier multipliers which are functions on $\mathbb{R}^n$). If $m = n$, then $\tilde{a}$ is bijective.
6. *Let $L > \frac{n}{2}$ be an integer and $\rho \in L^2(\mathbb{R}^n, \mathbb{C})$ and $D^\alpha \rho \in L^2$ for $|\alpha| = L$. Then for $1 \le p \le \infty$, $\rho \in \mathfrak{M}_p$ and*
$$\|\rho\|_{\mathfrak{M}_p} \le C \|\rho\|^{1-\theta}_{L^2} (\sup_{|\alpha| = L} \|D^\alpha \rho\|_{L^2})^\theta,$$
where $\theta = \frac{n}{2L}$.
7. *We have $\mathfrak{M}_2 = L^\infty$ and for every $\rho_1, \rho_2 \in \mathfrak{M}_p$, $\rho = \rho_1 \rho_2 \in L^\infty$ and*
$$\|\rho\|_{\mathfrak{M}_p} \le \|\rho_1\|_{\mathfrak{M}_p} \|\rho_2\|_{\mathfrak{M}_p}.$$

Proof. for 1.: For $f \in L^p$, $g \in L^{p'}$ and $\rho \in \mathfrak{M}_p$ we have

$$\begin{aligned}\int_{\mathbb{R}^n} |\mathcal{F}^{-1}\rho\mathcal{F}g(x) \cdot f(x)| \, \mathrm{d}x &= \int_{\mathbb{R}^n} \check{\rho} * g \cdot f(x) \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \check{\rho} * f \cdot g(x) \, \mathrm{d}x \\ &= \|\mathcal{F}^{-1}\rho\mathcal{F}f\|_{L^p} \|g\|_{L^{p'}} \\ &\leq \|\rho\|_{\mathfrak{M}_p} \|f\|_{L^p} \|g\|_{L^{p'}}.\end{aligned}$$

The second statement follows from the fact that the translation operators commute with T_ρ .
The third statement is a consequence of the Riesz-Thorin theorem. It directly implies the fourth statement.
Regarding 5.: Ok to see: $\mathfrak{M}_p$ isometrically invariant under non-singular linear coordinate transforms. After that, choose coordinates in a good way and look at the integrals (See [3], p. 134).
Proof of 6.: From 4. and 2. we may deduce that it suffices to show the estimate in the case $p = 1$, where

$$\|\rho\|_{\mathfrak{M}_1} = \int_{\mathbb{R}^n} |\check{\rho}(x)| \, \mathrm{d}x.$$

We consider first

$$\begin{aligned}\int_{|x|>t} |\check{\rho}(x)| \, \mathrm{d}x &\leq \int_{|x|>t} |x|^{-L} |x|^L |\check{\rho}(x)| \, \mathrm{d}x \\ &\leq Ct^{-L+n/2} \sup_{|\alpha|=L} \|D^\alpha \rho\|_{L^2}.\end{aligned}$$

Then we see

$$\int_{|x|<t} |\check{\rho}(x)| \, \mathrm{d}x \leq Ct^{n/2} \|\rho\|_{L^2}.$$

We can choose t such that $\|\rho\|_{L^2} = t^{-L} \sup_{|\alpha|=L} \|D^\alpha \rho\|_{L^2}$ to get

$$\|\rho\|_{\mathfrak{M}_p} \leq \|\rho\|_{\mathfrak{M}_1} \leq C\|\rho\|_{L^2}^{1-\theta} (\sup_{|\alpha|=L} \|D^\alpha \rho\|_{L^2})^\theta.$$

Proof of 7.: Clearly, if $\rho \in L^\infty$, then $\rho \in \mathfrak{M}_2$ with equal norm. Conversely, if $\rho \in \mathfrak{M}_2$, then

$$\sup_{\|f\|_2=1} \|\mathcal{F}^{-1}\rho\mathcal{F}f\|_{L^2} = \sup_{\hat{f}\in L^2} \|\rho\hat{f}\|_{L^2} = \|\rho\|_{L^\infty} \|\hat{f}\|_{L^2}.$$

□

We turn now to the proof of Theorem 8.2. We reduce it to the case where $0 \leq s < 1$ and let $m = 1$, $N = 0$. As an exercise, one can consider the situation when additionally, $p = q$.

