Lecture Notes
on
Calculus of Variations I

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Contents

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1 Introduction

1.1 Mission statement

Consider (non)linear functionals

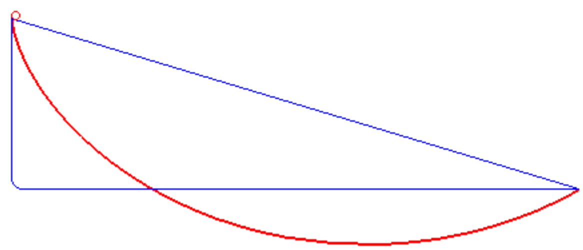
$$I: M \to \mathbb{R} \cup \{+\infty\},$$

on a set $M \subset X$, where X is an infinite-dimensional Banach space.

Find (local or global) minimizers of I.

1.2 Examples: sketches and pictures

1.2.1 Brachistochrone curve – the *fastest* marble run



1.2.2 Catenoid – the *laziest* chain

1.2.3 Soap bubble – the *smallest* surface, the *largest* volume

1.2.4 Elasticity – the *most relaxed* sponge

Note: (almost) every solid material is somehow elastic!

1.3 Examples: famous functionals and some modelling

1.3.1 Brachistochrone curve

Find a curve $\mathcal{S}$ that starts at $A = (0, h)$ and ends at $B = (l, 0)$ such that the traveling time of a marble running along $\mathcal{S}$ is minimal (no friction!). The curve $\mathcal{S} = (x, u(x))$ is given as the graph of a function $u\colon [0, l] \to \mathbb{R}$ with $u(0) = h$ the initial height and $u(l) = 0$. The total energy of the marble at position x is given by the sum of its *potential energy*

$$E_{pot}(x) = \mathrm{mg}u(x),$$

where m is its mass and g the gravitation constant, and its *kinetic energy,*

$$E_{kin}(x) = \frac{1}{2}\mathrm{m}|v(x)|^2,$$

where $v(x)$ is the velocity of the marble's center of gravity. Here, as is customary for this problem in mathematics, we disregard the rotational energy

$$E_{rot}(x) = \frac{1}{5}\mathrm{m}|v(x)|^2,$$

of the marble. It would only modidy the factor "2″ that appears in the following to be "$\frac{10}{7}$″. By conservation of energy, for all x, we obtain the equation

$$E(0) = \mathrm{m}\mathrm{g}h + \frac{1}{2}\mathrm{m}|v(0)|^2 = \mathrm{m}\mathrm{g}u(x) + \frac{1}{2}\mathrm{m}|v(x)|^2 = E(x).$$

Since the marble starts at zero velocity, $v(0) = 0$, this gives

$$|v(x)|^2 = 2\mathrm{g}(h - u(x)). \tag{1.1}$$

At the same time, the velocity of the curve is

$$v(t) = \begin{pmatrix} \dot{x}(t) \\ \frac{d}{dt}u(x(t)) \end{pmatrix} \overset{\text{chain rule}}{=} \dot{x}(t) \begin{pmatrix} 1 \\ u'(x(t)) \end{pmatrix},$$

so

$$|v(t)|^2 = \dot{x}(t)^2(1 + u'(x(t))^2).$$

Combining this with (1.1), we get

$$|\dot{x}(t)| = \sqrt{\frac{2\mathrm{g}(h - u(x(t)))}{1 + u'(x(t))^2}}.$$

Hence, the Brachistochrone curve is given by the function u with $u(0) = h$ and $u(l) = 0$ that minimizes the functional

$$\begin{aligned} T(u) &= \int_0^T \mathrm{d}t \overset{t\mapsto x(t)}{=} \int_0^l t'(x)\,\mathrm{d}x \overset{t'(x)=\frac{1}{\dot{x}(t(x))}}{=} \int_0^l \frac{1}{\dot{x}(t(x))}\,\mathrm{d}x \\ &= \int_0^l \sqrt{\frac{1 + u'(x)^2}{2\mathrm{g}(h - u(x))}}\,\mathrm{d}x. \end{aligned}$$

The problem of finding this curve was first solved by Jakob Bernoulli in 1696. The solution will be discussed in Subsection 2.7.2.

1.3.2 Fermat's principle

"Light takes the path of shortest time." Pierre de Fermat (1607 – 1665) proved this principle using the calculus of variations. A possible formalization of this prinicple is the following: Again, look for a parameterized curve $(x(t), u(x(t)))$ that is the graph of a function u and connects a point $A = (x_0, u_0)$ to a point $B = (x_1, u_1)$. Light should travel along this curve in a medium with index of refraction $n(x, u) > 0$. As before, we have

$$|v(t)| = |\dot{x}(t)|\sqrt{1 + u'(x(t))^2}.$$

In addition, a *constitutive equation* as in (1.1) is needed. In this case, it is given by the law of optics,

$$v(x, u) = \frac{c}{n(x, u)},$$

where c is the (constant) speed of light in vacuum. Thus, we need to find u such that $u(x_0) = u_0$, $u(x_1) = u_1$, and such that the total time

$$T(u) = \int_0^T \mathrm{d}t \overset{x=x(t)}{=} \int_{x_0}^{x_1} \frac{1}{\dot{x}(t(x))}\,\mathrm{d}x = \int_{x_0}^{x_1} \sqrt{1 + u'(x)^2}\,\frac{n(x, u(x))}{c}\,\mathrm{d}x$$

is minimal.

1.3.3 Minimal surface

Consider a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$. Given a function $u_0 \colon \overline{\Omega} \to \mathbb{R}$, find a function $u \colon \Omega \to \mathbb{R}$, $u \in C^1(\overline{\Omega})$ such that $u = u_0$ on $\partial\Omega$ and such that the *surface area*

$$I(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\,\mathrm{d}x$$

is minimal. (The surface area of the surface given by the graph of u over Ω is calculated by integrating the cross product of its tangential vectors in this parameterization:

Remark.

$$\| \begin{pmatrix} 1 \\ 0 \\ \frac{\partial}{\partial x_1} u(x) \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial}{\partial x_2} u(x) \end{pmatrix} \| = \| \begin{pmatrix} -\frac{\partial}{\partial x_1} u(x) \\ -\frac{\partial}{\partial x_2} u(x) \\ 1 \end{pmatrix} \| = \sqrt{1 + |\nabla u(x)|^2}.)$$

In this course, we study these and other examples with two different strategies.
In the first part of the lecture, in Chapter 2, we use *indirect* or *classical methods*. They are based on the idea of generalizing necessary and sufficient conditions for the existence of minimizers from the finite-dimensional to the infinite-dimensional situation. As a brief reminder of these conditions, let us look at the finite-dimensional case:

1.4 Reminder: the case $X = \mathbb{R}^n$, $I = F \in C^2(\mathbb{R}^n, \mathbb{R})$.

(For proofs see "Analysis II" and Exercise 1.IV.)

Necessary Conditions: If $x_0 \in X$ is a local minimizer of F, then

1. x_0 is a *critical point*, i.e. $\frac{\partial}{\partial x_i} F(x_0) = 0$ for all $i = 1, \ldots, n$, and,
2. the Hessian $H_F(x_0)$ is positive semi-definite.

Sufficient Condition: If x_0 is a critical point of F and $H_F(x)$ is positive definite, then x_0 is a local minimizer of F.

Aims of Chapter 2: Generalize and adapt these ideas to infinite-dimensional X and specific types of I.

In the second part of the course, Chapter 3 in these notes, we get to know the *direct method in the calculus of variations.* Roughly speaking, it is an adaptation of Weierstraß' Principle to specific infinite-dimensional settings. In the following three short sections, we look at some aspects and a rough guideline for this adaptation.

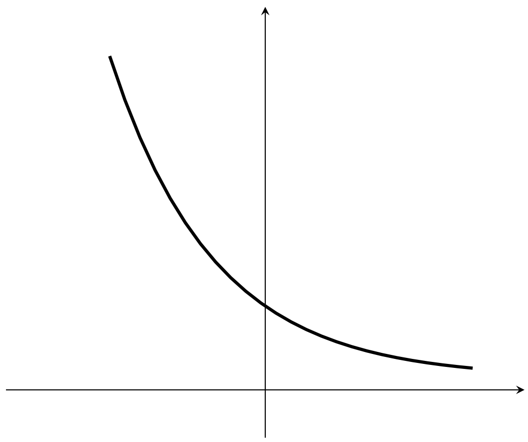
1.5 Reminder: Weierstraß' principle.

(Karl Weierstraß (1815-1897), for proofs recall "Analysis I and/or II":) A real-valued continuous function on a compact set attains its maximum and minimum.

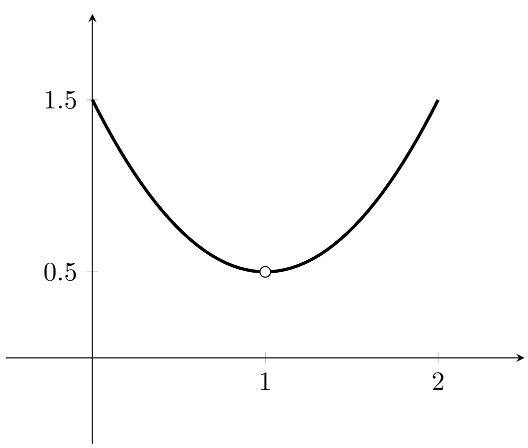
Aims of Chapter 3: For the problems we would like to solve, the assumptions of Weierstraß' Principle (compactness and continuity) do not fully apply. Thus, we try to modify them by using more specific information on $M \subset X$ and I.

Simple examples: Non-compact $X \subset Y = \mathbb{R}$ and non-continuous, bounded F – what can got wrong?

Problem 1. $M = X = \mathbb{R}$ (non-bounded) and $F(x) = e^{-x}$, then there is no minimizer of F:



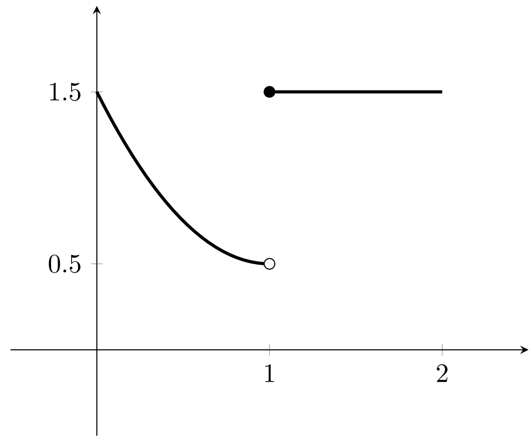
Problem 2. $M = [0, 1) \cup (1, 2]$ (non-closed) and $F(x) = 0.5 + (x-1)^2$, then $\inf_{x \in X} F(x) = 0.5$, but there is no minimizer in M:



Problem 3. $M = [0, 2]$ and

$$F(x) = \begin{cases} 0.5 + (x-1)^2, & x \in [0, 1), \\ 2, & x \in [1, 2], \end{cases}$$

(non-continuous), then again $\inf_{x \in X} F(x) = 1$, but there is no minimizer:



1.6 Strategy for Chapter 3 – how can this be fixed?

Let X be a Banach space, $M \subset X$, and let $I\colon M \to \mathbb{R}$ be a functional that is bounded from below. Then there is an infimizing sequence $(\bar{u}_n)_n \subset M$ for I, i.e.

$$\lim_n I(\bar{u}_n) = \inf_{u \in X} I(u) = \bar{I}.$$

Now adopt the following strategy:

Step 1. Assume that I is *coercive,* i.e. for every sequence $(u_n)_n \subset M$, if $\|u_n\|_X \to \infty$, then $I(u_n) \to \infty$. This implies that $(\bar{u}_n)_n$ is bounded without assuming that M is bounded. In particular, this assumption deals with Problem 1.

Step 2. Assume that M is *(weakly sequentially) closed* in X or that, for some other reason, there is a (weakly) convergent subsequence of $(\bar{u}_n)_n$ with (weak) limit $M \ni \bar{u} = \lim_k \bar{u}_{n_k}$. This assumption addresses Problem 2 – except that the assumption of weak closedness is *stronger* than the assumption of closedness of M.

Step 3. Now if I is *(weakly sequentially) continuous* on X, then $\lim_k I(\bar{u}_{n_k}) = I(\bar{u})$, so $\bar{u}$ is a minimizer and we are done. However, for the applications we have in mind, this assumption is *far too strong.* If, instead, we assume that I is *(weakly sequentially) lower semicontinuous*, then still

$$\overline{I} = \lim_k I(\overline{u}_{n_k}) \geq \liminf_k I(\bar{u}_{n_k}) \geq I(\bar{u}) \geq \overline{I},$$

so $\bar{u}$ is a minimizer. In particular, this assumption is sufficient for avoiding Problem 3 and can be shown to hold in relevant cases (think of the norm on a Hilbert space as a first example of a weakly sequentially lower semicontinuous functional).

1.7 Outlook and Motivation

This Introduction shows a little bit of everything we will encounter during this lecture. In particular,

- we will need
  - (multidimensional) calculus, and
  - some functional analysis, and,
- we will get to know
  - some convex analysis,
  - some PDE (Partial Differential Equations) theory, and
  - some modelling, and,
- the problems, theorems and examples in this lecture are strongly inspired by and connected to many other interesting topics in
  - analysis,
  - geometry,
  - (nonlinear) optimization,
  - mechanics, and, more generally, physics,
  - ...

2 Classical Methods in the Calculus of Variations

2.1 Problem set-up

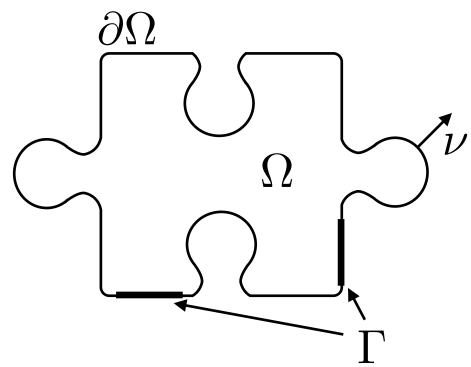
Let $n, m \in \mathbb{N}$ be (spatial) dimensions for our problem (e.g. "$n = 1, m = 1$" for the Brachistochrone curve, "$n = 2, m = 1$" for the minimal surface, "$n = 3, m = 3$" for the sponge) and let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ (see Remark below) and outer normal vector field ν. Now consider functionals of the form

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x + \int_{\partial\Omega} g(x, u(x))\,\mathrm{d}x' \tag{2.1}$$

and find (differentiable) functions $u\colon \Omega \to \mathbb{R}^m$ that minimize I.

In this Chapter,

- the function $f\colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$, $(x, u, A) \mapsto f(x, u, A)$ is called a *Lagrange function* or *(energy) density* and satisfies $f \in C^2(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times n}; \mathbb{R})$, and
- the function $g\colon \partial\Omega \times \mathbb{R}^m \to \mathbb{R}$ is called a *boundary density* and satisfies $g \in C^2(\partial\Omega \times \mathbb{R}^m; \mathbb{R})$.



It often happens that the sought-for function $u \in C^1(\overline{\Omega};\mathbb{R}^m) =: X$ should be restricted to fixed values u_0 on some part $\Gamma \subset \partial\Omega$ of the boundary (e.g. $u(0) = h, u(l) = 0$ for the Brachistochrone curve). Hence, for given $u_0 \in X$, we define the set

$$M := \{v \in X : v|_\Gamma = u_0|_\Gamma\}.$$

Then we denote by

$$X_0 := \{v \in X : v|_\Gamma = 0\}$$

the corresponding *linear* space of *test functions* with $M = u_0 + X_0$ and consider the functional I as a map

$$I\colon X \supset M \to \mathbb{R}. \tag{2.2}$$

Remark. The assumption on the smoothness of the boundary $\partial\Omega$ is "technical" – it can be relaxed and it doesn't really have an impact on what we would like to do. We will discuss this point a bit more in Chapter 3. Here, we just need to know that we can integrate over $\partial\Omega$ in a reasonable way. The notation $\mathrm{d}x'$ is used for this integral.

The examples in Chapter 1 fit into the form of (2.2) with suitable X, M and I as in (2.1). You can check this as an exercise even before we discuss them again.

2.2 (Multidimensional) notation

In this Section, we collect specific multidimensional notation used in this course.

In the previous Section and in the following, $\nabla u(x) \in \mathbb{R}^{m\times n}$ denotes the (transposed) gradient and the Jacobian matrix of u,

$$(\nabla u(x))_{ij} = \frac{\partial u_i}{\partial x_j}(x), \qquad i \in \{1,\dots,m\}, j \in \{1,\dots,n\}.$$

We adopt this (transposed) notation because it is used frequently in the literature and in applications.

On the set $\mathbb{R}^{m\times n}$ of real $m \times n$-matrices we use the scalar product

$$B\colon C = \mathrm{trace}(B^T C) = \Sigma_{i=1}^m \Sigma_{j=1}^n B_{ij} C_{ij}$$

for two matrices $B, C \in \mathbb{R}^{m\times n}$ and the *Frobenius norm* $|B| = \sqrt{B\colon B}$.

For f as before and $x \in \Omega$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{m\times n}$, we write:

$$\partial_x f(x,u,A) = \left(\frac{\partial}{\partial x_j} f(x,u,A)\right)_{1\le j\le n} \in \mathbb{R}^n,$$
$$\partial_u f(x,u,A) = \left(\frac{\partial}{\partial u_i} f(x,u,A)\right)_{1\le i\le m} \in \mathbb{R}^m,$$
$$\partial_A f(x,u,A) = \left(\frac{\partial}{\partial A_{ij}} f(x,u,A)\right)_{1\le i\le m, 1\le j\le n} \in \mathbb{R}^{m\times n}.$$

For second derivatives, and $v, w \in \mathbb{R}^m$, $A, B, C \in \mathbb{R}^{m\times n}$, we write

$$\partial_u^2 f(x,u,A)v\cdot w = \sum_{i,k=1}^m \frac{\partial^2}{\partial u_i \partial u_k} f(x,u,A) v_i w_k,$$
$$\partial_A^2 f(x,u,A)[B,C] = \sum_{1\le i,k\le m, 1\le j,l\le n} \frac{\partial^2}{\partial A_{ij}\partial A_{kl}} f(x,u,A) B_{ij} C_{kl},$$
$$\partial_A \partial_u f(x,u,A)[w,C] = \sum_{1\le i,k\le m, 1\le l\le n} \frac{\partial^2}{\partial A_{kl}\partial u_i} f(x,u,A) w_i C_{kl}.$$

For a differentiable function $A\colon \Omega \to \mathbb{R}^{m\times n}$, the divergence operator "acts on rows":

$$(\operatorname{div} A(x))_i = \Sigma_{j=1}^n \frac{\partial}{\partial x_j} A_{ij}(x).$$

For a k-times continuously differentiable function $v \in C^k(\overline{\Omega};\mathbb{R}^m)$, the corresponding *norm* $\|v\|_{C^k}$ of v is given by

$$\|v\|_{C^k} = \sum_{l=0}^k \max_{|\alpha|=k} \sup_{x\in\overline{\Omega}} |D^\alpha v(x)|. \tag{2.3}$$

2.3 The Euler-Lagrange Equations

In the following, let Ω, Γ, u_0, X, M, X_0 and f, g and I be as in the previous sections.

Definition 4. (First Variation) Let $u \in M$ and $v \in X_0$. Consider the map

$$\mathbb{R} \supseteq B_{\delta_u}(0) \ni t \mapsto I(u+tv) \in \mathbb{R},$$

with $B_{\delta_u}(0)$ an open interval such that $u + tv \in M$ for all $t \in B_{\delta_u}(0)$ and $v \in X_0$. Then the expression

$$DI(u)[v] := \frac{\mathrm{d}}{\mathrm{d}t} I(u+tv)|_{t=0}$$

is called the *first variation* of I at u in the direction of v. Moreover, we have

$$DI(u)[v] = \int_\Omega \partial_u f(x,u(x),\nabla u(x))\cdot v(x) + \partial_A f(x,u(x),\nabla u(x))\colon \nabla v(x)\,\mathrm{d}x$$
$$+ \int_{\partial\Omega} \partial_u g(x,u(x))\cdot v(x)\,\mathrm{d}x'$$

The first variation can be considered as a *directional derivative* of I and it is equal to the *Gâteaux differential* of I. The expression for $DI(u)[v]$ contains the definition (first $:=$) which works for general functionals, and it contains a small Proposition (second $=$) that follows from the chain rule if we differentiate the integral form (2.1) of I. We can interchange differentiation and integration as f, g are continuously differentiable. Note that the directions v need to be in the *linear* space of test functions X_0.

With the first variation at hand, it is clear how to define critical points of I. As in finite dimensions, critical points are candidates for local minimizers.

Definition 5. (Critical points of I) A point $u \in M$ is called a *critical point* of I, if for all $v \in X_0, DI(u)[v] = 0$.

From this definition, it is usually not clear how to find critical points of I. In particular, we cannot reduce the problem to finitely many equations. The aim of this Section is to show a necessary criterion for u being a critical point of I, namely being a solution of the *Euler-Lagrange Equations.* We start with two important preliminary results.

Theorem 6. *(Version of Gauss' Theorem* (Gauss)*) Let $u \in X$. Then for all $1 \leq i \leq m$, $1 \leq j \leq n$,*

$$\int_\Omega \frac{\partial u_i}{\partial x_j}(x)\,\mathrm{d}x = \int_{\partial\Omega} u_i(x)\nu_j(x)\,\mathrm{d}x'.$$

Proof. cf. Analysis II/IV! □

Theorem 7. *(Fundamental Lemma of the Calculus of Variations* (FL)*) If $a\colon \Omega \to \mathbb{R}^m$ and $b\colon \partial\Omega \to \mathbb{R}^m$ are continuous and for all $v \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$ such that $v|_\Gamma = 0$ we have*

$$\int_\Omega a(x) \cdot v(x)\,\mathrm{d}x + \int_{\partial\Omega} b(x) \cdot v(x)\,\mathrm{d}x' = 0,$$

then $a(x) = 0$ and $b(x') = 0$ for all $x \in \Omega$ and $x' \in \partial\Omega \setminus \Gamma$.

Proof. We use an indirect proof. Assume that $x_0 \in \Omega$ is such that $a(x_0) \neq 0$. Since a is continuous, then there exists $\delta > 0$ such that $B_\delta(x_0) \subset \Omega$ and $a(x) \cdot a(x_0) \geq \frac{1}{2}|a(x_0)|^2$ for all $x \in B_\delta(x_0)$. Here and in the following, $B_\delta(x_0) = \{x \in \mathbb{R}^n\colon |x - x_0| < \delta\}$ is the open ball of radius δ with center x_0. Define $\chi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ by

$$\chi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

so that $\operatorname{supp}\chi = \{x \in \Omega : \chi(x) \neq 0\} = B_1(0)$. Let $v(x) = a(x_0)\chi(\frac{x-x_0}{\delta})$. Then $v \in C^\infty(\overline{\Omega}; \mathbb{R}^m)$, $v|_{\partial\Omega} = 0$ and

$$\begin{aligned} 0 = \int_\Omega a(x) \cdot v(x)\,\mathrm{d}x &= \int_{B_\delta(x_0)} a(x) \cdot a(x_0)\chi(\frac{x - x_0}{\delta})\,\mathrm{d}x \\ &\geq \frac{1}{2}|a(x_0)|^2 \int_{B_\delta(x_0)} \chi(\frac{x - x_0}{\delta})\,\mathrm{d}x > 0, \end{aligned}$$

a contradiction. It follows that $a(x) = 0$ for all $x \in \Omega$. Using $a = 0$ and a similar argument, we can also show that $b = 0$ on $\partial\Omega \setminus \Gamma$. □

With these two tools at hand, we can prove the main result of this Section.

Theorem 8. *(Euler-Lagrange Equations* (ELE)*) If $u \in M$ is a critical point of I and $u \in C^2(\overline{\Omega};\mathbb{R}^m)$, then it satisfies the* Euler-Lagrange Equations

$$\begin{cases} -\mathrm{div}\,(\partial_A f(x,u(x),\nabla u(x))) + \partial_u f(x,u(x),\nabla u(x)) = 0, & x \in \Omega, \\ \partial_A f(x,u(x),\nabla u(x))\nu(x) + \partial_u g(x,u(x)) = 0, & x \in \partial\Omega \setminus \Gamma, \\ u(x) = u_0(x), & x \in \Gamma. \end{cases}$$

In this case, u is called a *classical solution* of the *variational problem* $DI(u)[v] = 0$.

The proof will also show the converse: if $u \in C^2(\overline{\Omega};\mathbb{R}^m) \cap M$ satisfies the ELEs, then it is a critical point of I.

Proof. By Definition 5 of critical points,

$$\begin{aligned} 0 = DI(u)[v] &= \int_\Omega \partial_u f(x,u(x),\nabla u(x)) \cdot v(x) + \partial_A f(x,u(x),\nabla u(x))\colon \nabla v(x)\,\mathrm{d}x \\ &+ \int_{\partial\Omega} \partial_u g(x,u(x)) \cdot v(x)\,\mathrm{d}x', \\ &\overset{\text{short}}{=} \int_\Omega \partial_u f \cdot v + \partial_A f\colon \nabla v\,\mathrm{d}x + \int_{\partial\Omega} \partial_u g \cdot v\,\mathrm{d}x', \end{aligned}$$

for every $v \in X_0$ (recall that $X_0 = \{v \in C^1(\overline{\Omega};\mathbb{R}^m) : v|_\Gamma = 0\}$). Here, the last line is just for introducing a short way of writing the integrands. Using Gauss' Theorem 6 and the definition of the divergence operator in Subsection 2.2, we have

$$\begin{aligned} &\int_{\partial\Omega} \sum_{1\le i\le m,} \sum_{1\le j\le n} (\partial_A f)_{ij} v_i \nu_j\,\mathrm{d}x' \\ &\overset{\text{Gauss}}{=} \int_\Omega \sum_{1\le i\le m} \sum_{1\le j\le n} \frac{\partial}{\partial x_j}\Big((\partial_A f)_{ij}\, v_i\Big)\,\mathrm{d}x \\ &\overset{\text{Chain Rule}}{=} \int_\Omega \sum_{1\le i\le m} \mathrm{div}\,(\partial_A f)_i\, v_i\,\mathrm{d}x + \int_\Omega \sum_{1\le i\le m} \sum_{1\le j\le n} (\partial_A f) : \nabla v\,\mathrm{d}x. \end{aligned}$$

Understanding this index notation and how Gauss' Theorem needs to be used is the most difficult part of the proof. We then have

$$0 = \int_\Omega (\partial_u f - \mathrm{div}(\partial_A f)) \cdot v\,\mathrm{d}x + \int_{\partial\Omega} ((\partial_A f)\nu + \partial_u g) \cdot v\,\mathrm{d}x', \tag{2.4}$$

for every $v \in X_0$. Since $u \in C^2(\overline{\Omega};\mathbb{R}^m)$, $f \in C^2(\overline{\Omega} \times \mathbb{R}^m \times \mathbb{R}^{m\times n};\mathbb{R})$, and $g \in C^2(\partial\Omega \times \mathbb{R}^m;\mathbb{R})$, and the normal vector field ν is (at least) continuous on $\partial\Omega \setminus \Gamma$, the functions

$$\Omega \ni x \mapsto (\partial_u f - \mathrm{div}(\partial_A f))(x,u(x),\nabla u(x))$$

and

$$\partial\Omega \ni x \mapsto ((\partial_A f)\nu + \partial_u g)(x,u(x))$$

are continuous as well. The theorem now follows directly from (2.4) and the Fundamental Lemma (FL). □

Of course, the question is: how do the ELEs help us find critical points? Solving the ELEs may also be a difficult task. For varying dimensions $m, n \in \mathbb{N}$, the Euler-Lagrange Equations are:

For $n = 1, m = 1$: an *ordinary differential equation*,

for $n = 1, m \geq 2$: a *system* of ordinary differential equations,

for $n \geq 2, m = 1$: a *partial differential equation*, and

for $n \geq 2, m \geq 2$: a *system* of partial differential equations.

Next, we look at two typical examples of Euler-Lagrange Equations that are a PDE. The following subsection contains a lot of terms from PDE theory. They appear here so that if you have seen them before, you can make the connection – but it is not necessary that you know them already.

2.3.1 The Dirichlet integral.

Here, $n \geq 2$, $m = 1$, the functional I is *quadratic* and the corresponding ELEs are a *linear* partial differential equation. For a matrix-valued (coefficient) function

$$K \in C^2(\overline{\Omega}; \mathbb{R}^{n\times n})$$

and functions $f_0 \in C^2(\overline{\Omega}; \mathbb{R}), g_0 \in C^0(\partial\Omega; \mathbb{R})$, define

$$\begin{aligned} f(x, u, A) &= \frac{1}{2} A^T K(x) A - f_0(x) u, \\ g(x', u) &= -g_0(x') u, \end{aligned}$$

where $x \in \Omega, x' \in N := \partial\Omega \setminus \Gamma$, $u \in \mathbb{R}$ and $A \in \mathbb{R}^{1\times n} = \mathbb{R}^n$ so that

$$I(u) = \int_\Omega \frac{1}{2} (\nabla u)^T(x) K(x) \nabla u(x) - f_0(x) u(x) \, \mathrm{d}x + \int_{\partial\Omega} g_0(x) u(x) \, \mathrm{d}x'.$$

Then the ELEs for the critical points of I are

$$\begin{cases} -\operatorname{div}\left(\frac{1}{2}(K(x) + K(x)^T)\nabla u(x)\right) = f_0(x), & \textit{in } \Omega, \\ \frac{1}{2}(K(x) + K(x)^T)\nabla u(x) \cdot \nu(x) = g_0(x), & \textit{on } N, \\ u(x) = u_0(x), & \textit{on } \Gamma. \end{cases}$$

(Exercise!) This is a *linear elliptic* PDE in u if $K(x)$ is positive definite, with *Dirichlet boundary conditions* on Γ and *Neumann* or 'natural' *boundary conditions* on N. If

$$K(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$-\operatorname{div}\left(\frac{1}{2}(K(x) + K(x)^T)\nabla \cdot\right) = -\Delta \cdot,$$

the *Laplace operator*. In this case, the ELEs are called the *Poisson equation* ($-\Delta u = f_0$) or the *Laplace equation* ($\Delta u = 0$) on Ω.

2.3.2 Minimal surfaces

As a second important example, we check the ELEs for the minimal surface problem from Subsection 1.3.3. Recall that we have a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$ and a function $u_0 \in C^0(\overline{\Omega}; \mathbb{R})$ is given. The graph of this function on $\partial\Omega$ is a frame for our soap film. The soap film itself is given by the graph $F = \{(x, u(x)) \in \mathbb{R}^3 : x \in \Omega\}$ of a function $u\colon \Omega \to \mathbb{R}$, such that $u = u_0$ on $\partial\Omega$ and such that F has minimal surface area, i.e.

$$I(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2}\,\mathrm{d}x$$

should be minimal. In this setting, the energy density is $f(x, u, A) = \sqrt{1 + |A|^2}$ with $A \in \mathbb{R}^2$ and $M = \{u \in C^1(\overline{\Omega}; \mathbb{R}) : u|_{\partial\Omega} = u_0|_{\partial\Omega}\}$. Since

$$\partial_A f(x, u, A) = \frac{1}{\sqrt{1 + |A|^2}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix},$$

the ELEs are

$$\begin{cases} -\mathrm{div}\left(\frac{\nabla u(x)}{\sqrt{1+\partial_1 u(x)^2+\partial_2 u(x)^2}}\right) = 0, & \text{in } \Omega, \\ u(x) = u_0(x), & \text{on } \partial\Omega. \end{cases}$$

Again, we have a PDE because $n = 2, m = 1$, but now it is *nonlinear* in u. This distinction is important as, like in the finite-dimensional case, the theory for solving nonlinear equations is much less complete than the theory for solving linear problems (solving nonlinear PDE is a key application of the Calculus of Variations). Here, the expression $\mathrm{div}\left(\frac{\nabla u(x)}{\sqrt{1+\partial_1 u(x)^2+\partial_2 u(x)^2}}\right)$ is the *mean curvature* of F. So the ELEs show that a minimal surface has zero mean curvature.

More examples will appear soon: an ODE in the next Section, systems of PDEs at a later point in time...

2.4 Different types of extrema

We have seen that the solutions of the ELEs are critical points of the functional I. But do critical points provide extrema? This need not be true, even in finite dimensions. Critical points may be saddle points. The aim of this Section is to derive necessary and sufficient conditions for critical points to be minimizers. As a first step, we need to define different types of extrema and will give an extensive, but simple example that motivates the distinction (and that will be used again).

Remark. We only talk about *minima* and *minimizers,* but everything could be adopted to the problem of finding *maxima* and *maximizers*, for example, by changing the sign of I and its boundary conditions.

Definition 9. (Global, strong, and weak minimizers) Let $M \subset C^1(\overline{\Omega}; \mathbb{R}^m)$ be the set of admissible functions and let $I\colon M \to \mathbb{R} \cup \{+\infty\}$ be a functional (slightly more general than the setting from Section 2.1). Then a point $u \in M$ is called

1. a *weak local minimizer,* if

$$(\exists \delta > 0)(\forall v \in M)\|u - v\|_{C^1} < \delta \Rightarrow I(u) \leq I(v),$$

2. a *strong local minimizer,* if

$$(\exists \delta > 0)(\forall v \in M)\|u - v\|_{C^0} < \delta \Rightarrow I(u) \leq I(v),$$

3. a *global minimizer*, if

$$(\forall v \in M) I(u) \leq I(v).$$

Here, the norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^0}$ are defined as in (2.3).

Remark. In finite dimensions, there would not be any distinction between weak and strong local minimizers, because all norms on the space of admissible vectors would be equivalent. The relevance of this distinction in the infinite-dimensional setting derives from the examples that we look at and from the applications that we have in mind (in other words: it need not/should not be clear now, but will become clear as we move on).

As a first context for the definition, consider the following implications:

$$u \text{ global min.} \overset{1)}{\Rightarrow} u \text{ strong loc. min.} \overset{2)}{\Rightarrow} u \text{ weak loc. min.} \overset{3)^*}{\Rightarrow} u \text{ crit. pt.},$$

and their converse implications:

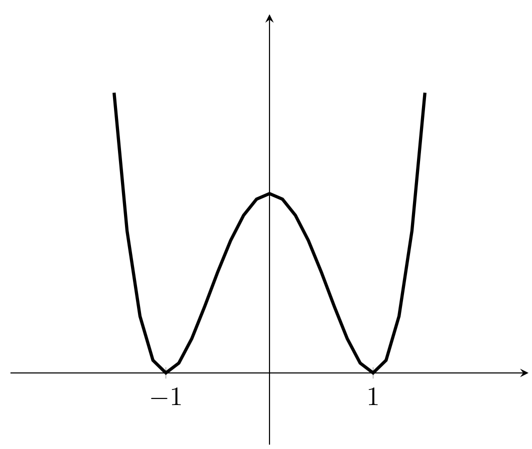
$$u \text{ global min.} \overset{4)}{\not\Leftarrow} u \text{ strong loc. min.} \overset{5)^*}{\not\Leftarrow} u \text{ weak loc. min.} \overset{6)}{\not\Leftarrow} u \text{ crit. pt.}$$

Then 1) and 2) follow directly from the definition, and we will show $3)^*$ in Theorem 13. 4) is known already from finite-dimensional counterexamples (not every minimum is a global minimum), $5)^*$ will be shown in the Example below and 6) is again obvious, because a critical point may also be a maximum. To summarize, the "unstarred" implications are trivial, but we will have a second look at the "starred" ones.

To illustrate the definition, we use the following example:

2.4.1 The double-well functional

Let $\Omega = (0, 1)$ and define the energy density $f(a) = (1 - a^2)^2$ (double-well potential).



Then consider the functional

$$I_\alpha(u) = \int_0^1 f(u'(x))\,\mathrm{d}x = \int_0^1 (1-u'(x)^2)^2\,\mathrm{d}x$$

on the set

$$M_\alpha = \{u \in C^1([0,1];\mathbb{R}) : u(0) = 0 \text{ and } u(1) = \alpha\}$$

of functions with fixed/clamped/Dirichlet boundary conditions $u(0) = 0$ and $u(1) = \alpha$ that vary with some parameter $\alpha \geq 0$.

Of course, the question is: What are minimizers of I_α in M_α?

Claim 10. We will show the following:

1. The only critical points of I_α in M_α are u_α defined by $u_\alpha(x) = \alpha x$.
2. If $\alpha = 1$, then u_1 is a global minimizer.
3. If $\alpha < 1$, then $\inf_{v\in M_\alpha} I_\alpha(v) = 0$, but there is no global minimizer.
4. If $\alpha \geq 1$, then u_α is a global minimizer.
5. If $\frac{1}{\sqrt{3}} < \alpha < 1$, then u_α is weak local minimizer, but not a strong local minimizer.
6. If $0 \leq \alpha < \frac{1}{\sqrt{3}}$, then there is no weak local minimizer.
7. If $\alpha = \frac{1}{\sqrt{3}}$, then .... Exercise!

Remark. This example also clearly illustrates the importance of boundary conditions!

Proof. First, we observe that regardless of the choice of α and u, $I_\alpha(u) \geq 0$. In the proof of 1., we use the following result related to the Fundamental Lemma and proved in the second Exercise.

Lemma 11. *(Lemma of du Bois-Reymond) Consider*

$$X_0 = \{v \in C^1([a,b];\mathbb{R}^m) : v(a) = v(b) = 0\}$$

and $h, H \in C^0([a,b];\mathbb{R}^m)$. *Then if*

$$\int_a^b h(x)\cdot v(x) + H(x)\cdot v'(x)\,\mathrm{d}x = 0$$

for all $v \in X_0$ *, then* $H \in C^1([a,b];\mathbb{R}^m)$ *and* $H'(x) = h(x)$ *for all* $x \in (\alpha,\beta)$.

1. For all $a \in \mathbb{R}$, we have $f'(a) = -4a(1-a^2)$, so by definition, a critical point u must satisfy

   $$DI_\alpha(u)[v] = -\int_0^1 4u'(x)(1-u'(x)^2)v'(x)\,\mathrm{d}x$$

   for all $v \in X_0$. By the du Bois-Reymond Lemma,

   $$2u'(x)^3 - 2u'(x) + c = 0 \tag{2.5}$$

for all $x \in (0,1)$ and a constant $c \in \mathbb{R}$ (plug in $h = 0$, $H(x) = f'(u'(x))$). Since (2.5) is a third-order polynomial in $u'(x)$, it has at most three distinct real roots. If $u'(x)$ takes on the values of two different roots at two different places, then since u' is continuous it must take on all values that lie inbetween, a contradiction. It follows that $u'(x) = \text{const}$. From the boundary conditions for $u \in M_\alpha$, it follows that $u_\alpha(x) = \alpha x$ is the only possible choice as a critical point. Note that by solving the corresponding ELEs

$$(4u'(1-(u')^2))'(x) = 0, \ \text{ for all } x \in (0,1),$$

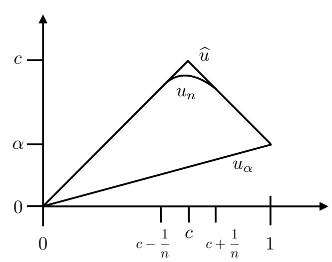
with boundary conditions $u(0) = 0, u(1) = \alpha$, we also turn up u_α, but we have not proved that it is the only critical point in M_α.

2. In this case, $I_1(u_1) = 0$, so u_1 is a global minimizer because $I_1(u) \geq 0$ for all $u \in M_1$.

3. In order to show the first statement, we construct a sequence of functions $u_n \in M_\alpha$ such that $\lim_{n\to\infty} I_\alpha(u_n) = 0$. A first observation is that if u can be constructed such that $u'(x) = \pm 1$ for all $x \in (0,1)$, then $I_\alpha(u) = 0$. For example, the function $\widehat{u}$ with

$$\widehat{u}(x) = \begin{cases} x, & 0 \leq x \leq c, \\ -x + 2c & c \leq x \leq 1, \end{cases}$$

and $c = \frac{\alpha+1}{2}$ satisfies $I_\alpha(\widehat{u}) = 0$ and $\widehat{u}(0) = 0$, $\widehat{u}(1) = \alpha$. However, $\widehat{u}$ is not continuously differentiable, its derivative has a jump, and thus $\widehat{u} \notin M_\alpha$ (note that we have $\widehat{u} \in PC^1([0,1];\mathbb{R})$, the space of *piecewise C^1-functions)*. So, the idea is to approximate $\widehat{u}$ by a sequence $u_n \in M_\alpha$ for which the "tip" is smoothened by a parabola on smaller and smaller intervals around $x = c$.



In particular, we set

$$u_n'(x) = \begin{cases} 1, & 0 \leq x \leq c - \frac{1}{n}, \\ -1, & c + \frac{1}{n} \leq x \leq 1, \\ -n(x-c), & c - \frac{1}{n} \leq x \leq c + \frac{1}{n}, \end{cases}$$

with

$$u_n(x) = \begin{cases} x, & 0 \leq x \leq c - \frac{1}{n}, \\ -x + 2c, & c + \frac{1}{n} \leq x \leq 1, \\ -\frac{n}{2}(x-c)^2 - \frac{1}{2n} + c, & c - \frac{1}{n} \leq x \leq c + \frac{1}{n}. \end{cases}$$

Then, $u_n \in M_\alpha$, there is pointwise convergence of $u_n(x)$ to $\widehat{u}(x)$ for all $x \in [0,1]$, and

$$I_\alpha(u_n) = \int_{c-1/n}^{c+1/n} (1 - n^2(x-c)^2)^2 \, \mathrm{d}x \le \int_{c-1/n}^{c+1/n} 1 \, \mathrm{d}x = \frac{2}{n} \overset{n\to\infty}{\to} 0.$$

Thus, we have shown that $\inf_{u \in M_\alpha} I_\alpha(u) = 0$. The fact that there is no global minimizer follows from 5. and 6., so we do not give a separate proof here.