With this simplification, it remains to prove that

$$\|f\|_{B^s_{p,q}} \sim \|f\|_{L^p} + n\left(\int_0^\infty (t^{-s}\sup_{|y|<t}\|f(\cdot)-f(\cdot+y)\|_{L^p})^q\,\frac{\mathrm{d}t}{t}\right)^{1/q}.$$

We can immediately guess that the proof requires finding the right Fourier multipliers in order to move from one norm to the other. The first observation is, however, that instead of the right hand side, we may consider equivalently the norm

$$\|f\|_{L^p} + n\left(\sum_{i=-\infty}^{\infty}(2^{is}\sup_{|y|<2^{-i}}\|\Delta^1_y f\|_{L^p})^q\right)^{1/q}. \tag{8.3}$$

The basic argument is that $\omega_p^1 = \omega_p$ is monotonely increasing in t. We set $(0,\infty) = \cup_{i\in\mathbb{Z}}(2^{-i}, 2^{-i+1})$ to get

$$\int_0^\infty t^{-sq}\omega_p^q(t,f)\,\frac{\mathrm{d}t}{t} = \sum_{i=-\infty}^{\infty}\int_{2^{-i}}^{2^{-i+1}} t^{-sq}\omega_p^q(t,f)\,\frac{\mathrm{d}t}{t}.$$

The integral on the right hand side can then easily be estimated from above and from below by

$$2^{-sq-1}2^{isq}\omega_p^q(2^{-i},f) \le \int_{2^{-i}}^{2^{-i+1}} t^{-sq-1}\omega_p^q(t,f)\,\mathrm{d}t \le 2^{sq}2^{(i-1)sq}\omega_p^q(2^{-i+1},f).$$

We now assume $f\in B^s_{p,q}$ and want to estimate f in the Norm in (8.3). Note that for fixed $0\ne y\in\mathbb{R}^n$, we can write

$$\begin{aligned} f(x+y) = \mathcal{F}^{-1}\mathcal{F}f(x+y) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(x+y)\cdot\eta}e^{-i\xi\cdot\eta}f(\xi)\,\mathrm{d}\eta\mathrm{d}\xi \\ &= \mathcal{F}^{-1}e^{iy\cdot}\mathcal{F}f(x) \end{aligned}$$

and set $\rho_y(\xi) = 1-e^{i\xi\cdot y}$ to get

$$\begin{aligned} f(x)-f(x+y) &= \sum_{k=0}^{\infty}\mathcal{F}^{-1}\varphi_k\mathcal{F}f(x) - \mathcal{F}^{-1}\varphi_k\mathcal{F}f(x+y) \\ &= \sum_{k=0}^{\infty}\mathcal{F}^{-1}\varphi_k\cdot\rho_y\mathcal{F}f(x). \end{aligned}$$

We must now consider the multipliers φ_k and ρ_y to show the estimate

$$\|\mathcal{F}^{-1}(\varphi_k\cdot\rho_y\cdot\mathcal{F}f)\|_{L^p} \le C\min(1,|y|2^k)\|\mathcal{F}^{-1}(\varphi_k\cdot\mathcal{F}f)\|_{L^p}. \tag{8.4}$$

Since $|\mathcal{F}^{-1}\rho_y\mathcal{F}f(0)| = f(y) \le \|f\|_{L^\infty}$, we have $\|\rho_y\|_{\mathfrak{M}_1} = 1$. As an exercise, we can show that $r_y\in\mathfrak{M}_1$, where $r_y(\xi) = \frac{1}{y\cdot\xi}\rho_y(\xi)$. From Theorem 8.3.6, we get that

$$\|(\frac{y}{|y|},\cdot)\varphi(\cdot)\|_{\mathfrak{M}_1} < C,$$

which implies

$$\|(y,\cdot)\varphi(2^{-k}\cdot)\|_{\mathfrak{M}_1(\mathbb{R}^n)} = \|y\cdot\varphi(2^{-k}\cdot)\|_{\mathfrak{M}_1(\mathbb{R})} < C|y|2^k,$$

if we use the calculation

$$\begin{aligned}
|\mathcal{F}^{-1}y\cdot\varphi(2^{-k}\cdot)\mathcal{F}f(0)| \quad &= \quad \int_{\mathbb{R}}\int_{\mathbb{R}} y\xi\varphi(2^{-k}\xi)e^{-i\xi\eta}f(\eta)\,\mathrm{d}\eta\mathrm{d}\xi \\
&\overset{x=2^{-k}\xi}{=} \quad 2^k\int_{\mathbb{R}}\int_{\mathbb{R}} y2^k x\varphi(x)e^{-i2^k x\eta}f(\eta)\,\mathrm{d}x\mathrm{d}\eta \\
&\overset{z=2^k\eta}{=} \quad 2^k\int_{\mathbb{R}}\int_{\mathbb{R}} yx\varphi(x)e^{-izx}f(2^{-k}z)\,\mathrm{d}x\mathrm{d}z \\
&\leq \quad 2^k\|f\|_{L^\infty}\int_{\mathbb{R}}\mathcal{F}(y\cdot\varphi(\cdot))(z)\,\mathrm{d}z \\
&\leq \quad C2^k\|f\|_{L^\infty}.
\end{aligned}$$

In conclusion, we have

$$\|\mathcal{F}^{-1}\rho_y\varphi_k\mathcal{F}f\|_{L^p} \leq C\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p}$$

as well as

$$\begin{aligned}
\|\mathcal{F}^{-1}\rho_y\varphi_k\mathcal{F}f\|_{L^p} \quad &\leq \quad \|\mathcal{F}^{-1}\rho_y(\cdot)\frac{1}{(y,\cdot)}\mathcal{F}\mathcal{F}^{-1}(y,\cdot)(\varphi(2^{-k}\cdot)-\varphi(2^{-k+1}\cdot))\mathcal{F}f\|_{L^p} \\
&\leq \quad C|y|2^k\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p},
\end{aligned}$$

so that in conclusion, we get (8.4). The remaining step in the proof of the first inclusion is to see why this estimate is helpful, in particular, why the case $|y|2^k < 1$ is important. We get that

$$\begin{aligned}
2^{is}\omega_p(2^{-i},f) \quad &\leq \quad C2^{is}\sum_{k=0}^{\infty}\sup_{|y|<2^{-i}}\|\mathcal{F}^{-1}\rho_y\varphi_k\mathcal{F}f\|_{L^p} \\
&\leq \quad C\sum_{k=0}^{\infty}2^{(i-k)s}\min(1,2^{-i+k})2^{sk}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p}.
\end{aligned}$$

The right hand side is a convolution of the two sequences $b_k = 2^{ks}\min(1,2^{-k})$, and $a_k = 2^{sk}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p}$, $k\in\mathbb{Z}$, where $a_k = 0$ for $k<0$. We can use that

$$\sum_{k=-\infty}^{\infty} b_k = \sum_{k=-\infty}^{\infty} 2^{sk}\min(1,2^{-k}) < \infty$$

to get

$$\left(\sum_{i=-\infty}^{\infty}(2^{is}\omega_p(2^{-i},f))^q\right)^{1/q} \leq C\left(\sum_{k=0}^{\infty}a_k^q\right)^{1/q} = C\|f\|_{B^s_{p,q}}.$$

It remains to prove the converse estimate, by using a similar technique. We set

$$\rho_{jk}(\xi) = \rho_{(2^{-k}e_j)}(\xi),$$

where e_j is the jth unit vector. The crucial part is to prove the estimate

$$\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f\|_{L^p} \le C\sum_{j=1}^{n}\|\mathcal{F}^{-1}\rho_{jk}\mathcal{F}f\|_{L^p}. \tag{8.5}$$

It then follows that

$$\begin{aligned}
\|f\|_{B^s_{p,q}} &\le \|f\|_{L^p} + C\left(\sum_{k=1}^{\infty}(2^{ks}\sum_{j=1}^{n}\|\mathcal{F}^{-1}\rho_{jk}\mathcal{F}f\|_{L^p})^q\right)^{1/q} \\
&\le \|f\|_{L^p} + C\sum_{j=1}^{n}\left(\sum_{k=1}^{\infty}(2^{ks}\sup_{|y|<2^{-k}}\|f(\cdot)-f(\cdot+y)\|_{L^p})^q\right)^{1/q} \\
&\le \|f\|_{L^p} + Cn\left(\int_0^{\infty}(t^{-s}\omega_p(t,f))^q\,\frac{\mathrm{d}t}{t}\right)^{1/q}.
\end{aligned}$$