4. This follows from the fact that if $\alpha \ge 1$, then for all $b \in \mathbb{R}$,

$$f(b) \ge f(\alpha) + f'(\alpha)(b - \alpha).$$

Hence, for all $u \in M_\alpha$,

$$\begin{aligned} I_\alpha(u) = \int_0^1 f(u'(x)) \, \mathrm{d}x &\ge \int_0^1 f(\alpha) + f'(\alpha)(u'(x) - \alpha) \, \mathrm{d}x \\ &= I_\alpha(u_\alpha) + f'(\alpha) \left(\int_0^1 u'(x) \, \mathrm{d}x - \alpha \right) = I_\alpha(u_\alpha). \end{aligned}$$

5. To show that u_α is a weak local minimizer, note that if $\frac{1}{\sqrt{3}} < \alpha < 1$, then $f''(\alpha) > 0$, so there exists a $\delta_\alpha > 0$ such that for all $|b - \alpha| < \delta_\alpha$,

$$f(b) \ge f(\alpha) + f'(\alpha)(b - \alpha),$$

as in 4. above. So for all $u \in M_\alpha$ such that

$$\sup_{x \in [0,1]} |u'(x) - \alpha| = \|u' - \alpha\|_{C^0} \le \|u - u_\alpha\|_{C^1} < \delta_\alpha,$$

we have

$$I_\alpha(u) \ge I_\alpha(u_\alpha)$$

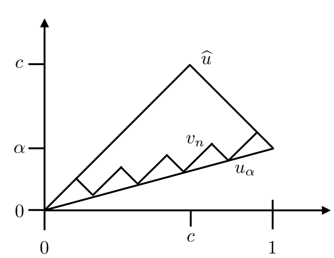
as in 4.

To show that u_α is *not* a strong minimizer, the idea is the following: construct a sequence of zig-zag (highly oscillating) functions $v_n \in M_\alpha$, such that

(a) $\|v_n - u_\alpha\|_{C^0} \to 0$ as $n \to \infty$, and

(b) $I_\alpha(v_n) = 0$.

Then v_n approximates u_α in C^0 but $I_\alpha(v_n) < I_\alpha(u_\alpha)$, so u_α cannot be a strong local minimizer. To get (b), we must have $v_n'(x) = \pm 1$ for all $x \in (0,1)$. We set $v_1 = \widehat{u}$ and then make v_n "more zig-zag" (more changes in the sign of v_n') as n increases so that the difference to u_α becomes smaller:



More precisely, let

$$w_1(x) = \begin{cases} (1-\alpha)x, & 0 \le x < c, \\ 2c(1-x), & c \le x \le 1, \end{cases}$$

so that $\widehat{u} = u_\alpha + w_1$. Extend w_1 periodically to be a continuous function on all of $\mathbb{R}$. Then set

$$w_n(x) = \frac{1}{n} w_1(nx), \text{ so that } w_n'(x) = w_1'(nx),$$

and set $v_n \simeq u_\alpha + w_n$, where $\simeq$ means that to get $v_n \in M_\alpha$, we need it to be smoothed at the tips, as in the parabola-construction in 3.! We ignore this "technical" detail in the following. We get that

$$\|v_n - u_\alpha\|_{C^0} = \sup_{x\in[0,1]} \|w_n(x)\| \le \frac{1}{n}(1-\alpha)c \overset{n\to\infty}{\to} 0$$

and that

$$\begin{aligned} I_\alpha(v_n) &= \int_0^1 f(v_n'(x))\,\mathrm{d}x \\ &= \int_0^1 f(\alpha + w_1'(nx))\,\mathrm{d}x \\ &\overset{y=nx}{=} \frac{1}{n}\int_0^n f(\alpha + w_1'(y))\,\mathrm{d}y \\ &\overset{w_1 \text{ is } 1-\text{periodic}}{=} \int_0^1 f(\alpha + w_1'(y))\,\mathrm{d}y = I_\alpha(v_1) = 0, \end{aligned}$$

which proves the claim.

6. If $0 \le \alpha < \frac{1}{\sqrt{3}}$, then $f''(\alpha) < 0$, i.e. f is locally at α strictly concave. As in the proof of 5., where we showed that u_α is a weak local minimizer, we can now show that u_α is a weak local *maximizer*. Thus, it is not a weak local minimizer. By Theorem 13 (next Section!), it is necessary for a weak local minimizer to be a critical point, so by 1., there can be no other weak minimizer.

□

2.5 Some necessary and sufficient conditions

In the finite-dimensional case, the definiteness of the second derivative or of the Hessian give us necessary and sufficient conditions for critical points to be extrema, cf. Section 1.4 and Exercise 1.IV. Here, the idea is to modify these conditions to apply to weak local minima of the functional I.

Definition 12. (Second variation) Let $u \in M$ and $v, w \in X_0$. Then the map

$$D^2 I(u)[v,w] := \frac{\mathrm{d}}{\mathrm{d}t} DI(u+tw)[v]|_{t=0}$$

is called the *second variation* of I at u along v, w. If I is as before with integral densities f, g, we have

$$\begin{aligned}
&D^2I(u)[v,w] \\
&= \int_\Omega \partial_u^2 f(x,u(x),\nabla u(x))v(x)\cdot w(x) \\
&\quad + \partial_A^2 f(x,u(x),\nabla u(x))[\nabla v(x),\nabla w(x)] \\
&\quad + \partial_u\partial_A f(x,u(x),\nabla u(x))[v(x),\nabla w(x)] \\
&\quad + \partial_u\partial_A f(x,u(x),\nabla u(x))[w(x),\nabla v(x)]\,\mathrm{d}x \\
&\quad + \int_{\partial\Omega} \partial_u^2 g(x,u(x))v(x)\cdot w(x)\,\mathrm{d}x'
\end{aligned}$$

(notation from Section 2.2!).

Proof. As for the first variation, we use the chain rule and the fact that differentiation and integration can be interchanged. This follows from the fact that f and g are twice continuously differentiable. □

Theorem 13. *Let $\Omega, \Gamma, u_0, M, X_0, f, g$ as before. Then the following holds:*

1. *If $u \in M$ is a weak local minimizer, then*
   (a) *for all $v \in X_0$, $DI(u)[v] = 0$, i.e. u is a critical point of I,*
   (b) *for all $v \in X_0$, $D^2I(u)[v,v] \geq 0$, i.e. the second variation is positive semi-definite.*
2. *If $u \in M = u_0 + X_0$ is a critical point and there exists $\gamma > 0$ such that for all $v \in X_0$,*

$$D^2I(u)[v,v] \geq \gamma\Big(\int_\Omega |v(x)|^2 + |\nabla v(x)|^2\,\mathrm{d}x + \int_{\partial\Omega} |v(x)|^2\,\mathrm{d}x'\Big),$$

   then u is a weak local minimizer.

Proof. To prove the two necessary conditions, we observe that a minimizer of a functional should be a minimizer along every direction and thus reduce the problem to a one-dimensional situation:

1. For $u \in M$, $v \in X_0$ define $\varphi_v\colon \mathbb{R} \to \mathbb{R}, t \mapsto I(u + tv)$. Since u is a weak local minimizer, there is a $\delta > 0$ such that for all $w \in M$,

$$\|w - u\|_{C^1} \leq \delta \Rightarrow I(u) \leq I(w).$$

   In particular, for all $|t| \leq \frac{\delta}{\|v\|_{C^1}}$, $\|u - (u + tv)\|_{C^1} \leq \delta$, so

$$\varphi_v(t) = I(u + tv) \geq I(u) = \varphi_v(0).$$

   Hence, 0 is a local minimizer of φ_v. By Exercise 1.IV (or Analysis II), necessarily,

$$\varphi_v'(0) = 0 \text{ and } \varphi_v''(0) \geq 0.$$

Hence, necessarily,

$$DI(u)[v] = \varphi_v'(0) = 0 \text{ and } D^2I(u)[v,v] = \varphi_v''(0) \geq 0.$$

Note, however, that being a minimizer for every φ_v does not imply being a local minimizer for I, not even in the finite-dimensional situation, cf. Exercise 1.II!

2. Let $u \in M$ be a critical point of I and $w \in X_0$. Then by Taylor's Theorem in one dimension, there is a $\theta \in [0,1]$ such that

$$\begin{aligned} I(u+w) = \varphi_w(1) &= \varphi_w(0) + \varphi_w'(0) + \frac{1}{2}\varphi_w''(\theta) \qquad (2.6)\\ &= I(u) + DI(u)[w] + \frac{1}{2}D^2I(u+\theta w)[w,w] \\ &= I(u) + \frac{1}{2}D^2I(u)[w,w] \\ &\quad + \frac{1}{2}(D^2I(u+\theta w)[w,w] - D^2I(u)[w,w]), \qquad (2.7) \end{aligned}$$

where for the last equality, we added a suitable zero and used that u is a critical point. To prove the claim, it is now sufficient to show that the sum of all terms adding to $I(u)$ in the last equality is ≥ 0 for sufficiently small $\|w\|_{C^1}$. We use Definition 12 to obtain

$$\begin{aligned} &D^2I(u+\theta w)[w,w] - D^2I(u)[w,w] \qquad (2.8)\\ &= \int_\Omega (\partial_u^2 f(x,u+\theta w,\nabla u+\theta\nabla w) - \partial_u^2 f(x,u,\nabla u))w\cdot w \qquad (2.9)\\ &\quad + 2(\partial_u\partial_A f(x,u+\theta w,\nabla u+\theta\nabla w) - \partial_u\partial_A f(x,u,\nabla u))[w,\nabla w]\\ &\quad + (\partial_A^2 f(x,u+\theta w,\nabla u+\theta\nabla w) - \partial_A^2 f(x,u,\nabla u))[\nabla w,\nabla w]\,\mathrm{d}x\\ &\quad + \int_{\partial\Omega} (\partial_u^2 g(x,u+\theta w) - \partial_u^2 g(x,u))w\cdot w\,\mathrm{d}x'. \end{aligned}$$

Now we use that the function

$$\begin{aligned} \partial_u^2 f^u \colon \overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times n} &\to \mathbb{R}^{m\times m}\\ (x,v,A) &\mapsto \partial_u^2 f(x,u(x)+v,\nabla u(x)+A) \end{aligned}$$

is uniformly continuous on compact subsets of $\overline{\Omega}\times\mathbb{R}^m\times\mathbb{R}^{m\times n}$, so that there exists a $\delta_1^\gamma > 0$ such that for all $w \in X_0$ and $x \in \overline{\Omega}$, if $\|w\|_{C^1} \leq \delta_1^\gamma$, then

$$|\partial_u^2 f(x,u(x)+\theta w(x),\nabla u(x)+\theta\nabla w(x)) - \partial_u^2 f(x,u(x),\nabla u(x)))| \leq \frac{\gamma}{4}.$$

In the same way, we choose $\delta_2^\gamma, \delta_3^\gamma, \delta_4^\gamma > 0$ for the other terms that appear in (2.8). Adding up, there exists a $\delta = \min(\delta_1^\gamma,\delta_2^\gamma,\delta_3^\gamma,\delta_4^\gamma) > 0$ such that for all $w \in X_0$, if $\|w\|_{C^1} < \delta$, then

$$\begin{aligned} &D^2I(u+\theta w)[w,w] - D^2I(u)[w,w]\\ &\leq \frac{\gamma}{4}\int_\Omega |w(x)|^2 + 2|w(x)||\nabla w(x)| + |\nabla w(x)|^2\,\mathrm{d}x + \frac{\gamma}{4}\int_{\partial\Omega}|w(x)|^2\,\mathrm{d}x'\\ &\leq \frac{\gamma}{2}\int_\Omega |w(x)|^2 + |\nabla w(x)|^2\,\mathrm{d}x + \frac{\gamma}{4}\int_{\partial\Omega}|w(x)|^2\,\mathrm{d}x'. \end{aligned}$$

We insert this estimate in (2.6). Together with the assumption on $D^2 I(u)[v,v]$, we get that

$$I(u+w) \geq I(u)$$

for all $\|w\|_{C^1} < \delta$, i.e. u is a weak local minimizer for I.

□

As a first application of Theorem 13, we look at the double-well functional I_α from the last section. We have shown that u_α are the only critical points, so they are necessarily the only candidates for weak local (and thus strong local or even global) minimizers. We have $f''(a) = 4(3a^2 - 1)$ for the double-well density f, so the second variation of I_α at u_α is

$$D^2 I_\alpha(u_\alpha)[v,v] = \int_0^1 4(3\alpha^2 - 1)(v'(x))^2 \,\mathrm{d}x.$$

Clearly, $D^2 I_\alpha(u_\alpha)[v,v] \geq 0$ if and only if $|\alpha| \geq \frac{1}{\sqrt{3}}$, so there are no weak local minimizers for $0 < \alpha < \frac{1}{\sqrt{3}}$. This is an alternative proof of 6. in the last Section. Moreover, if $\alpha > \frac{1}{\sqrt{3}}$, then $4(3\alpha^2 - 1) = \gamma' > 0$ and so

$$D^2 I_\alpha(u_\alpha)[v,v] = \gamma' \int_\Omega v'(x)^2 \,\mathrm{d}x \geq \gamma \int_\Omega v(x)^2 + v'(x)^2 \,\mathrm{d}x$$

for some $\gamma > 0$ by the *Poincaré Inequality* (the Poincaré Inequality may be known from functional analysis or some PDE or geometry course. If not, don't worry, it will be discussed in more detail later!) In particular, by Theorem 13, if $\alpha > \frac{1}{\sqrt{3}}$, then u_α is a weak local minimizer (compare 3. and 4. in the last Section).

We have seen that the ELEs (Theorem 8) provide a necessary condition for the necessary condition of u being a critical point for being a local minimizer (Theorem 13). Similarly, with the next result, we replace the second necessary condition in Theorem 13 on the *second* variation by a weaker, but pointwise necessary condition on f, the *Legendre-Hadamard* (LH) condition. The idea is the following:

1. In the second variation, the first term

$$D^2 I(u)[v,v] = \int_\Omega \partial_A^2 f(x, u(x), \nabla u(x))[\nabla v(x), \nabla v(x)] \,\mathrm{d}x + \cdots \geq 0$$

   is the *principal part*, the most relevant part. The LH-condition is a necessary condition that asks for the non-negativity of this part only.

2. The condition

$$\partial_A^2 f(x, u(x), \nabla u(x))[B,B] \geq 0 \tag{2.10}$$

   for all matrices $B \in \mathbb{R}^{m \times n}$ would be sufficient for the non-negativity of this part, but it would also be too strong to be necessary, as the integral term has more structure. Instead, the LH-condition replaces $B \in \mathbb{R}^{m \times n}$ with *rank-1-matrices* $\xi \otimes \eta \in \mathbb{R}^{m \times n}$ *for* $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$.

Theorem 14. *(Legendre-Hadamard condition* (LH)*) If $u \in X$ and*

$$D^2 I(u)[v,v] \geq 0$$

for all $v \in X_0$, then for all $x \in \Omega$, for all $\xi \in \mathbb{R}^m$ and for all $\eta \in \mathbb{R}^n$,

$$\sum_{i,k=1}^{m} \sum_{j,l=1}^{n} \frac{\partial^2}{\partial A_{ij} \partial A_{kl}} f(x, u(x), \nabla u(x)) \xi_i \xi_k \eta_j \eta_l \geq 0,$$

in short:

$$\partial_A^2 f(x, u(x), \nabla u(x))[\xi \otimes \eta, \xi \otimes \eta] \geq 0.$$

In particular, if u is a weak local minimizer of I, then the Legendre-Hadamard condition holds.

Proof. Fix $\xi \in \mathbb{R}^m$ and $\eta \in \mathbb{R}^n$. The idea is to consider the limit

$$\lim_{\delta \searrow 0} D^2 I(u)[v_\delta, v_\delta],$$

where

- $\nabla v_\delta \simeq \xi \otimes \eta$ – not exactly equal, but highly oscillating,
- $v_\delta \in X_0$ and $\operatorname{supp} v_\delta \subset B_\delta(x_0)$ for given $x_0 \in \Omega$,
- $v_\delta = O(\delta)$, so that lower-order terms in $D^2 I(u)[v_\delta, v_\delta]$ disappear in the limit.

Step 1: *"Freeze coefficients in the integral and get rid of lower-order terms"*: we show that if $D^2 I(u)[v,v] \geq 0$ for all $v \in X_0$, then for all $x_0 \in \Omega$ and $w \in C_0^1(B_1(0)) := \{w \in C^1(B_1(0)) : w|_{\partial B_1(0)} = 0\}$,

$$\int_{B_1(0)} \partial_A^2 f(x_0, u(x_0), \nabla u(x_0))[\nabla w(y), \nabla w(y)] \,\mathrm{d}y \geq 0. \tag{2.11}$$

Fix $w \in C_0^1(B_1(0))$, $x_0 \in \Omega$ and $\delta_0 > 0$ such that $B_{\delta_0}(x_0) \subset \Omega$. For all $0 < \delta < \delta_0$ define

$$w_\delta(x) = \delta w(\frac{x - x_0}{\delta}).$$

Then $\nabla w_\delta(x) = \nabla w(\frac{x-x_0}{\delta})$ and $w_\delta \in X_0$. By assumption, for all $0 < \delta \leq \delta_0$, we have

$$\begin{aligned}
0 \leq \frac{1}{\delta^n} \int_{B_\delta(x_0)} & \partial_A^2 f(x, u(x), \nabla u(x))[\nabla w(\frac{x-x_0}{\delta}), \nabla w(\frac{x-x_0}{\delta})] \\
&+ 2\delta \partial_A \partial_u f(x, u(x), \nabla u(x))[w(\frac{x-x_0}{\delta}), \nabla w(\frac{x-x_0}{\delta})] \\
&+ \delta^2 \partial_u^2 f(x, u(x), \nabla u(x))[w(\frac{x-x_0}{\delta}), w(\frac{x-x_0}{\delta})] \,\mathrm{d}x \\
= \int_{B_1(0)} & \partial_A^2 f(x_0 + \delta y, u(x_0 + \delta y), \nabla u(x_0 + \delta y))[\nabla w(y), \nabla w(y)] \,\mathrm{d}y \\
&+ O(\delta),
\end{aligned}$$

with the substitution $y = \frac{x-x_0}{\delta}$ in the last equality. In this estimate, we now apply the limit $\delta \searrow 0$ and use the boundedness and continuity of

$$y \mapsto \partial_A^2 f(x_0 + \delta y, u(x_0 + \delta y), \nabla u(x_0 + \delta y))$$

to obtain (2.11).

Step 2: *"Construction of v_δ"*: Fix $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$. Let $\chi \in C_0^\infty(B_1(0); \mathbb{R})$ with $\chi > 0$. For example, choose

$$\chi(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

as in the proof of (FL). Then define

$$v_\delta(y) := \delta \chi(y) \cos\left(\frac{\eta \cdot y}{\delta}\right) \xi \qquad \in C_0^1(B_1(0)),$$

with $y \mapsto \cos(\frac{\eta \cdot y}{\delta})$ a suitable "Wellblech"-function. It follows that

$$\nabla v_\delta(y) = \delta \cos(\frac{\eta \cdot y}{\delta}) \xi \otimes \nabla \chi(y) - \chi(y) \sin(\frac{\eta \cdot y}{\delta}) \xi \otimes \eta.$$

We can consider the first term in the derivative as *of lower order* for $\delta \searrow 0$. Thus, plugging v_δ into (2.11), we get

$$0 \leq \int_{B_1(0)} \partial_A^2 f(x_0, u(x_0), \nabla u(x_0))[\xi \otimes \eta, \xi \otimes \eta] \chi^2(y) \sin^2(\frac{\eta \cdot y}{\delta}) \, \mathrm{d}y + O(\delta) + O(\delta^2).$$

Taking the limit $\delta \searrow 0$ in this equality, we obtain

$$0 \leq \partial_A^2 f(x_0, u(x_0), \nabla u(x_0))[\xi \otimes \eta, \xi \otimes \eta] \lim_{\delta \searrow 0} \int_{B_1(0)} \chi^2(y) \sin^2(\frac{\eta \cdot y}{\delta}) \, \mathrm{d}y.$$

Now note that if $\eta = 0$, then (LH) holds automatically. If $\eta \neq 0$, then (LH) now follows from the following Lemma.