In order to prove (8.5), we use the following construction: For $n \ge 2$ there exist functions $\chi_j \in \mathcal{S}(\mathbb{R}^n)$, such that

$$\begin{aligned}
\sum_{j=1}^{n}\chi_j &= 1 \quad \text{on } \operatorname{supp}\varphi_1 = B_2(0)\backslash B_{1/2}(0), \\
\operatorname{supp}\chi_j &\subset \{\xi \in \mathbb{R}^n \| \xi_j| \ge \frac{1}{3\sqrt{n}}\}
\end{aligned}$$

for $1 \le j \le n$. Idea, why this is true: consider $k \in \mathcal{S}(\mathbb{R})$ satisfying $\operatorname{supp} k = \{\xi \in \mathbb{R}\||\xi| \ge \frac{1}{3\sqrt{n}}\}$ and taking positive values. Choose $l \in \mathcal{S}(\mathbb{R}^{n-1})$ such that $\operatorname{supp} l = \{\xi \in \mathbb{R}^{n-1}\||\xi| \le 3\}$ taking positive values. Then with $\tilde{\xi^j} = (\xi_1, \dots, \xi_{j-1}, \xi_{j+1}, \dots, \xi_n)$, the function

$$\chi_j(\xi) = \frac{k(\xi_j)l(\tilde{\xi}_j)}{\sum_{j=1}^{n}k(\xi_j)l(\tilde{\xi}_j)}$$

works.
By Theorem 8.3.5, we see that

$$m : \xi \mapsto \chi_j(\xi)\varphi_1(\xi)\frac{1}{e^{i\xi_j}-1}$$

is a map in L^2 and similarly, its derivatives, as we only need to consider the last multiplier for $0 < \frac{1}{3\sqrt{n}} \leq \xi_j \leq 2 < 2\pi$. It follows that $m \in \mathfrak{M}_p$, $1 \leq p \leq \infty$ and so

$$\begin{aligned} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f\|_{L^p} &\leq C\|\mathcal{F}^{-1}\varphi_1(2^{-k}\cdot)\mathcal{F} f\|_{L^p} \\ &\leq C\sum_{j=1}^{n} \|\mathcal{F}^{-1} m(2^{-k})\mathcal{F}\mathcal{F}^{-1}\rho_{jk}\mathcal{F} f\|_{L^p} \\ &\leq C\sum_{j=1}^{n} \|\mathcal{F}^{-1}\rho_{jk}\mathcal{F} f\|_{L^p}. \end{aligned}$$

Now that we have a new characterization of the Besov norm from Theorem 8.2, we are much closer to proving Step 2 in the Trace Theorem 8.1.

We need the estimate

$$\|Tf\|_{L^p(\mathbb{R}^{n-1})} \leq C\|f\|_{B^{1/p}_{p,1}(\mathbb{R}^n)} \tag{8.6}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$. From the Embedding Theorem 7.19, we know that $B^{1/p}_{pq}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ for every $1 \leq q \leq \infty$. It follows that

$$|Tf(x')| \leq C\|f(\cdot, x')\|_{B^{1/p}_{p,1}(\mathbb{R})} \tag{8.7}$$

for all $x' \in \mathbb{R}^{n-1}$. To deduce (8.6) from this, we use the following characterization given in [1, Theorem 7.47], which is not trivial and is shown by taking the $B^s_{p,q}(\mathbb{R}^n)$ norm from real interpolation: for $0 < s < 1$, $1 < p < \infty$, $1 \leq q < \infty$,

$$\|\cdot\|_{B^s_{p,q}} \sim \|\cdot\|_{L^p} + \int_0^\infty [t^{-s}\omega^1_p(\cdot, t)]^q \frac{\mathrm{d}t}{t} \sim \|\cdot\|_{L^p} + \int_{\mathbb{R}^n} [|h|^{-s}\|\Delta_h f\|_{L^p(\mathbb{R}^n)}]^q \frac{\mathrm{d}h}{|h|^n}.$$