□

Lemma 15. *Let $\chi \in C_0^1(\Omega; \mathbb{R})$ and $\eta \in \mathbb{R}^n \setminus \{0\}$. Then*

$$\lim_{\delta \searrow 0} \int_{B_1(0)} \chi(y) \sin^2(\frac{\eta \cdot y}{\delta}) \, \mathrm{d}y = \frac{1}{2} \int_{B_1(0)} \chi(y) \, \mathrm{d}y.$$

(Later, we will show a general result that provides: $\sin^2(\frac{\eta \cdot y}{\delta}) \overset{L^2(B_1(0))}{\rightharpoonup} \frac{1}{2}$)

Proof. Note that for all $\alpha \in \mathbb{R}$, $\sin^2(\alpha) = \frac{1}{2}(1 - \cos(2\alpha))$, so

$$\int_{B_1(0)} \chi(y) \sin^2(\frac{\eta \cdot y}{\delta}) \, \mathrm{d}y = \frac{1}{2} \int_{B_1(0)} \chi(y) \, \mathrm{d}y - \frac{1}{2} \int_{B_1(0)} \chi(y) \cos(\frac{2\eta \cdot y}{\delta}) \, \mathrm{d}y.$$

We use Gauss' Theorem to deal with the second term on the right-hand side. By the chain rule,

$$\operatorname{div}_y(\sin(\frac{2\eta\cdot y}{\delta})\eta) \overset{\operatorname{div}_y(\eta)=0}{=} \eta\cdot\nabla_y \sin(\frac{2\eta\cdot y}{\delta}) = \frac{2}{\delta}\cos(\frac{2\eta\cdot y}{\delta})|\eta|^2.$$

It follows that

$$\begin{aligned}
&\lim_{\delta\searrow 0}\int_{B_1(0)} \chi(y)\cos(\frac{2\eta\cdot y}{\delta})\,\mathrm{d}y \\
&= \lim_{\delta\searrow 0}\frac{\delta}{2|\eta|^2}\int_{B_1(0)} \chi(y)\operatorname{div}_y\left(\sin(\frac{2\eta\cdot y}{\delta})\right)\,\mathrm{d}y \\
&\overset{\text{Gauss}}{=} -\lim_{\delta\searrow 0}\frac{\delta}{2|\eta|^2}\int_{B_1(0)} \nabla\chi(y)\cdot\eta\sin(\frac{2\eta\cdot y}{\delta})\,\mathrm{d}y = 0,
\end{aligned}$$

where in the last line, we have used that $\chi(y) = 0$ on $\partial B_1(0)$ and that the remaining integral is bounded independently of $\delta > 0$. □

Remark 16. Some special cases and applications of the (LH)-condition:

- If $m = n = 1$, then clearly, the (LH)-condition is equivalent to

  $$\partial_a^2 f(x, u(x), u'(x)) \geq 0,$$

  the second derivative of f with respect to the last component being non-negative. As an example, we look at the double-well-potential:

  $$\partial_a^2 f(\alpha) = 12\alpha^2 - 4 \geq 0 \Leftrightarrow \alpha \geq \frac{1}{\sqrt{3}}.$$

  This is a quick proof of the necessity of $\alpha \geq \frac{1}{\sqrt{3}}$ for u_α to be a weak local minimizer.

- If $n = 1, m \geq 1$, or $n \geq 1, m = 1$, then the (LH)-condition asks that the A-component of the Hessian of f,

  $$\partial_A^2 f(x, u(x), \nabla u(x)) \in \mathbb{R}^{k\times k}, \qquad k = m \text{ or } k = n,$$

  is positive semi-definite at every $x \in \Omega$. As an example, for the *Dirichlet integral*, this means that the symmetric part of $K(x)$,

  $$\frac{1}{2}(K(x) + K(x)^T),$$

  needs to be positive semi-definite at every $x \in \Omega$.

- More generally, for $m, n \geq 2$, and quadratic functionals I, where the ELEs are a linear system of PDEs, the (LH)-condition is a particular type of *ellipticity condition.*

- The (LH)-condition states that, roughly speaking, f should be *rank-one-convex* in the gradient component, cf. the next Section.

As a next step, we introduce more explicit necessary conditions on f for u to be a *strong* local minimizer of I. This leads to the concept of *quasiconvexity* of f. Recall that, roughly speaking, being a *strong* local minimizer means being a minimizer *in spite of oscillations,* as highly oscillating functions drop out of the C^1-neighbourhood of a potential minimizer, but may still be in a C^0-neighbourhood. In the following, we "quantify" this idea by looking at C^0-perturbations $u + w_\varepsilon$ of u that satisfy $\|w_\varepsilon\|_{C^0} \sim \varepsilon$ and $\|\nabla w_\varepsilon\|_{C^0} = O(1)$.

Let $u \in M$ be a strong local minimizer of I. In particular, there exists a $\delta_0 > 0$ such that for every $w \in C^1_0(\Omega; \mathbb{R}^m)$,

$$\|w\|_{C^0} \leq \delta_0 \Rightarrow u + w \in M \text{ and } I(u + w) \geq I(u).$$

We derive a *localized* necessary condition. Fix $x_0 \in \Omega$ and $\delta \leq \delta_0$ such that $B_\delta(x_0) \subset \Omega$. Now for any $w \in C^1_0(B^n_1(0); \mathbb{R}^m)$ and $\varepsilon \leq \min\{\delta, \frac{\delta}{\|w\|_{C^0}}\} = \tilde{\delta}$, define

$$w_\varepsilon(x) := \varepsilon w(\frac{x - x_0}{\varepsilon}).$$

Then $\nabla w_\varepsilon(x) = \nabla w(\frac{x-x_0}{\varepsilon})$, $u + w_\varepsilon \in M$ and $\|u - (u + w_\varepsilon)\|_{C^0} = \varepsilon\|w\|_{C^0} \leq \delta$ so

$$I(u + w_\varepsilon) \geq I(u).$$

It follows that

$$\begin{aligned}
0 &\leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n}(I(u + w_\varepsilon) - I(u)) \\
&= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^n} \int_{B_\varepsilon(x_0)} f(x, u + w_\varepsilon(x), \nabla u(x) + \nabla w(\frac{x - x_0}{\varepsilon})) - f(x, u(x), \nabla u(x)) \,\mathrm{d}x \\
&\overset{y=\frac{x-x_0}{\varepsilon}}{=} \lim_{\varepsilon \to 0} \int_{B_1(0)} f(x_0 + \varepsilon y, u(x_0 + \varepsilon y) + \varepsilon w(y), \nabla u(x_0 + \varepsilon y) + \nabla w(y)) \\
&\qquad - f(x_0 + \varepsilon y, u(x_0 + \varepsilon y), \nabla u(x_0 + \varepsilon y)) \,\mathrm{d}y \\
&= \int_{B_1(0)} f(x_0, u(x_0), \nabla u(x_0) + \nabla w(y)) - f(x_0, u(x_0), \nabla u(x_0)) \,\mathrm{d}y \\
&= \int_{B_1(0)} f(x_0, u(x_0), \nabla u(x_0) + \nabla w(y)) \,\mathrm{d}y - \mathrm{vol}(B^n_1(0)) f(x_0, u(x_0), \nabla u(x_0)).
\end{aligned}$$

This motivates the following definition.

Definition 17. (quasiconvexity, Morrey 1952) A function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *quasiconvex* in $A_0 \in \mathbb{R}^{m\times n}$, if for all $w \in PC^1_0(B^n_1(0); \mathbb{R}^m)$ ("piecewise C^1_0"),

$$\int_{B_1(0)} F(A_0 + \nabla w(y)) \,\mathrm{d}y \geq \int_{B_1(0)} F(A_0) \,\mathrm{d}y = \mathrm{vol}(B^n_1(0)) F(A_0).$$

We say that F *is quasiconvex*, if F is quasiconvex in every $A_0 \in \mathbb{R}^{m\times n}$.

Remark 18. There are several (equivalent) versions of this definition and we will use them as needed, without proof:

- $B^n_1(0)$ could be replaced by other suitable types of bounded domains D, for example, cubes.

- $w \in PC_0^1(D)$ can be replaced by $w \in C_0^1(D)$ or even $w \in C_c^\infty(D)$.

A direct consequence of our considerations is the following necessary condition for strong local minimizers:

Theorem 19. *(Morrey 1952) Let $u^* \in M$ be a strong local minimizer of I. Then the map*

$$f(x_0, u^*(x_0), \cdot) \colon \mathbb{R}^{m\times n} \to \mathbb{R}$$

is quasiconvex in $\nabla u^*(x_0)$ for all $x_0 \in \Omega$.

Example 20. We look at the (standard) double-well example and claim that f is quasiconvex in α if and only if $|\alpha| \geq 1$. In particular, this reproves that u_α is no strong local minimizer of I_α if $\alpha < 1$.

Proof. To check quasiconvexity of $f \colon \mathbb{R} \ni a \mapsto (1-a^2)^2 \in \mathbb{R}$ in $\alpha \in \mathbb{R}$, let $w \in PC_0^1([-1,1])$. By definition, we need to verify or disprove

$$\int_{-1}^{1} f(\alpha + w'(y))\,\mathrm{d}y \overset{!}{\geq} 2f(\alpha).$$

If $|\alpha| \geq 1$, then, cf. the proof of Claim 10.4., for all $\beta \in \mathbb{R}$,

$$f(\alpha+\beta) \geq f(\alpha) + f'(\alpha)\beta.$$

It follows that

$$\int_{-1}^{1} f(\alpha + w'(y))\,\mathrm{d}y \geq \int_{-1}^{1} f(\alpha)\,\mathrm{d}y + f'(\alpha)\int_{-1}^{1} w'(y)\,\mathrm{d}y = 2f(\alpha),$$

using $\int_{-1}^{1} w'(y)\,\mathrm{d}y = 0$.

If $|\alpha| < 1$, construct $w_\alpha \in PC^1([-1,1])$ such that

$$\int_{-1}^{1} f(\alpha + w_\alpha'(y))\,\mathrm{d}y = 0:$$

choose w_α as a piecewise affine “hat”, so that for all $y \in [-1,1]$, either $\alpha + w_\alpha'(y) = 1$ or $\alpha + w_\alpha'(y) = -1$. More precisely, set

$$w_\alpha(x) = \begin{cases} (1-\alpha)(x+1), & -1 \leq x \leq \alpha, \\ (-1-\alpha)(x-1), & \alpha \leq x \leq 1. \end{cases}$$

Then

$$\int_{-1}^{1} f(\alpha + w_\alpha'(y))\,\mathrm{d}y = \int_{-1}^{\alpha} f(1)\,\mathrm{d}y + \int_{\alpha}^{1} f(-1)\,\mathrm{d}y = 0,$$

and, at the same time, $2f(\alpha) > 0$, so f is not quasiconvex in any $|\alpha| < 1$. By Morrey's Theorem, in this case, u_α is no strong local minimizer of I_α. This quickly reproves some of the results in Claim 10.5. □

2.6 Types of Convexity

The conditions derived in the last Section suggest that we look more closely at different notions of convexity for the "A-"component of the density function f for given x, u:

$$F\colon \mathbb{R}^{m\times n} \to \mathbb{R}, \qquad F(A) = f(x,u,A).$$

Since it will be helpful later, we allow the function F to take on the value "$+\infty$", if it doesn't interfere with the definition.

Definition 21. (Types of Convexity)

1. $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ is called *convex*, if for all $A, B \in \mathbb{R}^{m\times n}$ and $\theta \in [0,1]$,

$$F(\theta A + (1-\theta)B) \le \theta F(A) + (1-\theta)F(B).$$

2. A locally bounded, measurable function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is called *quasiconvex* (qc), if for all $A \in \mathbb{R}^{m\times n}$ and for all $w \in PC^1_0(B^n_1(0);\mathbb{R}^m)$,

$$\int_{B_1(0)} F(A + \nabla w(y))\,\mathrm{d}y \ge \mathrm{vol}(B^n_1(0))F(A).$$

3. $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}\cup\{+\infty\}$ is called *rank-1-convex* (r1c), if for all $\theta \in [0,1]$ and for all $A, B \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(B-A) = 1$,

$$F(\theta A + (1-\theta)B) \le \theta F(A) + (1-\theta)F(B).$$

Even now, the relations between these different notions of convexity are not fully understood. We know the following.

Theorem 22. *(Types of Convexity) Let $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, then*

1. *F convex $\Rightarrow F$ quasiconvex $\Rightarrow F$ rank-1-convex.*
2. *If* $\min(m,n) = 1$*, then*

$$F \text{ convex } \Leftrightarrow F \text{ quasiconvex } \Leftrightarrow F \text{ rank-1-convex.}$$

3. *If $F \in C^2(\mathbb{R}^{m\times n})$, then*
F rank-1-convex $\Leftrightarrow$ F satisfies (LH) *in every $A \in \mathbb{R}^{m\times n}$, i.e. (*

$$(\forall A \in \mathbb{R}^{m\times n})(\forall \xi \in \mathbb{R}^m)(\forall \eta \in \mathbb{R}^n) \quad \partial^2_A F(A)[\xi\otimes\eta, \xi\otimes\eta] \ge 0.$$

Before we proceed to the proof, we discuss Statement 1. in the Theorem in some more detail: In general, we can say that the converse implications fail,

$$F \text{ convex } \not\Leftarrow F \text{ quasiconvex } \not\Leftarrow F \text{ rank-1-convex.}$$

More precisely,

- If $m = n = 2$, then the determinant $\det\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ with $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$ is quasiconvex, but not convex.

Proof. We show first that det is quasiconvex: For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{R}^{2\times 2}$, define $\operatorname{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ the *adjugate* of A. Then for all $A, \tilde{A} \in \mathbb{R}^{2\times 2}$,

$$\begin{aligned}\det(A + \tilde{A}) &= ad - bc + \tilde{a}\tilde{d} - \tilde{b}\tilde{c} + \tilde{a}d + a\tilde{d} - \tilde{b}c - b\tilde{c} \\ &= \det A + \det \tilde{A} + \operatorname{adj}(A)^T : \tilde{A}.\end{aligned}$$

Let now $w \in PC_0^1(B_1(0); \mathbb{R}^2)$, then, using this equality,

Proof.

$$\begin{aligned}&\int_{B_1(0)} \det(A + \nabla w(y))\, \mathrm{d}y \\ &= \operatorname{vol}(B_1(0)) \det A \\ &+ \int_{B_1(0)} \det(\nabla w(y))\, \mathrm{d}y + \operatorname{adj}(A) : \int_{B_1(0)} \nabla w(y)\, \mathrm{d}y. \end{aligned} \tag{2.12}$$

For the last term, we get

$$\operatorname{adj}(A) : \int_{B_1(0)} \nabla w(y)\, \mathrm{d}y \overset{\text{Gauss}}{=} \operatorname{adj}(A) : \int_{\partial B_1(0)} w(y) \otimes \nu\, \mathrm{d}y' \overset{w|_{\partial B_1(0)}=0}{=} 0.$$

For the middle term on the right-hand-side of (2.12), we get

$$\begin{aligned}\int_{B_1(0)} \det(\nabla w(y))\, \mathrm{d}y &= \int_{B_1(0)} \partial_1 w_1(y) \partial_2 w_2(y) - \partial_1 w_2(y) \partial_2 w_1(y)\, \mathrm{d}y \\ &\overset{\text{Gauss}}{=} - \int_{B_1(0)} \partial_2 \partial_1 w_1(y) w_2(y)\, \mathrm{d}y \\ &\quad + \int_{\partial B_1(0)} \partial_1 w_1(y) w_2(y) \nu_2(y)\, \mathrm{d}y' \\ &\quad + \int_{B_1(0)} \partial_1 \partial_2 w_1(y) w_2(y)\, \mathrm{d}y \\ &\quad - \int_{\partial B_1(0)} \partial_2 w_1(y) w_2(y) \nu_1(y)\, \mathrm{d}y' \\ &\overset{w|_{\partial B_1(0)}=0}{=} 0.\end{aligned}$$

Here, we have actually used $w \in C_0^2(B_1(0); \mathbb{R}^2)$ to apply Gauss' Theorem. It remains to use Remark 18 to argue that this is sufficient.

Now we show by counterexample that det is not convex: Let $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then for $\theta = \frac{1}{2}$,

$$\theta A + (1-\theta)B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so

$$\det A = \det B = \theta \det A + (1-\theta)\det B = -1 < \det(0) = 0.$$

□

□

- If $m \geq 3$, $n \geq 2$, then there is an example of a function $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ that is rank-1-convex, but not quasiconvex [Dac08, Sect 5.3.7]
- If $m = 2$, $n \geq 3$, then it is not known whether

$$F \text{ quasiconvex } \not\Leftarrow F \text{ rank-1-convex.}$$

Proof. (of Theorem 22)

To prove 2., assume 1.: Then it is sufficient to show that

$$F \text{ rank-1-convex } \Rightarrow F \text{ convex},$$

but this is also trivially satisfied since all $A, B \in \mathbb{R}^{1\times n}$ or $A, B \in \mathbb{R}^{m\times 1}$ have rank one. To prove 3., first note that for all $C \in \mathbb{R}^{m\times n}$,

$$\operatorname{rank}(C) = 1 \Leftrightarrow (\exists \xi \in \mathbb{R}^m)(\exists \eta \in \mathbb{R}^n) \quad C = \xi \otimes \eta.$$

To see this, let C be given such that $\operatorname{Im}(C) = \operatorname{span}(\xi)$. Then for all canonical basis vectors $e^j \in \mathbb{R}^n$, $e^j_i := \delta_{ij}$, there exists $\eta_j \in \mathbb{R}$ such that $Ce^j = \eta_j \xi$. In particular, $Ce^j = (\xi \otimes \eta)e^j$. Now proceed through the following equivalences:

$$\begin{aligned}
F \text{ rank-1-convex } &\overset{I}{\Leftrightarrow} (\forall \xi \in \mathbb{R}^m)(\forall \eta \in \mathbb{R}^n)(\forall A \in \mathbb{R}^{m\times n}) \text{ the map } \varphi_A^{\xi,\eta}\colon \mathbb{R} \to \mathbb{R}, \\
&\quad \text{given by } \varphi_A^{\xi,\eta}(t) = F(A + t\xi \otimes \eta), \text{ is convex}, \\
&\overset{II}{\Leftrightarrow} (\forall \xi \in \mathbb{R}^m)(\forall \eta \in \mathbb{R}^n)(\forall A \in \mathbb{R}^{m\times n})\forall t \in \mathbb{R}, \\
&\quad \partial_A^2 F(A + t\xi \otimes \eta)[\xi \otimes \eta, \xi \otimes \eta] = (\varphi_A^{\xi,\eta})''(t) \geq 0, \\
&\overset{III}{\Leftrightarrow} (\forall \xi \in \mathbb{R}^m)(\forall \eta \in \mathbb{R}^n)(\forall A \in \mathbb{R}^{m\times n}), \\
&\quad \partial_A^2 F(A)[\xi \otimes \eta, \xi \otimes \eta] \geq 0
\end{aligned}$$

$\overset{I}{\Rightarrow}$: Given $t_1, t_2 \in \mathbb{R}$ and $\theta \in [0,1]$, we have

$$\begin{aligned}\varphi_A^{\xi,\eta}(\theta t_1 + (1-\theta)t_2) &= F(A + \theta t_1 \xi \otimes \eta + (1-\theta) t_2 \xi \otimes \eta) \\ &= F(\theta(A + t_1 \xi \otimes \eta) + (1-\theta)(A + t_2 \xi \otimes \eta)) \\ &\overset{F \text{ r1c}}{\leq} \theta F(A + t_1 \xi \otimes \eta) + (1-\theta) F(A + t_2 \xi \otimes \eta) \\ &= \theta \varphi_A^{\xi,\eta}(t_1) + (1-\theta)\varphi_A^{\xi,\eta}(t_2).\end{aligned}$$

$\overset{I}{\Leftarrow}$: Given $\theta \in [0,1]$ and $A, B \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(B - A) = 1$, there exist $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$ such that $B - A = \xi \otimes \eta$, so

$$\begin{aligned}F(\theta B + (1-\theta)A) = F(A + \theta(B - A)) &= \varphi_A^{\xi,\eta}(\theta) \\ &\overset{\varphi_A^{\xi,\eta} \text{ convex}}{\leq} \theta \varphi_A^{\xi,\eta}(1) + (1-\theta)\varphi_A^{\xi,\eta}(0) \\ &= \theta F(B) + (1-\theta) F(A).\end{aligned}$$

$\overset{II}{\Leftrightarrow}$: This follows from the fact that for all $\varphi \in C^2(\mathbb{R};\mathbb{R})$,

$$\varphi \text{ convex } \Leftrightarrow \forall a \in \mathbb{R} \varphi''(a) \geq 0.$$

$\overset{III}{\Rightarrow}$: This is clear, just set $t = 0$.

$\overset{III}{\Leftarrow}$: This is clear since $A + t\xi \otimes \eta \in \mathbb{R}^{m\times n}$ holds for all $A \in \mathbb{R}^{m\times n}$, $\xi \in \mathbb{R}^m$, $\eta \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

It remains to prove 1.: To show that convex F are quasiconvex, we use Jensen's inequality (for a proof, see Exercise 3):

Lemma 23. *(Jensen's Inequality) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in $\mathbb{R}^n$, let $N \in \mathbb{N}$ and let $F\colon \mathbb{R}^N \to \mathbb{R}$ be convex, then for all $u \in C^0(\overline{\Omega};\mathbb{R}^N)$ and for all $u \in L^1(\Omega;\mathbb{R}^N)$,*

$$F\Big(\frac{1}{\operatorname{vol}(\Omega)} \int_\Omega u(y)\,\mathrm{d}y\Big) \leq \frac{1}{\operatorname{vol}(\Omega)} \int_\Omega F(u(y))\,\mathrm{d}y.$$

Let now $w \in PC_0^1(B_1(0);\mathbb{R}^m)$ and let $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be convex. In particular, F is locally bounded and measurable. Then by Gauss, $\int_{B_1(0)} \nabla w(y)\,\mathrm{d}y = 0$, so for all $A \in \mathbb{R}^{m\times n}$,

$$\frac{1}{\operatorname{vol}(B_1(0))} \int_{B_1(0)} F(A + \nabla w(y))\,\mathrm{d}y \overset{\text{Jensen}}{\geq} F\Big(\frac{1}{\operatorname{vol}(B_1(0))} \int_{B_1(0)} A + \nabla w(y)\,\mathrm{d}y\Big) = F(A),$$

and thus F is quasiconvex.
We "extensively sketch" the proof of the fact that quasiconvex functions F are rank-1-convex. A full proof is in [Dac08, Chapter 3.]