We now take the p-th power of both sides of (8.7), integrate over $\mathbb{R}^{n-1}$ and take the $1/p$-th root to get that by the Minkowski inequality,

$$\begin{aligned} &\|Tf\|_{L^p(\mathbb{R}^{n-1})} \\ \leq\ & C\left(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} |f(x_1, x')|^p \,\mathrm{d}x_1 \,\mathrm{d}x'\right)^{1/p} + C\left(\int_{\mathbb{R}^{n-1}}\left(\int_0^\infty t^{-1/p}\omega^1_p(f(\cdot, x'), t)\,\frac{\mathrm{d}t}{t}\right)^p\right)^{1/p} \\ \leq\ & C\|f\|_{L^p(\mathbb{R}^n)} + C\left(\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}} |h_1|^{-1/p}\|\Delta_{h_1} f(\cdot, x')\|_{L^p(\mathbb{R})}\,\frac{\mathrm{d}h_1}{|h_1|}\right)^p\right)^{1/p} \\ \overset{M.I.}{\leq}\ & C\|f\|_{L^p(\mathbb{R}^n)} + C\int_{\mathbb{R}} |h_1|^{-1/p}\left(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} |f(x_1, x') - f(x_1 + h_1, x')|^p \,\mathrm{d}x_1 \,\mathrm{d}x'\right)^{1/p} \frac{\mathrm{d}h_1}{|h_1|} \\ \overset{t=\pm h_1}{\leq}\ & C\|f\|_{L^p(\mathbb{R}^n)} + C\int_0^\infty t^{-1/p}\left(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} |f(x_1, x') - f(x_1 + t, x')|^p \,\mathrm{d}x_1 \,\mathrm{d}x'\right)^{1/p} \frac{\mathrm{d}t}{t} \\ \leq\ & C\|f\|_{L^p(\mathbb{R}^n)} + C\int_0^\infty t^{-1/p} \sup_{|y|\leq t} \|\Delta_y f\|_{L^p(\mathbb{R}^n)}\,\frac{\mathrm{d}t}{t} \\ \leq\ & C\|f\|_{B^{1/p}_{p,1}(\mathbb{R}^n)}, \end{aligned}$$

which proves the claim.

9 More Spaces

In this last chapter and in the last lecture, we want to briefly introduce and consider a few function spaces which have not appeared before, but which are at the same time somehow related to our previous topics. There is no time for proofs, but they can be found in the references.

9.1 Quasi-norms

In contrast to a *norm,* a *quasi-norm* does not fulfill the triangle inequality, but the following more general condition. There exists a constant $C > 0$, such that for every elements x, y of the quasi-normed space $(X, \|\cdot\|)$,

$$\|x + y\| \leq C(\|x\| + \|y\|).$$

Typical examples of quasi-normed spaces are the L^p-spaces, when $0 < p < 1$, which are defined as in the $p \geq 1$-case.

We have met a quasi-norm in Section 6.2. For $1 \leq p, q, p_0, p_1 \leq \infty$, and $0 < \theta < 1$ appropriately, the spaces $(L^{p,q}, \|\cdot\|_{L^{p,q}})$ are quasi-Banach spaces, whereas $(L^{p,q}, \|\cdot\|_{(L^{p_1}, L^{p_0})_{\theta,q}})$ are Banach spaces. We showed that they have equivalent quasi-norms.

Remark 9.1. For quasi-Banach spaces, the K-method still works as before, giving a quasi-Banach space as the interpolation space. One can choose $0 < q \leq \infty$ as interpolation parameter, [7, Remark 2, p. 27].

9.2 Semi-norms and Homogeneous Spaces

In contrast to a norm $\|\cdot\|$, a *semi-norm* $[\cdot]$ does not satisfy

$$\|x\| = 0 \Rightarrow x = 0.$$

Typical examples of semi-normed function spaces are those which only care about the properties of (higher order) derivatives of a function. These spaces are called *homogeneous.*

Example 9.2. For a domain $\Omega \subset \mathbb{R}^n$, $m \in \mathbb{N}$, $1 \leq q < \infty$, we set

$$D^{m,q}(\Omega) = \{f \in L^1_{\text{loc}}(\Omega) : D^l f \in L^q(\Omega), |l| = m\}.$$

It follows that $D^{m,q}(\Omega) \hookrightarrow W^{m,q}_{\text{loc}}(\Omega)$ by Poincare Inequality. If Ω is a local Lipschitz domain, then

$$D^{m,q}(\Omega) \hookrightarrow W^{m,q}_{\text{loc}}(\overline{\Omega}).$$

We look at the semi-norm

$$[f]_{D^{m,q}} = \left(\sum_{|l|=m} \int_\Omega |D^l f|^q \right)^{1/q}.$$

We have that $[f]_{D^{m,q}} = 0$ if and only if f is a polynomial of degree at most $m-1$, $f \in \mathcal{P}_{m-1}$, $D^{m,q}$ modulo $\mathcal{P}_{m-1}$ is a Banach space.