Let $A, B \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(B - A) \leq 1$. Then for all $\theta \in [0,1]$, we need to show

$$F(A + \theta(B - A)) \leq (1-\theta)F(A) + \theta F(B).$$

Rough idea: Using quasiconvexity of F, with $B_1(0)$ replaced by cubes

$$D_1(0) = \{x \in \mathbb{R}^n : 0 < x_i < 1\}, \quad \mathrm{vol}(D_1(0)) = 1,$$

we would get

$$\begin{aligned} F(A + \theta(B - A)) &\leq \int_{D_1(0)} F(A + \theta(B - A) + \nabla w(y))\,\mathrm{d}y \\ &= \int_{\Omega_A} F(A)\,\mathrm{d}y + \int_{\Omega} F(B)\,\mathrm{d}y \\ &= (1 - \theta)F(A) + \theta F(B), \end{aligned} \tag{2.13}$$

if $w \in PC_0^1(D_1(0); \mathbb{R}^m)$ can be constructed suitably, i.e. such that there exist $\Omega_A, \Omega_B \subset D_1(0)$ with $\mathrm{vol}(\Omega_A) = 1 - \theta$, $\mathrm{vol}(\Omega_B) = \theta$ and

$$\nabla w(y) = \begin{cases} -\theta(B - A) =: R_A, & y \in \Omega_A, \\ (1 - \theta)(B - A) =: R_B, & y \in \Omega_B. \end{cases}$$

Clearly, these properties of w can only be achieved approximately. The following lemma shows how this can be done.

Lemma 24. *(Dacorogna: Lemma 3.11) Let $R_A, R_B \in \mathbb{R}^{m \times n}$ such that*

$$\mathrm{rank}(R_A - R_B) \leq 1$$

and such that $0 \in \mathrm{co}(R_A, R_B)$, where

$$\mathrm{co}(R_A, R_B) = \{C \in \mathbb{R}^{m \times n} : \exists s \in [0, 1]\, C = sR_A + (1 - s)R_B\}$$

is the line segment joining R_A and R_B. Then for all $\varepsilon > 0$, there exists a

$$w^\varepsilon \in PC_0^1(D_1(0); \mathbb{R}^m)$$

and $\Omega_A^\varepsilon, \Omega_B^\varepsilon \subset D_1(0)$ such that

1. $1 - \theta - \varepsilon < |\mathrm{vol}\,\Omega_A^\varepsilon| < 1 - \theta$ and $\theta - \varepsilon < |\mathrm{vol}\,\Omega_B^\varepsilon| < \theta$,
2. $\|w^\varepsilon\|_{C^0} < \varepsilon$,
3. $\nabla w(y) = \begin{cases} R_A, & y \in \Omega_A^\varepsilon, \\ R_B, & y \in \Omega_B^\varepsilon. \end{cases}$
4. for all $y \in D_1(0)$, $\mathrm{dist}(\nabla w^\varepsilon(y), \mathrm{co}(R_A, R_B)) < \varepsilon$.

Using this Lemma, we can prove rank-1-convexity of F in the following way: we apply the Lemma to R_A, R_B defined as above. This is possible, as we have $0 \in \mathrm{co}(R_A, R_B)$

since $(1-\theta)R_A + \theta R_B = 0$. Following (2.13), for all $\varepsilon > 0$, we have

$$\begin{aligned}
&F(A+\theta(B-A)) \\
&\le \int_{D_1(0)} F(A+\theta(B-A)+\nabla w^\varepsilon(y))\,\mathrm{d}y \\
&= \int_{D_1(0)\setminus(\Omega_A^\varepsilon\cup\Omega_B^\varepsilon)} F(A+\theta(B-A)+\nabla w^\varepsilon(y))\,\mathrm{d}y \\
&\qquad + \operatorname{vol}(\Omega_A^\varepsilon)F(A) + \operatorname{vol}(\Omega_B^\varepsilon)F(B) \\
&\le \varepsilon C + (1-\theta)F(A) + \theta F(B),
\end{aligned}$$

where in the last inequality, we have used 4. from Lemma 24 and the local boundedness of F. Taking the infimum over all $\varepsilon > 0$ then finishes the proof of Theorem 22. □

Before we sketch the proof of Lemma 24, here is a brief discussion:

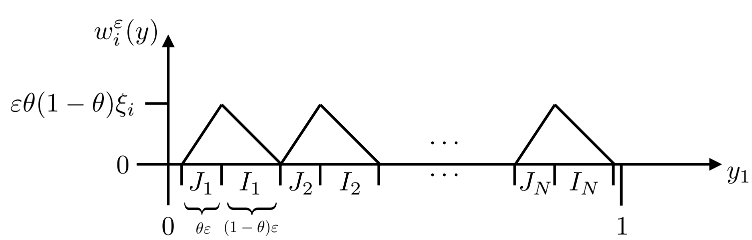
- There is a heuristics, why $\operatorname{rank}(R_B - R_A) \le 1$ is (somewhat) necessary for a construction of this kind: Let $\nabla w(y) = \begin{cases} L_A, & y \in \Omega_A, \\ L_B, & y \in \Omega_B. \end{cases}$ and $\Gamma = \partial\Omega_A \cap \partial\Omega_B$, where, in general Γ may contain n linearly independent vectors $x^j \in \Omega$. Then for all $x \in \Gamma$, as w is continuous,
$$w(x) = L_A x + c_A = L_B x + c_B,$$
so for all x^j, $(L_B - L_A)x^j = c_A - c_B$ and thus $\operatorname{rank}(L_B - L_A) \le 1$.

- In fact, the functions w^ε given in Lemma 24 are piecewise *affine* (see sketch of proof).

- The Lemma holds not only for $D_1(0)$, but for bounded open sets (see Step 2 in the sketch of proof)

Proof. (Sketch of the proof of Lemma 24)

Step 1: First, we assume that $R_B - R_A = \begin{pmatrix} \xi & 0 & \dots & 0 \end{pmatrix}$ for some $\xi \in \mathbb{R}^m$, i.e. $\eta = e^1$. Then all components w_i^ε of w^ε should be constant except in e^1-direction with slopes $-\theta\xi_i$ and $(1-\theta)\xi_i$. In dependence of y_1, we thus construct w_i^ε as a *Sägezahn*-function:



For all $i = 1, \dots, m$, we set

$$\partial_1 w_i^\varepsilon(y) \simeq \begin{cases} -\theta\xi_i, & y_1 \in \bar{I} := \cup_{k=1}^{N:=\lfloor \frac{1}{\varepsilon} \rfloor} I_k, \\ (1-\theta)\xi_i, & y_1 \in \bar{J} := \cup_{k=1}^{N} J_k, \\ 0, & \text{otherwise,} \end{cases}$$

where I_k, J_k are defined as in the picture. Here, $\simeq$ is used as we must decrease the height of the "Säge" suitably near the boundary $D_1(0)$ to get $w^\varepsilon|_{\partial D_1(0)} = 0$, in accordance with 4. For this, the assumption $0 \in \mathrm{co}(R_A, R_B)$ is used. Set

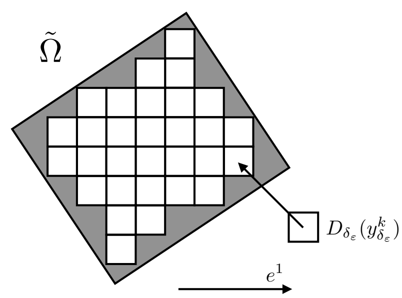
$$\Omega_A^\varepsilon := \{y \in D_1(0) : y_1 \in \bar{I}\} \text{ and } \Omega_B^\varepsilon := \{y \in D_1(0) : y_1 \in \bar{J}\}.$$

It follows that $1 - \theta - \varepsilon < |\mathrm{vol}\, \Omega_A^\varepsilon| < 1 - \theta$ and $\theta - \varepsilon < |\mathrm{vol}\, \Omega_B^\varepsilon| < \theta$ (1.),

$$\nabla w^\varepsilon(y) \simeq \begin{cases} -\theta(B-A), & y \in \Omega_A^\varepsilon, \\ (1-\theta)(B-A), & y \in \Omega_B^\varepsilon, \\ 0, & \text{otherwise,} \end{cases}$$

(3.) with w^ε piecewise affine, and $\|w^\varepsilon\|_{C^0} < \varepsilon \max_i |\xi_i|$ (see picture)(2.).

Step 2: It remains for us to see that the situation with general $\eta \in \mathbb{R}^n$ in the Lemma can be reduced to the case $\eta = e^1$ treated in Step 1:



Set $\tilde{\xi} := |\eta|\xi$ and $\tilde{\eta} := \frac{\eta}{|\eta|}$, so that $\tilde{\xi} \otimes \tilde{\eta} = \xi \otimes \eta$ and thus w.l.o.g. $|\tilde{\eta}| = 1$. Then we can choose $\mathcal{O} \in SO(n)$ orthogonal such that $\tilde{\eta} = e^1\mathcal{O}$ and $e^1 = \tilde{\eta}\mathcal{O}^T$. Set $\tilde{\Omega} = \mathcal{O}D_1(0)$ and $\tilde{R}_A = R_A\mathcal{O}^T$, $\tilde{R}_B = R_B\mathcal{O}^T$, then

$$\tilde{R}_B - \tilde{R}_A = (R_A - R_B)\mathcal{O}^T = \xi \otimes e^1.$$

Now we split $\tilde{\Omega}$ into many cubes $D_{\delta_\varepsilon}(y^k_{\delta_\varepsilon})$ with suitable length and midpoints $y^k_{\delta_\varepsilon}$, as in the picture, to get that the remaining grey part of $\tilde{\Omega}$ is small. Then we apply Step 1 to $\tilde{R}_B, \tilde{R}_A$ on each of the $D_{\delta_\varepsilon}(y^k_{\delta_\varepsilon})$ — after rescaling and translating to $D_1(0)$ – and glue together the resulting functions $w^{\varepsilon,k}$ – after re-rescaling and translating back – to get a suitable function $\tilde{w}^\varepsilon$ on $\tilde{\Omega}$ (note that we used $w^{\varepsilon,k}(\partial D_{\delta_\varepsilon}(y^k_{\delta_\varepsilon})) = 0$ for the glueing). The function

$$w^\varepsilon : D_1(0) \to \mathbb{R}^m, \quad w^\varepsilon(y) = \tilde{w}^\varepsilon(\mathcal{O}y),$$

should then be the right one.

□

After this extensive discussion and sketch of the proof of the general Theorem 22, we now look at the much more simple case of quadratic F.

Lemma 25. *Let $M \in \mathbb{R}^{(m\times n)\times(m\times n)}$ be symmetric and $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ given by $F(A) = MA : A$, i.e.*

$$F(A) = \sum_{i,k\in\{1,\dots,m\},j,l\in\{1,\dots,n\}} M_{(ij)(kl)} A_{ij} A_{kl}.$$

Then

$$F \text{ quasiconvex } \Leftrightarrow F \text{ rank-1-convex.}$$

Proof. "⇒" follows from Theorem 22. To show "⇐", we first observe that if F is quadratic, then for all $A \in \mathbb{R}^{m\times n}$ and $w \in PC_0^1(B_1^n(0);\mathbb{R}^m)$,

$$\begin{aligned}\int_{B_1^n(0)} F(A+\nabla w(y))\,\mathrm{d}y &\overset{M \text{ symmetric}}{=} |B_1^n(0)|MA : A + 2MA : \int_{B_1^n(0)} \nabla w(y)\,\mathrm{d}y \\ &\quad + \int_{B_1^n(0)} F(\nabla w(y))\,\mathrm{d}y \\ &= |B_1^n(0)|F(A) + \int_{B_1^n(0)} F(\nabla w(y))\,\mathrm{d}y,\end{aligned}$$

where we have used that by Gauss' Theorem, $\int_{B_1^n(0)} \nabla w(y)\,\mathrm{d}y = 0$. It follows that if F is quadratic, then

$$\begin{aligned}&F \text{ quasiconvex} \\ &\Leftrightarrow \forall w \in PC_0^1(B_1^n(0);\mathbb{R}^m) \quad \int_{B_1^n(0)} F(\nabla w(y))\,\mathrm{d}y \ge 0.\end{aligned} \tag{2.14}$$

On the other hand, if F is quadratic, in particular, $F \in C^2(\mathbb{R}^{m\times n})$ and by Theorem 22.3., if F is rank-1-convex, then for all $\xi \in \mathbb{R}^m, \eta \in \mathbb{R}^n$,

$$F(\xi\otimes\eta) = M(\xi\otimes\eta) : (\xi\otimes\eta) \overset{A\in\mathbb{R}^{m\times n}}{=} \partial_A^2 F(A)[\xi\otimes\eta,\xi\otimes\eta] \overset{\text{(LH)}}{\ge} 0. \tag{2.15}$$

It is interesting that (2.14) can now be shown via Fourier transform: for all $w \in PC_0^1(B_1^n(0);\mathbb{R}^m)$, define the extension $w_0 \in PC_0^1(\mathbb{R}^n;\mathbb{R}^m)$ by

$$w_0(y) = \begin{cases} w(y), & y \in B_1^n(0), \\ 0, & y \in \mathbb{R}^n \setminus B_1^n(0), \end{cases}$$

and let

$$\widehat{w_0}(\eta) = \int_{\mathbb{R}^n} w_0(y) e^{-2\pi i\langle y,\eta\rangle}\,\mathrm{d}y$$

be the Fourier transform of w_0. Then for $k = 1, \dots, m$, $l = 1, \dots, n$,

$$\begin{aligned}\widehat{(\nabla w_0)}_{kl}(\eta) &= \int_{\mathbb{R}^n} (\partial_l w_{0,k}(y)) e^{-2\pi i \langle y,\eta\rangle} \, \mathrm{d}y \\ &\overset{\text{Gauss}}{=} -\int_{\mathbb{R}^n} w_{0,k}(y)(\partial_l e^{-2\pi i\langle y,\eta\rangle}) \, \mathrm{d}y \\ &= 2\pi i \int_{\mathbb{R}^n} w_{0,k}(y)\eta_l e^{-2\pi i\langle y,\eta\rangle} \, \mathrm{d}y \\ &= 2\pi i \widehat{w_{0,k}}(y)\eta_l.\end{aligned}$$

Summing up, we get

$$\begin{aligned}\int_{B_1(0)} F(\nabla w(y)) \, \mathrm{d}y &= \int_{\mathbb{R}^n} M\nabla w_0(y) : \nabla w_0(y) \, \mathrm{d}y \\ &\overset{M \text{ symm.} + \text{Plancherel}}{=} \int_{\mathbb{R}^n} M\widehat{\nabla w_0}(\eta) : \overline{\widehat{\nabla w_0}(\eta)} \, \mathrm{d}\eta \\ &= 4\pi^2 \int_{\mathbb{R}^n} F(\widehat{w_0}(\eta) \otimes \eta) \, \mathrm{d}\eta \overset{(2.15)}{\geq} 0.\end{aligned}$$

By (2.14), this finishes the proof. □

Note that the function $\det\colon \mathbb{R}^{2\times 2} \to \mathbb{R}$ is *quadratic,* quasiconvex, but not convex, so the Lemma doesn't extend to convexity of F. Recall that if $f : \mathbb{R}^{m\times n} \to \mathbb{R}$ is quadratic, then the ELEs corresponding to $I(u) = \int_\Omega f(\nabla u(x)) \, \mathrm{d}x$ are *linear.*

We now go back to studying the functional I. Combining the results of this Section and of Section 2.5, we have seen how quasiconvexity of the "A-"component of the density function f, $F\colon \mathbb{R}^{m\times n} \to \mathbb{R}, F(A) = f(x, u, A)$, is necessary for the existence of strong local minimizers, and how rank-1-convexity of F is necessary for the existence of weak local minimizers. Now we look at the situation when I or f are convex. "As usual", we say that

$$I\colon M \ni u \mapsto I(u) \in \mathbb{R} \cup \{\infty\}$$

is *convex*, if for all $u, v \in M$, for all $\theta \in [0, 1]$,

$$I(\theta u + (1 - \theta)v) \leq \theta I(u) + (1 - \theta)I(v).$$

If we can put $<$ instead of $\leq$ for $u \neq v$, then I is *strictly convex.*

Theorem 26. *(Convexity of I)*

1. *If I is convex, then every critical point of I is a global minimizer.*
2. *If I is strictly convex, then there is only one minimizer and it is global.*
3. *If for all $x \in \Omega$, $x' \in \partial\Omega$, $f(x, \cdot, \cdot)\colon \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}$ and $g(x', \cdot)\colon \mathbb{R}^m \to \mathbb{R}$ are convex, then I is convex.*

Proof. The proof is straightforward:

1. If I is convex, then for all $\theta \in [0,1]$ and $u, v \in M$,

$$\theta(I(v) - I(u)) \geq -I(u) + I(u + \theta(v-u)).$$

For $\theta > 0$, we get

$$\begin{aligned} I(v) &\geq I(u) + \frac{1}{\theta}[I(u+\theta(v-u)) - I(u)] \\ &\overset{\theta \searrow 0}{\to} I(u) + DI(u)[v-u] \\ &\overset{u \text{ crit. pt.}}{=} I(u). \end{aligned}$$

2. Since all weak local minimizers are critical points, by 1., they are all global minimizers. If there are two distinct global minimizers, $u, \bar{u}$, then for all $\theta \in (0,1)$, by strict convexity,

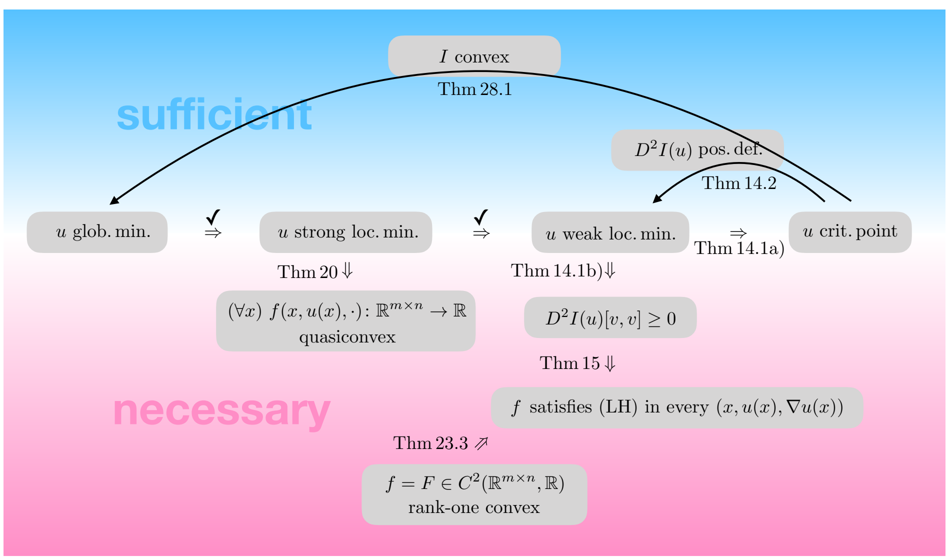
$$I(\theta u + (1-\theta)\bar{u}) < \theta I(u) + (1-\theta) I(\bar{u}) = \min_{v \in M} I(v),$$

a contradiction.

3. This follows directly by applying the definition(s).

□

The following figure wraps up some of the results of this Chapter:



2.7 Examples

How do the abstract results of this chapter apply? Here, a brief discussion of the minimal surface problem and a longer discussion of the Brachistochrone:

2.7.1 Minimal surface

As before, given a bounded domain $\Omega \subset \mathbb{R}^2$ and $u_0 \in C^0(\overline{\Omega}; \mathbb{R})$, set

$$M = \{u \in C^1(\overline{\Omega}; \mathbb{R}) : u|_{\partial\Omega} = u_0|_{\partial\Omega}\}$$

and $X_0 = C_0^1(\Omega; \mathbb{R})$, so $n = 2$, $m = 1$. Consider the functional

$$I \colon M \to \mathbb{R}, \quad I(u) = \int_\Omega \sqrt{1 + |\nabla u(x)|^2} \, \mathrm{d}x.$$

In Section 2.3.2 we derived the ELEs for this problem − but what are the extremal properties of its solutions? Are critical points minimizers? Here, Theorem 26.3 applies. Since $g = 0$, it suffices to show that $f \colon \mathbb{R}^2 \ni A \to f(A) = \sqrt{1 + |A|^2}$ is convex. This follows if $\partial_A^2 f(A)$ is positive semi-definite. For $i, j = 1, 2$, we calculate

$$\begin{aligned}
\partial_A f(A)_i &= \frac{1}{\sqrt{1 + |A|^2}} A_i, \\
\partial_A^2 f(A)_{ij} &= \left(1 + |A|^2\right)^{-\frac{1}{2}} \delta_{ij} - \left(1 + |A|^2\right)^{-\frac{3}{2}} A_j A_i,
\end{aligned}$$

with the Kronecker symbol $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. To see that the second derivative is positive semi-definitie, for all $\eta \in \mathbb{R}^2$, just apply Cauchy-Schwartz to get

$$\begin{aligned}
\partial_A^2 f(A)[\eta, \eta] &= \left(1 + |A|^2\right)^{-\frac{1}{2}} |\eta|^2 - \left(1 + |A|^2\right)^{-\frac{3}{2}} |A \cdot \eta|^2 \\
&\geq \left(1 + |A|^2\right)^{-\frac{3}{2}} \left(\left(1 + |A|^2\right) |\eta|^2 - |A|^2 |\eta|^2\right) \geq 0.
\end{aligned}$$

It follows that f is convex, so all critical points of I are *global* minimizers.