Example 9.3. In the Fourier analysis context, we can define homogeneous Besov and Bessel potential spaces, see [3, Section 6.3]. We choose $\varphi \in \mathcal{S}(\mathbb{R}^n)$, such that

$$\begin{aligned} \operatorname{supp}\varphi &= \overline{B}_2(0) \backslash B_{\frac{1}{2}}(0), \\ \varphi(\xi) &> 0, \quad \frac{1}{2} < |\xi| < 2, \\ \sum_{k=-\infty}^{\infty} \varphi(2^{-k}\xi) &= 1, \quad \xi \neq 0. \end{aligned}$$

We define $\tilde{\varphi}_k(\xi) = \varphi(2^{-k}\xi)$ for all $k \in \mathbb{Z}$.

1. Then we consider for all $f \in S'(\mathbb{R}^n)$,

$$[f]_{\dot{B}^s_{p,q}} = \left(\sum_{k=-\infty}^{\infty} 2^{skq} \|\mathcal{F}^{-1}\tilde{\varphi}_k \mathcal{F} f\|^q_{L^p} \right)^{1/q}.$$

The space of all $f \in \mathcal{S}'$, for which $[f]_{\dot{B}^s_{p,q}}$ is finite is called the *homogeneous Besov space* $\dot{B}^s_{p,q}$. We have that $[\cdot]_{\dot{B}^s_{p,q}}$ is a semi-norm and that $[f]_{\dot{B}^s_{p,q}} = 0$ if and only if $\operatorname{supp} \mathcal{F} f = \{0\}$, which happens if and only if f is a polynomial.
We have that $\dot{B}^s_{p,q} = L^p \cap B^s_{p,q}$ if $s > 0$ and that

$$[f]_{\dot{B}^s_{p,q}} \sim \sum_{j=1}^{N} \left(\int_0^\infty (t^{N-s} \omega^m_p(t, \frac{\partial^N f}{\partial x_j^N}))^q \, \frac{\mathrm{d}t}{t} \right)^{1/q}$$

as in Theorem 8.2.

2. Similarly, the *homogeneous Bessel potential spaces* $\dot{H}^{s,p}$ are defined. We set $f \in \dot{H}^{s,p}$ for $f \in \mathcal{S}'$, if

$$[f]_{\dot{H}^{s,p}} = \| \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}(|\xi|^s \tilde{\varphi}_k)\mathcal{F}f\|_{L^p} < \infty,$$

meaning that the sum on the right hand side converges to an L^p-function. Then $(\dot{H}^{s,p}, [\cdot]_{\dot{H}^{s,p}})$ is also a semi-normed space and $[f]_{\dot{H}^{s,p}} = 0$ if and only if f is a polynomial.
It follows that $\dot{H}^{s,p} = L^p \cap H^{s,p}$ if $s > 0$ and that

$$[f]_{\dot{H}^{N,p}} \sim \sum_{j=1}^{n} \|\frac{\partial^N f}{\partial x_j^N}\|_{L^p}, \quad 1 < p < \infty,$$

if $\hat{f}$ vanishes in a neighbourhood of the origin.

Interpolation and embedding results from the usual (inhomogeneous) spaces mostly carry over to homogeneous spaces, e.g. Theorem 7.13 and Theorem 7.19.

9.3 Orlicz Spaces L_A

Orlicz spaces can be considered as a type of generalization of L^p-spaces. We only give a very brief introduction here following [1, Chapter 8] and also refer to [2, Section 4.8].

Idea: $f \in L^p$ if $\int |f(x)|^p \,\mathrm{d}x < \infty$, $f \in L_A$ if $\int A(|f(x)| \,\mathrm{d}x < \infty$.

Problem: If $mp = 1$, $p > 1$, then for a good domain $\Omega \subset \mathbb{R}^n$,

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \quad p \le q < \infty, \quad \text{but} \quad W^{m,p}(\Omega) \not\subseteq L^\infty(\Omega),$$

so there is no optimal target L^p-space for the embedding.