2.7.2 Brachistochrone curve

Recall the functional

$$I(u) = \int_0^l \sqrt{\frac{1 + u'(x)^2}{2g(h - u(x))}} \, \mathrm{d}x$$

associated to the Brachistochrone problem for

$$u \in M = \{w \in C^1([0, l], \mathbb{R}) : w(0) = h, w(l) = 0\}.$$

It turns out that the abstract results we obtained so far don't apply well in this case. But it is still interesting to see how they fail and to do some explicit calculations that quantify the fastest marble run:

1. *What are the ELEs?*

   The function

$$\begin{aligned}
f \colon \mathbb{R}^2 &\to \mathbb{R} \\
(u, a) &\mapsto \sqrt{\frac{1 + a^2}{2g(h - u)}}
\end{aligned}$$

is the energy density for the Brachistochrone problem. We have

$$\partial_u f(u,a) = \frac{1}{2\sqrt{2g}}(1+a^2)^{1/2}(h-u)^{-3/2},$$
$$\partial_a f(u,a) = \frac{a}{\sqrt{2g}}(1+a^2)^{-1/2}(h-u)^{-1/2}.$$

After taking out the factor $\frac{1}{\sqrt{2g}}$ and omitting (x) in $u(x)$, $u'(x)$, the ELEs are:

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(u'(1+u'^2))^{-1/2}(h-u)^{-1/2}\right) + \frac{1}{2}(1+u'^2)^{1/2}(h-u)^{-3/2} = 0,$$
$$u(0) = h,$$
$$u(l) = 0,$$

where the first equality should hold for all $x \in (0,l)$.

2. *Noether's Theorem and a simplification of the ELEs:*

   The ELEs look ugly and it is in fact not straightforward to solve them. The following result is a special case of Noether's Theorem that is very helpful in this and other cases.

   Proposition 27. *If $n = 1$ and f does not depend on x, $\partial_x f = 0$, then the function*

   $$E(u,u') = u' \cdot \partial_A f(u,u') - f(u,u')$$

   is constant along all solutions u of the ELEs.

   Proof. This follows by a direct calculation. In Exercise 4 on "Eshelby's tensor", a higher-dimensional result is shown, which includes Proposition 27 as a special case. □

   We apply this result:

   $$E(u,u') = \frac{1}{\sqrt{2g}}u'^2(1+u'^2)^{-1/2}(h-u)^{-1/2} - \frac{1}{\sqrt{2g}}(1+u'^2)^{1/2}(h-u)^{-1/2}$$
   $$= -\frac{1}{\sqrt{2g}}(1+u'^2)^{-1/2}(h-u)^{-1/2},$$

   so to find a solution of the ELEs, we can look for solutions of the simpler equation

   $$-\frac{\partial}{\partial x}\left((1+u'^2)^{-1/2}(h-u)^{-1/2}\right) = 0. \tag{2.16}$$

3. *Verification of the parameterized cycloid solution:*

   There are several strategies for solving the ELEs or (2.16). All of them involve several tricks and they get quite messy[1]. The solution is a *cycloid*, usually given in

[1] For a reference, see the manuscript "Variationsrechnung und Sobolevräume" by Professor Hans-Dieter Alber, p. 49,
https://www.mathematik.tu-darmstadt.de/media/analysis/lehrmaterial_anapde/alber/vari.pdf.

parameterized form through the angle $\varphi \in (\varphi_0, \varphi_l)$:

$$u(x(\varphi)) = y(\varphi) = h - \frac{1}{2}k'(\cos(2\varphi) + 1),$$
$$x(\varphi) = \frac{1}{2}k' \sin(2\varphi) + k'\varphi + k,$$

where $k, k', \varphi_0, \varphi_l$ are constants determined by the boundary conditions

$$\begin{aligned} x(\varphi_0) &= 0, \\ x(\varphi_l) &= l, \\ y(\varphi_0) &= h, \\ y(\varphi_l) &= 0. \end{aligned} \tag{2.17}$$

We verify that this gives a solution of the ELEs (this is messy already....): In order to calculate u', we use that by the chain rule,

$$u'(x) = \varphi'(x) y'(\varphi(x)),$$

where $\varphi = x^{-1}$ is the inverse function of $x(\varphi)$ with

$$\varphi'(x) = \frac{1}{x'(\varphi(x))} = \frac{1}{k'(\cos(2\varphi) + 1)} = \frac{1}{2(h - u(x))}.$$

Since $y'(\varphi) = k' \sin(2\varphi)$, we get

$$u'(x) = \frac{k' \sin(2\varphi(x))}{2(h - u(x))}.$$

Next, calculate and simplify

$$\begin{aligned} 1 + u'^2 &= \frac{4(h-u)^2 + k'^2 \sin^2(2\varphi)}{4(h-u)^2} \\ &= \frac{k'^2(\cos^2(2\varphi) + 2\cos(2\varphi) + 1) + k'^2 \sin^2(2\varphi)}{4(h-u)^2} \\ &= \frac{2k'^2(\cos^2(2\varphi) + 1)}{4(h-u)^2} = \frac{k'}{h-u}. \end{aligned}$$

It follows that

$$\begin{aligned} u'(1 + u'^2))^{-1/2}(h-u)^{-1/2} &= \frac{k' \sin(2\varphi)}{2(h-u)} \frac{(h-u)^{1/2}}{\sqrt{k'}} \frac{1}{(h-u)^{1/2}}. \\ &= \frac{\sqrt{k'}\sin(2\varphi)}{2(h-u)} = \frac{u'}{\sqrt{k'}}, \end{aligned}$$

hence, the first term in the ELE is

$$\begin{aligned} -\frac{1}{\sqrt{k'}} \frac{\mathrm{d}}{\mathrm{d}x} u'(x) &= -\frac{\varphi'(x)}{\sqrt{k'}} \frac{\mathrm{d}}{\mathrm{d}\varphi} \frac{k' \sin(2\varphi)}{2(h - y(\varphi))} \\ &= -\frac{\sqrt{k'}}{2} \frac{k' \left(\cos(2\varphi)(\cos(2\varphi) + 1) + \sin^2(2\varphi)\right)}{2(h - u(x))^3} \\ &= -\frac{\sqrt{k'}}{2} \frac{1}{(h - u(x))^2}. \end{aligned}$$

On the other hand, the second term in the ELE also simplifies to

$$\frac{1}{2}(1+u'^2)^{1/2}(h-u)^{-3/2} = \frac{\sqrt{k'}}{2}(h-u)^{-2}.$$

The boundary conditions are verified in the next paragraph.

Just for fun, we show that it is much simpler to see that u solves (2.16): In fact, by the above calculations,

$$E(u,u') = -\frac{1}{\sqrt{2g}}(1+u'^2)^{-1/2}(h-u)^{-1/2} = \frac{1}{\sqrt{2gk'}} = \text{const.}$$

4. *Determine parameters from boundary conditions:*

   We use (2.17) to determine $k, k', \varphi_0, \varphi_l$. We get

$$\begin{aligned} &\text{I)} & \tfrac{k'}{2}\sin(2\varphi_0) + k'\varphi_0 + k &= 0,\\ &\text{II)} & \tfrac{k'}{2}\sin(2\varphi_l) + k'\varphi_l + k &= l,\\ &\text{III)} & \tfrac{k'}{2}(\cos(2\varphi_0)+1) &= 0,\\ &\text{IV)} & \tfrac{k'}{2}(\cos(2\varphi_l)+1) &= h. \end{aligned}$$

   From III), we get $\varphi_0 = \frac{\pi}{2}$. Plugging this into I), we have $k = -\frac{\pi}{2}k'$. With this, II) is equivalent to

$$\sin(2\varphi_l) + 2\varphi_l - \pi = \frac{2l}{k'},$$

   and from IV), using $\cos(2\varphi_l) + 1 = 2\cos^2(\varphi_l)$, we get

$$k' = \frac{2h}{\cos(2\varphi_l)+1} = \frac{h}{\cos^2(\varphi_l)}. \tag{2.18}$$

   Combining these two, φ_l satisfies

$$g(\varphi_l) := \frac{\sin(2\varphi_l)+2\varphi_l-\pi}{\cos(2\varphi_l)+1} = \frac{\cos(\varphi_l)\sin(\varphi_l)+\varphi_l-\pi/2}{\cos^2(\varphi_l)} = \frac{l}{h}. \tag{2.19}$$

   In the range $\frac{\pi}{2} < \varphi_l < \frac{3\pi}{2}$, g is bijective. To see this, we show that g is monotone which follows from

$$g'(\varphi_l) = \frac{2\cos^3(\varphi_l) + 2\sin(\varphi_l)(\cos(\varphi_l)\sin(\varphi_l)+\varphi_l-\pi/2)}{\cos^3(\varphi_l)} \geq 0.$$

   For $\varphi_l \in (\pi/2, 3\pi/2)$, this is equivalent to the numerator being non-positive,

$$\begin{aligned} n(\varphi_l) &= 2\cos^3(\varphi_l) + 2\sin(\varphi_l)(\cos(\varphi_l)\sin(\varphi_l)+\varphi_l-\pi/2)\\ &= 2\cos(\varphi_l) + 2\sin(\varphi_l)(\varphi_l-\pi/2) \leq 0. \end{aligned}$$

   To see this, note that $n(\pi/2) = 0$ and

$$n'(\varphi_l) = 2\cos(\varphi_l)(\varphi_l - \pi/2) \leq 0$$

   in this range. Thus,

$$(\frac{\pi}{2}, \frac{3\pi}{2}) \ni \varphi_l = g^{-1}(\frac{l}{h}).$$

   In particular, $g(\frac{\pi}{2}) = 0$ and $\lim_{\varphi_l \to \frac{3\pi}{2}} g(\varphi_l) = +\infty$, so any positive ratio $\frac{l}{h}$ is covered. From there, we can also solve for k' and k.

5. Before we proceed, we should relate the solution x, y of the ELEs to the standard cycloid parameterization

$$\tilde{x}(t) = r(t - \sin t),$$
$$\tilde{y}(t) = r(1 - \cos t).$$

The curve $(\tilde{x}(t), \tilde{y}(t))$ is traced out by a fixed point on a turning wheel's rim. The wheel has radius r and $t = 0$ with $\tilde{x}(0) = 0 = \tilde{y}(0)$ is where the wheel starts turning. At $t = 2\pi$, the wheel has turned exactly once. The easiest way to see the cycloid is to write

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}(t) = r \begin{pmatrix} t \\ 1 \end{pmatrix} - r \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

where $m(t) = r \begin{pmatrix} t \\ 1 \end{pmatrix}$ is the position of the wheel's axis.
Now from solving the ELEs we have

$$x(\varphi) = \frac{1}{2}k'(\sin(2\varphi) + 2\varphi - \pi),$$
$$y(\varphi) = h - \frac{1}{2}k'(\cos(2\varphi) + 1).$$

This is a cycloid upside down, translated on the y-axis by h. So we set

$$\tilde{x}(t) = x(\varphi) = \frac{1}{2}k'(\sin(2\varphi) + 2\varphi - \pi) = \frac{k'}{2}(t - \sin t),$$
$$\tilde{y}(t) = -(y(\varphi) - h) = \frac{1}{2}k'(\cos(2\varphi) + 1) = \frac{k'}{2}(1 - \cos t),$$

for $t = 2\varphi - \pi$. We obtain $0 \le t \le t_l = 2\varphi_l - \pi$ and $r = \frac{k'}{2}$.
The calculations in 4. can thus be rephrased in the following way: Given h and l, find the radius r of the wheel and the time t_l such that

$$h = \tilde{y}(t_l) = r(1 - \cos t_l),$$
$$l = \tilde{x}(t_l) = r(t_l - \sin t_l).$$

(the calculations do not appear to become simpler though).

6. *Examples:*

   (a) Simple explicit values for φ_l, k', k are $\varphi_l = \frac{5}{4}\pi$ with $\frac{l}{h} = 1 + 1.5\pi \simeq 5.71$, $k' = 2h$, $k = -\pi h$, so $h = r$. Then

   $$u(x(\varphi)) = -h\cos(2\varphi).$$

   (b) Another example is the case $h = 0$: then with $\varphi_l = \frac{3}{2}\pi$, $k' = \frac{l}{\pi}$ (from II)) and $k = -\frac{l}{2}$, we get a solution

   $$u(x(\varphi)) = -\frac{l}{2\pi}(\cos(2\varphi) + 1),$$

   a symmetric curve with minimum $y(\pi) = -\frac{l}{\pi}$ (see 7.).

7. *When does u have a negative minimum?*

   It is the first striking feature of the Brachistochrone curve that its minimum may be below the target height $u(l) = 0$. When does this happen (we have some ideas about this as we know what the cycloid looks like)?

   To get local extrema of u, we check that $u'(x) = 0$ if

   $$y'(\varphi) = k' \sin(2\varphi) = 0.$$

   For $\pi/2 < \varphi < 3\pi/2$, this happens at $\varphi = \pi$. In fact,

   $$y''(\pi) = 2k' \cos(2\pi) > 0,$$

   so π gives a local minimum. The value is

   $$y(\pi) = h - k',$$

   as $k' = 2r = u_{\max} - u_{\min}$. By (2.18), $y(\pi) < 0$. In Example (a), $-h = -r$ is the minimum! The minimum is attained at $x(\pi) = \frac{\pi}{2} k' = \pi r$, after a half turn of the wheel. However, the negative local minimum is never reached if $\varphi_l \leq \pi$. Since g is monotone, this happens if $g(\pi) = \frac{\pi}{2} \geq \frac{l}{h}$, or, more intuitively, if the average slope $\frac{h}{l} \geq \frac{\pi}{2}$ is sufficiently large.

8. *What is the marble's speed?*

   Using the relations we determined when we modelled I, the marble's speed is

   $$|v(x)| = |\dot{x}|(1 + u'(x)^2) = \sqrt{2g(h - u(x))}.$$

   In particular, by assumption, $|v(0)| = 0$, and we see that $|v|$ is maximal where u is minimal (cf. 7.), and that $|v(l)| = \sqrt{2gh}$. This shows that in Example (b), as expected, the end point $(l, 0)$ is reached with zero velocity.
   Regarding the slope of u, we note that

   $$u'(0) = \lim_{\varphi \to \pi/2} \tan(\varphi) = +\infty,$$

   supporting the intuition that the marble should gain speed as quickly as possible.

9. *So, how long does it take?*

   With u the solution of the ELEs from above, and using the calculations from above,

   $$\begin{aligned} I(u) &= \int_0^l \sqrt{\frac{1 + u'(x)^2}{2g(h - u(x))}} \, \mathrm{d}x = \sqrt{\frac{k'}{2g}} \int_{\frac{\pi}{2}}^{\varphi_l} (h - u(x))^{-1} 2(h - u(x)) \, \mathrm{d}\varphi \\ &= \sqrt{\frac{2k'}{g}} (\varphi_l - \frac{\pi}{2}). \end{aligned}$$

10. *The Brachistochrone is also the Tautochrone.*

    The Tautochrone curve describes the marble run (more commonly, it is associated to a pendulum) with the property that no matter where the marble starts (with zero initial velocity), the time it takes to get to the end is always the same. We show that the cycloid Brachistochrone curve u is also the Tautochrone curve if the end point is the lowest point at $(x(\pi), y(\pi) = u_\pi)$, (cf. 7): The time it takes a marble that starts at any point $(\varphi_*, u(x_*) = h_* = y(\varphi_*))$ and runs along u to get to (π, u_π) is, using the calculations from above,

$$\begin{aligned}
I_*(u) &= \int_{l_*}^{x(\pi)} \sqrt{\frac{1+u'(x)^2}{2g(h_* - u(x))}}\,\mathrm{d}x \\
&= \sqrt{\frac{2k'}{g}} \int_{\varphi_*}^{\pi} (h_* - y(\varphi))^{-1/2}(h - y(\varphi))^{1/2}\,\mathrm{d}\varphi \\
&= \sqrt{\frac{2k'}{g}} \int_{\varphi_*}^{\pi} \frac{\sqrt{\cos(2\varphi)+1}}{\sqrt{\cos(2\varphi)+1+\kappa}}\,\mathrm{d}\varphi,
\end{aligned}$$

with $\kappa = 2\frac{h_*-h}{k'}$. We use the substitution $w(\varphi) = \sqrt{\cos(2\varphi)+1+\kappa}$. Then

$$\sqrt{\cos(2\varphi)+1} = \sqrt{y^2-\kappa}$$

and

$$\begin{aligned}
w(\varphi_*) &= \sqrt{\frac{2}{k'}}\sqrt{h_* - y(\varphi_*)} = 0, \\
w'(\varphi) &= -\frac{\sin(2\varphi)}{w(\varphi)}, \quad \mathrm{d}\varphi = -\frac{w}{\sin(2\varphi)}\,\mathrm{d}w, \\
2\varphi &= \arccos(w^2-\kappa-1), \\
\sin(2\varphi) &= \sin(\arccos(w^2-\kappa-1)) \\
&= \sqrt{1-(w^2-\kappa-1)^2} = \sqrt{2+\kappa-w^2}\sqrt{w^2-\kappa},
\end{aligned}$$

so

$$\begin{aligned}
I_*(u) &= -\sqrt{\frac{2k'}{g}} \int_0^{w(\pi)} \frac{1}{\sqrt{2+\kappa-w^2}}\,\mathrm{d}w \\
&= \sqrt{\frac{2k'}{g}} \left[\arctan\left(\frac{w\sqrt{2+\kappa-w^2}}{w^2-\kappa-2}\right)\right]_0^{w(\pi)}.
\end{aligned}$$

Since $\arctan(0) = 0$, it remains to insert $w(\pi)$. We get

$$w(\pi) = \sqrt{\frac{2}{k'}}\sqrt{h_* - y(\pi)} \overset{7}{=} \sqrt{\frac{2}{k'}}\sqrt{h_* - h + k'},$$

so $w(\pi)^2 - \kappa - 2 = 0$, so

$$\arctan\left(\frac{w(\pi)\sqrt{2+\kappa-w(\pi)^2}}{w(\pi)^2-\kappa-2}\right) = \arctan(+\infty) = \frac{\pi}{2}$$

independently of (φ_*, h_*). This shows that the time $I_*(u) = \sqrt{\frac{k'}{2g}}\pi$ it takes the marble to get "to the bottom" is the same, no matter where the marble starts on the Brachistochrone curve u.

11. *Are these cycloids minimizers?*

In the context of this course, the actual question we should ask is whether the cycloid solutions u of the ELEs are minimizers of I. We have seen two sufficient conditions for this:

- In Theorem 13: if $D^2I(u)[\cdot,\cdot]$ is positive definite on $H_0^1((0,l);\mathbb{R})$, then u is a weak local minimizer.
- In Theorem 26: if I were convex, then u would be a global minimizer.

Unfortunately, neither of these criteria seem to apply:

- To check the first condition, we calculate

$$D^2I(u)[v,v] = \int_0^l \partial_a^2 f(u,u')v'^2 + 2\partial_u\partial_a f(u,u')vv' + \partial_u^2 f(u,u')v^2 \,\mathrm{d}x.$$

for $v \in C_0^1((0,l);\mathbb{R})$, where we have

$$\begin{aligned}
\partial_a\partial_u f(u,a) &= \frac{a}{2\sqrt{2g}}(1+a^2)^{-1/2}(h-u)^{-3/2},\\
\partial_u^2 f(u,a) &= \frac{3}{4\sqrt{2g}}(1+a^2)^{1/2}(h-u)^{-5/2},\\
\partial_a^2 f(u,a) &= \frac{1}{\sqrt{2g}}\left((1+a^2)^{-1/2} - a^2(1+a^2)^{-3/2}\right)(h-u)^{-1/2}.
\end{aligned}$$

To have a chance of $D^2I(u)[v,v] \geq \gamma\|v\|_{H^1}$ for some $\gamma > 0$, we essentially need to look at $\partial_a^2 f(u,u')$. From the calculations in 2.,

$$\begin{aligned}
\partial_a^2 f(u(x),u'(x)) &= \frac{1}{\sqrt{2g}}(1+u'^2)^{-3/2}(h-u)^{-1/2}\\
&= \frac{1}{\sqrt{2gk'^3}}(h-u)\\
&= \frac{1}{2\sqrt{2gk'}}(\cos(2\varphi(x))+1) \overset{x\to 0}{\rightarrow} 0.
\end{aligned}$$

This shows that the estimate $\int_0^l \partial_a^2 f(u,u')v'^2\,\mathrm{d}x \geq \gamma'\|v'\|_{L^2}^2$ will fail as v' concentrates near 0.

- If we wanted to use Theorem 26 to show that I is convex, we would need to show that f is convex on $\mathbb{R}^2$. By Theorem 22.1, it would be necessary that f is rank-1-convex at every point, meaning that, also by Theorem 22.3, the Hessian

$$H_f = \begin{pmatrix} \partial_u^2 f & \partial_a\partial_u f \\ \partial_a\partial_u f & \partial_a^2 f \end{pmatrix}$$

is everywhere positive semi-definite. Using the calculations from above,

$$\begin{aligned}
&H_f(u,a)\\
&= \frac{1}{\sqrt{2g}}\begin{pmatrix} \frac{3}{4}(1+a^2)^{1/2}(h-u)^{-5/2} & \frac{a}{2}(1+a^2)^{-1/2}(h-u)^{-3/2} \\ \frac{a}{2}(1+a^2)^{-1/2}(h-u)^{-3/2} & (1+a^2)^{-3/2}(h-u)^{-1/2} \end{pmatrix}.\\
&= c(u,a)\begin{pmatrix} \frac{3}{4}(1+a^2)^2 & \frac{a}{2}(1+a^2)(h-u) \\ \frac{a}{2}(1+a^2)(h-u) & (h-u)^2 \end{pmatrix},
\end{aligned}$$

with

$$c(u,a) = \frac{(1+a^2)^{-3/2}(h-u)^{-5/2}}{\sqrt{2g}} \geq 0$$

for the most relevant $u \leq h$. So for all $w \in \mathbb{R}^2$,

$$\begin{aligned} &w^T \cdot H_f(u,a)w \\ &= c(u,a)\left(\frac{9}{16}w_1^2(1+a^2)^2 + w_1w_2a(1+a^2)(h-u) + w_2^2(h-u)^2\right) \\ &= c(u,a)\left(\left[\frac{3}{4}w_1(1+a^2) + \frac{2}{3}w_2a(h-u)\right]^2 + \left[1-\frac{4}{9}a^2\right]w_2^2(h-u)^2\right). \end{aligned}$$

Clearly, $H_f(u,a)$ is not positive semi-definite if $|a| > \frac{3}{2}$ (for a numerically simple example, set $a = 3, h-u = h/2, w_2 = 1/h$ and $w_1 = -2/15$ to get a value < 0).

12. *What about necessary conditions?*

It is straightforward to show that the LH-condition is satisfied at u: for all $x \in (0,l)$,

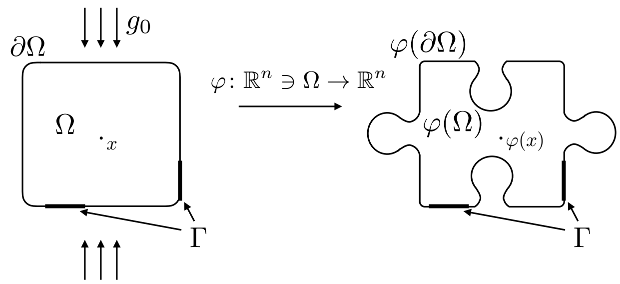
$$\partial_a^2 f(u(x), u'(x)) = (1+u'(x)^2)^{-3/2}(h-u(x))^{-1/2} \geq 0.$$

In fact, we have > 0, so the "negative" LH-condition for u is *not* satisfied. At least, we have thus shown that u is not a (weak local) maximizer. It seems considerably more difficult to check that, in fact, $D^2I(u)[v,v] \geq 0$, which is also necessary for u being a minimizer. So, in this sense, the LH-condition is helpful here.

13. *Conclusion:*

Many elementary, but not necessarily "straightforward" calculations reveal interesting facts about the Brachistochrone curve. Classical methods in the calculus of variations are about solving these types of problems. Even in this completely one-dimensional situation, they are hard to summarize in an abstract theory. We have seen a glimpse of further things that can be done: for example, Noether's Theorem and other ideas for gaining understanding of the ELEs.
At the same time, we have seen the limitations of the abstract criteria we proved in this chapter. They will be more relevant, in a slightly different sense, in the following chapter on the direct method, in particular, for multidimensional settings.

2.7.3 Linear elasticity



Basic strategies, thoughts and notation on/for modelling elastic behaviour of an n-dimensional body:

- aim: model the deformation of an elastic body subject to external forces
- The configuration $\varphi(\Omega)$ of the body is specified by: the open set $\Omega \subset \mathbb{R}^n$ as a *reference configuration* and $\varphi\colon \mathbb{R}^n \subset \Omega \to \mathbb{R}^n$ as its *deformation.* The function $F\colon \Omega \to \mathbb{R}^{n\times n}, x \mapsto \nabla\varphi(x)$ is called the *deformation gradient.*
- Depending on its deformation, the elastic body stores *elastic energy*

$$I(\varphi) = \int_\Omega W(x, \varphi(x), \nabla\varphi(x))\,\mathrm{d}x - \int_{\partial\Omega\setminus\Gamma} \varphi(x)\cdot g_0(x)\,\mathrm{d}x' - \int_\Omega \varphi(x)\cdot h(x)\,\mathrm{d}x,$$

  where

$$W\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}_\infty$$

  is a suitable energy density, g_0 is a boundary density for external loading, and h can be added as a volume density for volume forcings. At $\Gamma \subset \partial\Omega$, the body is fixed/clamped, so $\varphi(x) = x$ for $x \in \Gamma$.
- Idea: the elastic body is deformed in such a way that it stores the least amount of elastic energy $\to$ we need to model and then (locally) minimize I!
- in this Subsection, we focus on the *modelling* of W. This is different from what we did before: in previous examples from geometry and classical mechanics, the model was given, the mathematics was to be done.
- we should think about: how is *elastic* behaviour of a body different from *rigid, plastic, fluidic, gaseous, ...* behaviour?
- we use *continuum mechanics:* there are no distinguishable particles – it must be reasonable to associate a new position $\varphi(x) \in \mathbb{R}^n$ to every point $x \in \Omega \subset \mathbb{R}^n$.

Typcial (reasonable?) properties of W:

1. *translation invariance*: $W(x, \varphi, F) = W(x, F)$,
   i.e. F has all the relevant information on φ, translating the body doesn't take or produce elastic energy
2. *rotation invariance*: $(\forall F)\,(\forall Q \in SO(n)) \quad W(x, QF) = W(x, F)$,
   i.e. $I(\varphi) = I(Q\varphi)$, I is independent of an orthogonal change of coordinates. This is also called *objectivity.*

   From these two assumptions, it follows that (see [Ciarlet]),

   $W(x, F) = \tilde{W}(x, C)$, where $C = F^T F$ is the "right Cauchy-Green deformation tensor$''$.
3. no *self-interprenetation*: $\det F \le 0 \Rightarrow W(x, F) = \infty$.
   This is to avoid that the material folds in on itself. This is additionally supported by the condition

$$\det F_n > 0 \text{ with } \det F_n \overset{n\to\infty}{\to} 0 \Rightarrow W(x, F_n) \overset{n\to\infty}{\to} \infty.$$

Note that this condition implies *non-convexity* of W in F: For example, if

$$F_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } F_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then $\det F_1 = \det F_2 = 1$, but $\det(\frac{1}{2}F_1 + \frac{1}{2}F_2) = 0$, so $W(x, \frac{1}{2}F_1 + \frac{1}{2}F_2) = \infty$. On the other hand, $F_1^T F_1 = F_2^T F_2 = \mathrm{Id}_{n\times n}$, so $W(x, F_1) = W(x, F_2) = \tilde{W}(x, \mathrm{Id}_{n\times n})$ and we should have $\tilde{W}(x, \mathrm{Id}_{n\times n}) \neq \infty$.
As a consequence, non-convex energies are needed to model "large" deformations that may get close to $\det F = 0$. (What are typical situations/materials that allow/need $\det F = 0$?)

From here: we assume "small" deformation, and show how to linearize the model in this setting. The corresponding energy and material behaviour is called *Linear Elasticity:*

- we replace the deformation φ by the *displacement* $u\colon \Omega \to \mathbb{R}^n$, given by

$$\varphi(x) = x + u(x).$$

  Then
$$C = \nabla\varphi^T\nabla\varphi = \mathrm{Id}_{n\times n} + (\nabla u + \nabla u^T) + \nabla u^T \nabla u,$$
  with
$$e(u) := \frac{1}{2}(\nabla u + \nabla u^T)$$
  the *linearized* deformation tensor (with respect to the undeformed state $\varphi_0 = \mathrm{Id}_\Omega$.)

- A Taylor expansion of $\tilde{W}$ in $\mathrm{Id}_{n\times n}$ gives

$$\begin{aligned}\tilde{W}(\nabla\varphi^T\nabla\varphi) = {}& \tilde{W}(\mathrm{Id}_{n\times n}) + D\tilde{W}(\mathrm{Id}_{n\times n})[2e(u) + \nabla u^T\nabla u] \\ & + \frac{1}{2}D^2\tilde{W}(\mathrm{Id}_{n\times n})[2e(u) + \nabla u^T\nabla u, 2e(u) + \nabla u^T\nabla u] \\ & + \text{terms of higher order.}\end{aligned}$$

- now assume that the undeformed state $\varphi_0 = \mathrm{Id}_\Omega$ is a local minimizer of $\tilde{W}$ and thus, in particular, a critical point of I. Then with $\nabla\varphi_0 = \mathrm{Id}_{n\times n}$, we have $D\tilde{W}(\mathrm{Id}_{n\times n}) = 0$. In addition, assume that $\nabla u^T\nabla u$ is sufficiently small to be ignored. Then we may approximate $\tilde{W}$ by the quadratic functional

$$f_{el}(x, A) := 2D^2\tilde{W}(x, \mathrm{Id}_{n\times n})[A, A],$$

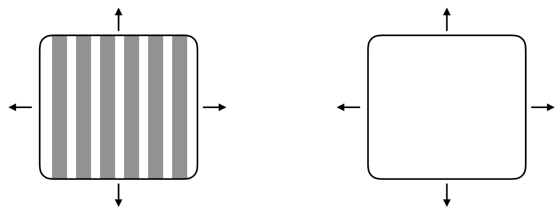
  defined on the *symmetric* tensors $A \in \mathbb{R}^{n\times n}_{sym}$. Recall that the fact that f_{el} is *quadratic* means that the ELEs are *linear*.

- we have modelled the *Problem of Linear Elasticity*: Minimize

$$I(u) = \int_\Omega 2D^2\tilde{W}(\mathrm{Id}_{n\times n})e(u) : e(u)\,\mathrm{d}x - \int_{\partial\Omega\setminus\Gamma} \tilde{g}\cdot u\,\mathrm{d}x' - \int_\Omega \tilde{h}\cdot u\,\mathrm{d}x$$

  on
$$M_0 = \{u \in C^1(\Omega;\mathbb{R}^n) : u|_\Gamma = 0\}.$$

As an important example, we consider linear elasticity models for *Isotropic material*: in this case, the material response to loading should not depend on the direction of the loading (see Picture):



anisotropic isotropic

Then (see [Ciarlet]):

$$\begin{aligned} f_{el}(x, A) &= \frac{\lambda(x)}{2}|\operatorname{tr} A|^2 + \frac{\mu(x)}{4}|A + A^T|^2 \\ &= \frac{\lambda(x)}{2}|\operatorname{tr} e(u)|^2 + \mu(x)|e(u)|^2, \end{aligned}$$

where $\lambda, \mu \in \mathbb{R}$ are called *Lamé-constants*. They depend on the elastic material properties – see the discussion below! As f_{el} is quadratic in A, we have

$$\begin{aligned} \partial_A f_{el}(A)[B] &= \lambda \operatorname{tr} A \operatorname{tr} B + \frac{1}{2}\mu(A + A^T) : (B + B^T), \\ \partial_A^2 f_{el}(A)[B, B] &= 2f(B). \end{aligned}$$

The ELEs are called *Lamé-Navier Equations*:

$$\begin{cases} \operatorname{div}(\sigma(u)) + \tilde{h} = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \Gamma, \\ \sigma\nu + \tilde{g} = 0, & \text{on } \partial\Omega \setminus \Gamma, \end{cases}$$

where

$$\sigma(u) = \partial_A f(x, \nabla u) = \lambda \operatorname{tr} \nabla u \, \mathrm{Id}_{n \times n} + 2\mu e(u)$$

is the corresponding *stress tensor*.

Now: isf_{el} convex in A?

Lemma 28. *(Convexity for linear isotropic elasticity) We fix $\mu, \lambda \in \mathbb{R}$. Then*

1. $f_{el}(A) = \frac{\lambda}{2}|\operatorname{tr} A|^2 + \frac{\mu}{4}|A + A^T|^2$ *satisfies* (LH) *if and only if $\mu \geq 0$ and $\lambda + 2\mu \geq 0$, and*

2. *f_{el} is convex if and only if $\mu \geq 0$ and $n\lambda + 2\mu \geq 0$.*

Since f_{el} is smooth and quadratic, the (LH)-condition is equivalent to rank-one convexity and rank-one convexity and quasiconvexity are the same (Lemma 25)! But clearly, also in this case, convexity is a *stronger* property than rank-one convexity (as long as $n \geq 2$).

Proof. (compare to Exercise 5 on Assignment 4.)

1. For all $B = \xi \otimes \eta$, $\xi, \eta \in \mathbb{R}^n$, we have

$$|\xi \otimes \eta + \eta \otimes \xi|^2 = 2|\xi|^2|\eta|^2 + 2|\xi \cdot \eta|^2,$$

hence

$$\begin{aligned} D^2 f(A)[B,B] = 2f(B) &= \frac{\lambda}{2}(\xi \cdot \eta)^2 + \frac{\mu}{4}|\xi \otimes \eta + \eta \otimes \xi|^2 \\ &= \frac{\lambda + \mu}{2}(\xi \cdot \eta)^2 + \frac{\mu}{2}|\xi|^2|\eta|^2. \end{aligned}$$

Moreover,

$$\begin{aligned} &\forall \xi, \eta \in \mathbb{R}^n \quad \frac{1}{2}\left((\lambda + \mu)(\xi \cdot \eta)^2 + \mu|\xi|^2|\eta|^2\right) \geq 0 \\ &\Leftrightarrow \forall \xi, \eta \in \mathbb{R}^n \setminus \{0\} \quad (\lambda + \mu)\frac{(\xi \cdot \eta)^2}{|\xi|^2|\eta|^2} + \mu \geq 0 \\ &\Leftrightarrow \forall \alpha \in (0,1] \quad (\lambda + \mu)\alpha + \mu \geq 0 \\ &\Leftrightarrow \mu \geq 0 \text{ and } \lambda + 2\mu \geq 0. \end{aligned}$$

2. First note that convexity of f_{el} is equivalent to $f_{el}(B) \geq 0$ for all $B \in \mathbb{R}^{n\times n}$. As a tool in the proof, we decompose

$$\mathbb{R}^{n\times n} = \mathbb{R}\mathrm{Id}_{n\times n} + \mathbb{R}^{n\times n}_{dev},$$

in the sense that

$$\mathbb{R}^{n\times n} \ni B = \frac{1}{n}\mathrm{tr}\, B\, \mathrm{Id}_{n\times n} + (B - \frac{\mathrm{tr}\, B}{n}\mathrm{Id}_{n\times n}).$$

Here, the scalar expression $\frac{\mathrm{tr}\, B}{n}$ can be seen as a linearized measure of the change of volume associated to B (see example below!), and the matrix

$$\mathrm{dev}(B) = (B - \frac{\mathrm{tr}\, B}{n}\mathrm{Id}_{n\times n}) \in \mathbb{R}^{n\times n}_{dev}$$

is called the *deviatoric part* of B. Note that

$$\mathrm{Id}_{n\times n} : \mathrm{dev}(B) = \mathrm{Id}_{n\times n} : B - \mathrm{tr}\, B = 0,$$

so $\mathbb{R}\mathrm{Id}_{n\times n} \perp \mathbb{R}^{n\times n}_{dev}$. Hence

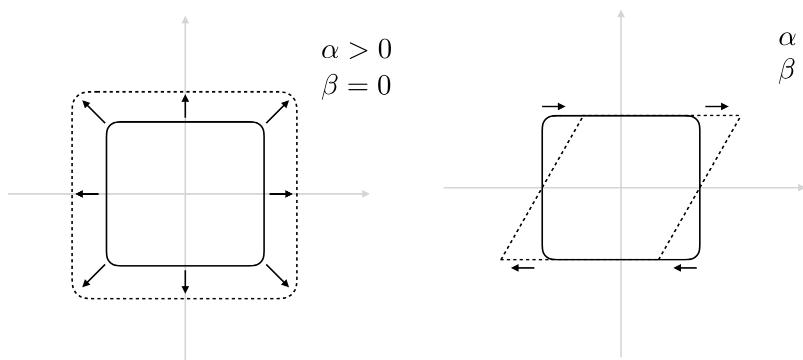
$$\begin{aligned} f_{el}(B) &= \frac{\lambda}{2}|\mathrm{tr}\, B|^2 + \frac{\mu}{4}|\frac{2}{n}\mathrm{tr}\, B\, \mathrm{Id}_{n\times n} + (B + B^T - \frac{2}{n}\mathrm{tr}\, B\, \mathrm{Id}_{n\times n})|^2 \\ &= \frac{n\lambda + 2\mu}{n}|\mathrm{tr}\, B|^2 + \frac{\mu}{4}|\mathrm{dev}(B) + \mathrm{dev}(B)^T|^2, \end{aligned}$$

so $f_{el}(B) \geq 0$ for all $B \in \mathbb{R}^{n\times n}$ holds if and only if $\mu \geq 0$ and $n\lambda + 2\mu \geq 0$.

□

We close this discussion of linear elasticity with two prototypes of deformations, their geometric/physical interpretation in dimension 2, and their representation in f_{el}.
Let $n = 2$ and $u(x) = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} x$, so that the gradient has the decomposition

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} = \alpha \mathrm{Id}_{2\times 2} + \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix} = \alpha \mathrm{Id}_{2\times 2} + \mathrm{dev}(A).$$



If $\alpha > 0$, $\beta = 0$, this corresponds to a stretching of the body in every direction, $\varphi(x) = x + \alpha x$, with a corresponding change in volume (Figure on the left). If $\alpha = 0$, $\beta > 0$, this corresponds to shearing of the body in e^1-direction, $\varphi(x) = x + \mathrm{dev}(A)x$, without any change in volume (Figure on the right). We have

$$f_{el}(A) = (\lambda + \mu)\alpha^2 + \frac{\mu}{2}\beta^2,$$

where the first part is the gain of energy due to volume change, and the second part is the energy absorbed due to shear stress.

3 The Direct Method in the Calculus of Variations

In this part of the course, we look at

(DP) (Direct Problem in the calculus of variations):
Let $(X, \|\cdot\|)$ be a real Banach space and $I\colon X \to \mathbb{R}_\infty$ a functional.
Can we find a global minimizer $u \in X$ such that $I(u) = \inf_{v\in X} I(v)$?

Here, $\mathbb{R}_\infty = \mathbb{R} \cup \{+\infty\}$ indicates that I can also take on the value $+\infty$. This can be used to model constraints: if actually we are looking for minimizers $u \in M \subset X$ of a functional $\tilde{I}\colon X \supset M \to \mathbb{R}$, then we can define

$$I(v) = \begin{cases} \tilde{I}(v), & v \in M, \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly, the minimizers and infimum/minimum values of $\tilde{I}$ and I in M coincide. If the set where $I(v) \neq +\infty$ is nonempty, we say that I is *proper*. We always assume (tacitly) that I is proper.

3.1 Abstract existence theorems from functional analysis

To solve (DP), we get a first existence theorem for reflexive Banach spaces X that follows from the strategy outlined in Section 1.6. In the Theorem, the functional I is assumed to be *coercive* and *weakly sequentially lower semicontinuous*. First, we define these notions.

Definition 29. A functional $I\colon X \to \mathbb{R}_\infty$ on a Banach space X is called *coercive*, if for all sequences $(v_n)_n \subset X$, if $\|v_n\| \to +\infty$, then $I(v_n) \to +\infty$.

Definition 30. A functional $I\colon X \to \mathbb{R}_\infty$ on a Banach space X is called

1. *sequentially weakly continuous*, if for all weakly converging sequences $(v_n)_n \subset X$, $v_n \rightharpoonup v$, we have
$$\lim_{n\to\infty} I(v_n) = I(v),$$

2. *sequentially weakly lower semicontinuous* ((s)wlsc), if for all weakly converging sequences $(v_n)_n \subset X$, $v_n \rightharpoonup v$, we have
$$\liminf_{n\to\infty} I(v_n) \geq I(v).$$

Remark 31. A short discussion and examples for Definition 30:

1. For brevity, we will omit "sequentially" in "sequentially weakly lower semicontinuous" and the first "s" in (swlsc). However, we always use this (sequential) definition.

2. Strongly converging sequences are also weakly converging, so clearly, the property of I being weakly continuous (or weakly lower semicontinuous) is stronger than the property of I being continuous (or lower semicontinuous).

3. Sequential weak continuity of a functional is a 'rare' property. We will come back to this later in the course.

4. The norm $\|\cdot\|\colon X \to \mathbb{R}$ is (wlsc).

   Proof. For all $x' \in X'$, the space of bounded linear functionals on X, and convergent sequences, $x_n \to x$ in X,

   $$\|x'\|_{X'}\|x_n\|_X \geq x'(x_n) \overset{n\to\infty}{\to} x'(x),$$

   so $\liminf_n \|x'\|_{X'}\|x_n\|_X \geq x'(x)$. Now choose (Hahn-Banach!) x' such that $\|x'\|_{X'} = 1$ and $x'(x) = \|x\|_X$. □

5. In many infinite-dimensional Banach spaces, the norm $\|\cdot\|\colon X \to \mathbb{R}$ is not sequentially weakly continuous, i.e. weak convergence does not imply norm convergence (Exercise: example + proof?)