Definition 9.4. Let $a : [0, \infty) \to \mathbb{R}$ be a function such that

1. $a(0) = 0$, $a(t) > 0$ if $t > 0$ and $\lim_{t\to\infty} a(t) = \infty$,
2. a is nondecreasing,
3. a is right continuous.

Then the function $A : [0, \infty) \to \mathbb{R}$, given by

$$A(t) = \int_0^t a(\tau) \,\mathrm{d}\tau$$

is called an $N-$*function.*

It follows that A is continuous, strictly increasing and convex, that $\lim_{t\to 0}\frac{A(t)}{t}=0$ and $\lim_{t\to\infty}\frac{A(t)}{t}=\infty$ and that $\frac{A(\cdot)}{\cdot}$ is strictly increasing.

Examples: $A(t)=t^p$ and $A(t)=e^{(t^p)}-1$ if $1<p<\infty$, $A(t)=e^t-t-1$ and $A(t)=(1+t)\log(1+t)-t$.

Definition 9.5. Let $\Omega\subset\mathbb{R}^n$ be a domain and A an N-function. The *Orlicz class* $K_A(\Omega)$ consists of all equivalence classes $[f]$ modulo equality a.e. on Ω of measurable functions f, such that

$$\int_\Omega A(|f(x)|)\,\mathrm{d}x<\infty.$$

The set $K_A(\Omega)$ is convex, but it may not be a vector space.

Definition 9.6. (and **Theorem**) The *Orlicz space* $L_A(\Omega)$ is the linear hull of $K_A(\Omega)$. The function

$$\|f\|_{L_A(\Omega)}=\inf\left\{k>0:\int_\Omega A\left(\frac{|f(x)|}{k}\right)\,\mathrm{d}x\le 1\right\}$$

is a norm on $L_A(\Omega)$. It is called the *Luxemburg norm.* $(L_A,\|\cdot\|_{L_A})$ is a Banach space.

Next, we look at some basic results.

Definition 9.7. Given a as in Definition 9.4 above, we consider

$$\tilde{a}(s)=\sup_{a(t)\le s}t$$

and $\tilde{A}(s)=\int_0^s\tilde{a}(\sigma)\,\mathrm{d}\sigma$. We have that $\tilde{a}$ also satisfies 1. - 3. in Definition 9.4 and

$$a(t)=\sup_{\tilde{a}(s)\le t}s,$$

so that $\tilde{A}$ is also an N-function. A and $\tilde{A}$ are said to be *complementary.*

For A and B given N-functions, we say that B *dominates A globally* if there exists a constant $C>0$ such that

$$A(t)\le B(Ct),\quad\text{for all } t\ge 0. \tag{9.1}$$

We say that B *dominates A near infinity* if there exists a constant $t_0>0$ such that (9.1) is satisfied for all $t\ge t_0$.

Examples: For $1<p<\infty,\frac{1}{p}+\frac{1}{p'}=1$, the functions A and $\tilde{A}$ given by $A(t)=\frac{t^p}{p}$ and $\tilde{A}(s)=\frac{s^{p'}}{p'}$ are complementary N-functions, as well as the functions B and $\tilde{B}$ given by $B(t)=e^t-t-1$ and $\tilde{B}(s)=(1+s)\log(1+s)-s$.

Definition 9.8. Hölder Inequality: If A and $\tilde{A}$ are complementary N-functions, then

$$\left|\int_\Omega u(x)v(x)\,\mathrm{d}x\right| \le 2\|u\|_{L_A}\|v\|_{L_{\tilde{A}}}.$$

Embedding: We have that

$$L_B(\Omega) \hookrightarrow L_A(\Omega)$$

if either B dominates A globally or B dominates A near infinity and $\mu(\Omega) < \infty$.

Theorem 9.9. *Let $\Omega \subset \mathbb{R}^n$ be a suitable bounded domain. Let $mp = n$ and $p > 1$ and set*

$$A(t) = e^{(t^{p/(p-1)})} - 1 = e^{(t^{n/(n-m)})} - 1.$$

Then the embedding

$$W^{m,p}(\Omega) \hookrightarrow L_A(\Omega)$$

holds true.

More: Orlicz-Sobolev spaces, trace theorems, interpolation.

$L^{p(x)}(\Omega)$-spaces: For a function $p : \Omega \to \mathbb{R}$, $1 \le p(x) < \infty$ and $\lambda > 0$ consider $\int_\Omega |\frac{f(x)}{\lambda}|^{p(x)}\,\mathrm{d}x$. Luxemburg norm $\rightsquigarrow L^{p(x)}(\Omega)$.

9.4 Hardy Spaces

Hardy spaces are complex Banach spaces of holomorphic functions, endowed with an L^p-type norm, usually defined on the unit disc or on the half plane. We consider the latter case, following the first part of [2, Section 5.6].

Definition 9.10. Let $\mathbb{C}_+ = \{x + iy : x \in \mathbb{R}, y > 0\}$ be the upper complex half plane. The Hardy space $\mathcal{H}^p(\mathbb{C}_+)$ is the Banach space of holomorphic functions F on $\mathbb{C}_+$ with finite norm

$$\|F\|_{\mathcal{H}^p(\mathbb{C}_+)} = \begin{cases} \sup_{y>0}\left(\int_{-\infty}^{\infty}|F(x+iy)|^p)\,\mathrm{d}x\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{z\in\mathbb{C}_+}|F(z)|, & p = \infty. \end{cases}$$

We know: There is a unique solution u for the Dirichlet problem on $\mathbb{C}_+$ if $f \in L^1 + L^\infty(\mathbb{R})$:

$$\begin{aligned} \Delta u &= 0, \quad \text{in } \mathbb{C}_+, \\ \lim_{t+iy\to x} u(t,y) &= f(x), \quad \text{on } \partial\mathbb{C}_+. \end{aligned}$$

It can be shown: if $F \in \mathcal{H}^p(\mathbb{C}_+)$, then its boundary value f satisfies $f \in L^p(\mathbb{R};\mathbb{C})$. Moreover, if F_1 and F_2 are functions in $\mathcal{H}^1(\mathbb{C}_+)$ which have the same real part f_R at the boundary, then by Cauchy-Riemann they can only differ by an imaginary constant, which must be zero for $F_1 - F_2 \in \mathcal{H}^1$. This implies: $\mathcal{H}^1$ as a real Banach space is isometrically isomorphic to the space $R(\mathcal{H}^1)$ of functions $f_R \in L^1$ which are real parts of boundary functions of functions $F \in \mathcal{H}^1$, where the norm is $\|f_R\|_{R(\mathcal{H}^1)} = \|F\|_{\mathcal{H}^1}$.

Philosophy: $R(\mathcal{H}^1) \subset L^1$, but nicer than L^1.

Theorem 9.11. *If $0 < \theta < 1$, $1 \leq q \leq \infty$, and $\theta = 1 - 1/p$, then*

$$(R(\mathcal{H}^1), L^\infty)_{\theta,q} = L^{p,q}.$$

9.5 The Space BMO

BMO means: bounded mean oscillation.

On a domain $\Omega \subset \mathbb{R}^n$, we consider the mean value $\overline{f}_A$ of a function $f \in L^1_{\text{loc}}(\Omega)$ on a (Lebesgue) measurable set A with $0 < \mu(A) < \infty$,

$$\overline{f}_A = \frac{1}{\mu(A)} \int_A f \, \mathrm{d}\mu.$$

Definition 9.12. The space $BMO(\Omega)$ is the space of functions $f \in L^1_{\text{loc}}(\Omega)$ which satisfy

$$[f]_{BMO} = \sup_{0<\mu(A)<\infty} \frac{1}{\mu(A)} \int_A |f(x) - \overline{f}_A| \, \mathrm{d}x < \infty,$$

where $[\cdot]_{BMO}$ is a semi-norm and $[f]_{BMO} = 0$ implies that f is constant.

The following holds true:

- if $\mu(\Omega) < \infty$, then $L^\infty(\Omega) \hookrightarrow BMO(\Omega) \hookrightarrow L^1(\Omega)$.
- if Ω is a bounded Lipschitz domain, then $BMO(\Omega)$ is contained in every $L^p(\Omega)$, $1 \leq p < \infty$.
- if Ω is a bounded Lipschitz domain, then for $0 < \theta < 1$, $1 \leq q \leq \infty$, we have

$$(L^1(\Omega), BMO(\Omega))_{\theta,q} = L^{\frac{1}{1-\theta},q}(\Omega).$$

- $BMO(\mathbb{R})$ modulo constants is isomomorphic to $R(\mathcal{H}^1)'$, [2, Theorem 6.17].

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