Theorem 32. *(Tonelli, "Adapted Weierstraß Principle") Let $(X, \|\cdot\|)$ be a reflexive real Banach space. Assume that the functional $I\colon X \to \mathbb{R}_\infty$ is bounded from below, coercive and weakly lower semicontinuous. Then* (DP) *has at least one solution.*

Proof. We follow the strategy from Section 1.6. Let $\inf_{v\in X} I(v) = \bar{I} \in \mathbb{R}$ (I is bounded from below). Then there is an infimizing sequence $(v_n)_n \subset X$ for I, i.e.

$$\lim_n I(v_n) = \inf_{v\in X} I(v) = \bar{I}.$$

Since I is coercive, $(v_n)_n$ is bounded. Since X is reflexive, $(v_n)_n$ has a weakly converging subsequence $v_{n_k} \rightharpoonup u \in X$. Since I is weakly lower semicontinuous,

$$\bar{I} = \lim_k I(v_{n_k}) \geq \liminf_k I(v_{n_k}) \geq I(u) \geq \bar{I},$$

so u is a global minimizer. □

Now, there are two natural questions:

1. How do we see if I is coercive?
2. How do we characterize I that are wlsc?

We will find answers to the first question in the more specific setting

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x + \int_{\partial\Omega\setminus\Gamma} g(x, u(x))\,\mathrm{d}x' \tag{3.1}$$

that we already used in the previous chapter. This is the content of Section 3.3 (after a brief reminder of Lebesgue and Sobolev spaces in Section 3.2). Regarding the second question, there are a few things that we can say now, in a general/abstract setting, before we reconsider this problem for I of type (3.1) in Section 3.4.

Lemma 33. *(Mazur) Let X be a real Banach space and let $M \subset X$ be closed and convex. Then M is weakly sequentially closed.*

Proof. This is a direct consequence of Separation Theorems/the Hahn-Banach Theorem from functional analysis. We recall one version that is suitable:

Theorem. *(Separation Theorem) Let X be a real Banach space, let $M \subset X$ be closed and convex and let $x_0 \in X \setminus M$. Then there exist $x' \in X'$ and $\alpha \in \mathbb{R}$ such that for all $x \in M$,*

$$x'(x) \leq \alpha \quad \textit{and} \quad x'(x_0) > \alpha.$$

Proof. See, for example, [Alt, p. 223, Section 6.11]. □

To prove Mazur's Lemma, let $(u_n)_n \subset M$, $u_n \rightharpoonup u \in X$ be weakly convergent. Assume $u \notin M$. Then M is closed and convex and by the Separation Theorem, there exist $x' \in X'$ and $\alpha \in \mathbb{R}$ such that

$$x'(u_n) \leq \alpha \quad \text{and} \quad x'(u) > \alpha.$$

Clearly, this is in contradiction to the weak convergence of u_n to u. □

Mazur's Lemma allows us to prove the following criterion for I being wlsc. Geometrically, we use that convexity of I implies convexity of its *sublevels*

$$L_\alpha(I) = \{u \in X : I(u) \leq \alpha\}.$$

Theorem 34. *Let X be a real Banach space and let $I\colon X \to \mathbb{R}_\infty$ be strongly lower semicontinuous and convex. Then I is* wlsc.

Proof. Let $(u_n)_n \subset X$, $u_n \rightharpoonup u$, be any weakly convergent sequence. We need to show that

$$I(u) \leq \liminf_n I(u_n) =: \alpha.$$

We distinguish three cases.

1. If $\alpha = +\infty$, then, clearly, $I(u) \leq \alpha$.

2. If $\alpha \in \mathbb{R}$, choose a subsequence $(u_{n_k})_k$ such that $\lim_k I(u_{n_k}) = \alpha$. For $\varepsilon > 0$, consider the sublevels

$$L_{\alpha+\varepsilon}(I) = \{v \in X : I(v) \leq \alpha + \varepsilon\}.$$

   By convexity of I, the $L_{\alpha+\varepsilon}(I)$ are convex. By strong lower semicontinuity of I, the $L_{\alpha+\varepsilon}(I)$ are strongly closed. Thus, by Mazur's Lemma, the sublevels are weakly closed. Moreover, for all $\varepsilon > 0$, there is a k_ε such that $u_{n_{k'}} \in L_{\alpha+\varepsilon}(I)$ for all $k' \geq k_\varepsilon$. Thus, $u \in L_{\alpha+\varepsilon}(I)$ for all $\varepsilon > 0$ and hence $I(u) \leq \alpha$.

3. If $\alpha = -\infty$, consider $L_{-R}(I)$ for $R \to +\infty$. For all $R > 0$, we find $u \in L_{-R}$ as in 2.. Hence $I(u) = -\infty$, a contradiction.

□

Now we briefly look at a related but different existence theorem for (DP) with "more convexity but different continuity". We start with a definition.

Definition 35. The functional $I\colon X \to \mathbb{R}_\infty$ is called *uniformly convex*, if there exists a monotone function $g : (0, \infty) \to (0, \infty]$ such that for all $x, y \in X$, $x \neq y$, $\theta \in (0, 1)$,

$$I(\theta x + (1-\theta)y) \leq \theta I(x) + (1-\theta)I(y) - \theta(1-\theta)g(\|x-y\|).$$

Theorem 36. *Let $I\colon X \to \mathbb{R}_\infty$ be uniformly convex and continuous (w.r.t. the norm in X). Then* (DP) *has a unique solution, i.e. a global minimizer.*

Proof. Let $(u_n)_n$ be an infimizing sequence for I and $\bar{I} = \inf_{v\in X} I(v) = \lim_n I(u_n)$. We show that $(u_n)_n$ is a Cauchy sequence. We have

$$g(\|u_m - u_n\|) \leq 2I(u_m) + 2I(u_n) - 4I(\frac{1}{2}u_m + \frac{1}{2}u_n)$$

and $I(\frac{1}{2}u_m + \frac{1}{2}u_n) \geq \overline{I}$, so

$$\limsup_{m,n} g(\|u_m - u_n\|) \leq 4\overline{I} - 4\overline{I} = 0. \tag{3.2}$$

Since g is positive and monotone, this implies that $(u_n)_n$ is a Cauchy sequence in X: If $(u_n)_n$ wasn't Cauchy, there would be $\varepsilon > 0$ with

$$\limsup_{m,n} \|u_m - u_n\| \geq \varepsilon.$$

By monotonicity and positivity of g, this would imply

$$\limsup_{m,n} g(\|u_m - u_n\|) \geq g(\varepsilon) > 0,$$

in contradiction to (3.2). We set $u = \lim_n u_n$. Since I is continuous,

$$\overline{I} = \lim_n I(u_n) = I(u),$$

so u is a global minimizer. Since I is uniformly convex, it is also strictly convex. Therefore, u is unique. □

To close this abstract Section, we summarize different notions of differentiability in (in)finite-dimensional Banach spaces.

Definition 37. (Gâteaux differential and Fréchet derivative) Let X, Y be Banach spaces and $F\colon X \to Y$.

1. The *Gâteaux differential* of F at $x_0 \in X$ in the direction of $v \in X$ is the limit

   $$DF(x_0)[v] = \lim_{t\searrow 0} \frac{F(x_0 + tv) - F(x_0)}{t} \in Y$$

   if it exists.

2. If for $x_0 \in X$, there exists $A_0 \in \mathcal{L}(X, Y)$ (the space of bounded linear operators from X to Y) such that for all $v \in X$,

   $$DF(x_0)[v] = A_0 v,$$

   then F is *Gâteaux differentiable at* x_0 with *Gâteaux derivative* A_0.

3. F is called *Fréchet differentiable at* $x_0 \in X$ if there exists $A \in \mathcal{L}(X, Y)$ such that for all $(v_n)_n \subset X$ with $\lim_{n\to\infty} v_n = 0$,

$$\lim_{n\to\infty} \frac{\|F(x_0 + v_n) - F(x_0) - Av_n\|_Y}{\|v\|_X} = 0.$$

   A is then called the *Fréchet derivative* of F at x_0.

4. As usual, F is Gâteaux/Fréchet differentiable if it is Gâteaux/Fréchet differentiable at every $x_0 \in X$.

Remark 38.

1. If F is Fréchet differentiable at x_0, then it is Gâteaux differentiable at x_0. The converse need not be true.

2. Even if all directional derivatives $DF(x_0)[v]$ exist, F need not be Gâteaux differentiable (counterexample for $X = \mathbb{R}^2, Y = \mathbb{R}$).

3. If $Y = \mathbb{R}$, then $DF(x_0)[v]$ is the first variation of F at x_0 in the direction of v. The Gâteaux derivative would be $DF(x_0) \in \mathcal{L}(X, \mathbb{R}) = X'$.

4. Recall that if $u_0 \in X$ is a solution of (DP) and $DI(u_0)[v]$ exists for all $v \in X$, then $DI(u_0)[v] = 0$ for all $v \in X$, i.e. I is Gâteaux differentiable at u_0 with derivative 0 and u_0 is a critical point.

5. Note that if $I\colon X \to \mathbb{R}$ is convex and Gâteaux differentiable, then for all $u, v \in X$,

$$I(u + v) \geq I(u) + DI(u)[v].$$

   Proof.

$$\begin{aligned} DI(u)[v] &= \lim_{t\searrow 0} \frac{I(u + tv) - I(u)}{t} = \lim_{t\searrow 0} \frac{I(t(u + v) + (1 - t)u) - I(u)}{t} \\ &\overset{I \text{ convex}}{\leq} \lim_{t\searrow 0} \frac{tI(u + v) + (1 - t)I(u) - I(u)}{t} = I(u + v) - I(u). \end{aligned}$$

   □

6. Note that even uniformly convex functionals need not be (Gâteaux) differentiable.

3.2 Reminder: Lebesgue and Sobolev spaces

We would like to apply the results of the last Section to functionals $I\colon X \to \mathbb{R}_\infty$ of type (3.1). Thus, we need that X is reflexive and contains functions that are differentiable (in some sense). Moreover, it may be useful if the norm of X also "quantifies" the derivatives. Sobolev spaces appear as a natural choice for X. In this Section, we recall properties of Lebesgue and Sobolev spaces that will be important for the remainder of the course that deals with the coercivity and weak lower semicontinuity of I. Compactness results will also be useful.

3.2.1 Lebesgue spaces

Let $\Omega \subset \mathbb{R}^n$ be a domain. Recall the Lebesgue spaces $L^p(\Omega) = L^p(\Omega;\mathbb{R})$ of (equivalence classes of) functions $f\colon \Omega \to \mathbb{R}$ for $p \in [1,\infty]$ with norms

$$\|f\|_p = \left(\int_\Omega |f(x)|^p \,\mathrm{d}x\right)^{1/p}, \quad p \neq \infty,$$
$$\|f\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} |f(x)| = \inf_{r>0}\{|\{x \in \Omega : |f(x)| > r\}| = 0\}.$$

Some facts we will use throughout this Section and the remainder of the course, see, e.g. [Alt]:

- For all $p \in [1,\infty]$, $L^p(\Omega)$ are *Banach spaces.*
- For $p \in (1,\infty)$, $L^p(\Omega)$ is *reflexive.*
- For $p = 2$, $L^2(\Omega)$ is a *Hilbert space.*
- For $p \in [1,\infty)$, $1 = \frac{1}{p} + \frac{1}{p'}$, $p' = \infty$ if $p = 1$, the *dual* $(L^p(\Omega))'$ of $L^p(\Omega)$ is isomorphic to $L^{p'}(\Omega)$, i.e. for all $\xi \in (L^p(\Omega))'$ there exists a unique $v_\xi \in L^{p'}(\Omega)$ such that for all $u \in L^p(\Omega)$,
$$\xi(u) = \int_\Omega v_\xi(x)u(x)\,\mathrm{d}x,$$
and $\|\xi\|_{(L^p(\Omega))'} = \|v_\xi\|_{L^{p'}(\Omega)}$. For this, recall *Hölder's inequality:* for all $f \in L^p(\Omega)$, $g \in L^{p'}(\Omega)$,
$$\int_\Omega |f(x)g(x)|\,\mathrm{d}x \leq \|f\|_p\|g\|_{p'}.$$
- For $p \in [1,\infty)$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.
- For $p \in [1,\infty]$, Ω bounded, *the span of step functions* is dense in $L^p(\Omega)$.

3.2.2 Convergence Theorems

- *Monotone convergence:* Let $u \in L^1(\Omega), (u_n)_n \subset L^1(\Omega)$ be such that
$$0 \leq u_1(x) \leq \cdots \leq u_n(x) \leq \cdots \nearrow u(x) \in L^1(\Omega) \quad a.e.,$$
then
$$u_n \overset{L^1}{\to} u \quad \text{and} \quad \lim_n \int_\Omega u_n(x)\,\mathrm{d}x = \int_\Omega u(x)\,\mathrm{d}x.$$
(Sketch of proof: $\int_\Omega u_n$ is monotone and bounded, so convergent. Hence for $k \geq l$,
$$\int_\Omega |u_k - u_l| = \int_\Omega u_k - \int_\Omega u_l \to 0,$$
so $(u_n)_n$ is Cauchy sequence in $L^1(\Omega)$.)

- *Fatous' Lemma:* Let $(u_n)_n$ be a sequence of measurable functions with $0 \le u_n$ a.e. Then
$$\liminf_{n\to\infty} \int_\Omega u_n(x)\,\mathrm{d}x \ge \int_\Omega \liminf_{n\to\infty} u_n(x)\,\mathrm{d}x.$$
(Sketch of proof: Apply monotone convergence to $f_n = \inf_{k\ge n} u_k \nearrow \liminf_n u_n$ yielding
$$\liminf_{n\to\infty} \int_\Omega u_n \ge \lim_n \int_\Omega \inf_{k\ge n} u_k = \int_\Omega \liminf_n u_n.)$$
- *Lebesgue dominated convergence:* Let $(u_n)_n$ be a sequence of measurable functions with $|u_n| \le h \in L^1(\Omega)$ and pointwise convergence $u_n(x) \to u(x)$ almost everywhere. Then $u \in L^1(\Omega)$ and $u_n \overset{L^1}{\to} u$.
(Sketch of proof: Apply Fatou's Lemma to $g_n := h - \frac{1}{2}|u_n - u| \ge 0$ to get
$$\limsup_{n\to\infty} \int_\Omega |u_n - u| = 0.)$$

Lemma 39. *(Weak convergence) Let $p \in [1,\infty)$, $u \in L^p(\Omega)$. Then the following are equivalent:*

- *$u_n \rightharpoonup u$ in $L^p(\Omega)$, and,*
- *Both*
  1. *$\exists c > 0 \quad \forall n \in \mathbb{N} \quad \|u_n\|_p < c$, and,*
  2. *for all sets $S \subset (L^p(\Omega))' = L^{p'}(\Omega)$ such that $\operatorname{span}(S)$ is dense in $L^{p'}(\Omega)$,*
$$\forall v \in S, \qquad \int_\Omega v u_n \to \int_\Omega v u.$$

In particular, we can choose $S = C_c^\infty(\Omega) \subset L^{p'}(\Omega)$ (asking $p \ne 1$), or $\operatorname{span}(S)$ the set of step functions (asking $p \ne 1$ if Ω is unbounded).

Proof. "$\Rightarrow$":

1. by Banach-Steinhaus
2. by definition of weak convergence.

"$\Leftarrow$": For $v \in L^{p'}(\Omega)$ and $\varepsilon > 0$, choose $v_\epsilon \in \operatorname{span}(S)$ such that $\|v - v_\varepsilon\|_{p'} < \varepsilon$. Set $c_0 := \|u\|_p$. Then, for sufficiently large n,
$$\begin{aligned}|\int_\Omega u_n v - \int_\Omega uv| &\le |\int_\Omega u_n(v - v_\varepsilon)| + |\int_\Omega u_n v_\varepsilon - \int_\Omega u v_\varepsilon| + |\int_\Omega u(v_\varepsilon - v)| \\ &\overset{\text{Hölder},1.}{\le} c\|v - v_\varepsilon\|_{p'} + |\int_\Omega (u_n - u)v_\varepsilon| + c_0\|v - v_\varepsilon\|_{p'} \overset{2.}{\le} C\varepsilon.\end{aligned}$$

□

Example 40. ("Concentrations") Let $\Omega = (-1,1)$, $\alpha \in \mathbb{R}$ and

$$u_n(x) = \begin{cases} n^\alpha, & x \in (0, \frac{1}{n}), \\ 0, & \text{otherwise.} \end{cases}$$

Then:

- for all $x \in \Omega$, $u_n(x) \to 0$, and
- $\|u_n\|_p^p = \int_0^{1/n} n^{\alpha p}\, \mathrm{dx} = n^{\alpha p - 1} < \infty$ for $p \in [1, \infty)$

Now distinguish four cases:

1. $\alpha p > 1$, then $\|u_n\|_p \to \infty$, so we have neither strong nor weak (Lemma 39) convergence in $L^p(\Omega)$,
2. $\alpha p < 1$, then $\|u_n\|_p \to 0$, so we have norm convergence and strong convergence in $L^p(\Omega)$,
3. $\alpha = \frac{1}{p}$, then $\|u_n\|_p = 1$, so we have boundedness. For $p \in (1, \infty)$, use Lemma 39: let $\varphi \in C_c^\infty(\Omega)$, then

$$|\int_{-1}^{1} u_n \varphi| = |\int_0^{1/n} n^{1/p} \varphi| = |n^{1/p} \int_0^{1/n} \varphi| \leq n^{1/p - 1} \|\varphi\|_\infty \overset{p>1}{\to} 0,$$

so $u_n \rightharpoonup 0$.

4. $\alpha = p = 1$, then $\|u_n\|_1 = 1$. Check what happens for $\varphi \in C_c^\infty(\Omega)$:

$$\int_{-1}^{1} u_n \varphi = n \int_0^{1/n} \varphi \to \varphi(0),$$

(Lebesgue differentiation), but there is no $u \in L^1(\Omega)$ such that $\varphi(0) = \int_{-1}^1 u\varphi$ for all $\varphi \in C_c^\infty(\Omega)$. We show that $(u_n)_n$ doesn't converge: Let $v(x) = \sum_{k=1}^\infty \chi_{[\frac{1}{4^k}, \frac{2}{4^k}]}$ with χ_M the characteristic function for the set M. Then $v \in L^\infty(\Omega) = (L^1(\Omega))'$ and

$$\int_{-1}^{1} u_n v \overset{n=\frac{4^k}{2}}{=} \int_0^{2/4^k} \frac{4^k}{2} v(x)\, \mathrm{d}x \geq \int_{1/4^k}^{2/4^k} \frac{4^k}{2} = \frac{1}{2},$$
$$\int_{-1}^{1} u_n v \overset{n=4^k}{=} \int_0^{1/4^k} 4^k v(x)\, \mathrm{d}x = \int_0^{2/4^{k+1}} 4^k v(x)\, \mathrm{d}x \leq 4^k \frac{2}{3} \frac{2}{4^{k+1}} = \frac{1}{3},$$

so $(u_n)_n$ does not converge weakly.

Example 41. (Oscillations) Let $\Omega = (0,1)$ and $u \in L^p(\Omega)$, $p \in [1, \infty)$. Extend u periodically to $\tilde{u}\colon \mathbb{R} \to \mathbb{R}$,

$$\tilde{u}(x+k) = u(x), \quad x \in (0,1), k \in \mathbb{Z}.$$

Consider the sequence $(u_n)_n \subset L^p(\Omega)$ with

$$u_n(x) = \tilde{u}(nx), \quad x \in (0,1), n \in \mathbb{N}.$$

Then $(u_n)_n$ converges weakly to its average, $u_n \overset{L^p}{\rightharpoonup} \bar{u} \colon x \mapsto \int_0^1 u(x)\,\mathrm{d}x$.

Proof. Note that the set of step functions on $(0,1)$ is given by

$$\operatorname{span}(\{\chi_{[0,\alpha]} : 0 < \alpha < 1\}),$$

and it is dense in $L^{p'}(0,1)$ for $p \in [1,\infty)$. Thus, by Lemma 39, to show weak convergence, it is sufficient to show:

1. the sequence $(u_n)_n$ is bounded, as by a direct calculation,

$$\|u_n\|_p^p = \int_0^1 |u(nx)|^p\,\mathrm{d}x \overset{y=nx}{=} \frac{1}{n}\int_0^n |\tilde{u}(y)|^p\,\mathrm{d}y = \|u\|_p^p,$$

and,

2. for all $\alpha \in (0,1)$, we have

$$\int_0^1 \chi_{[0,\alpha]} u_n \to \int_0^1 \chi_{[0,\alpha]} \bar{u} = \alpha\bar{u}.$$

This follows from

$$\begin{aligned}\int_0^1 \chi_{[0,\alpha]}(x) u_n(x)\,\mathrm{d}x &\overset{y=nx}{=} \frac{1}{n}\int_0^{n\alpha} \tilde{u}(y)\,\mathrm{d}y \\ &= \frac{1}{n}\int_0^{\lfloor n\alpha\rfloor} \tilde{u}(y)\,\mathrm{d}y + \frac{1}{n}\int_{\lfloor n\alpha\rfloor}^{n\alpha} \tilde{u}(y)\,\mathrm{d}y \to \alpha\bar{u},\end{aligned}$$

where

$$\frac{1}{n}\int_0^{\lfloor n\alpha\rfloor} \tilde{u}(y)\,\mathrm{d}y = \frac{\lfloor n\alpha\rfloor}{n}\bar{u} \to \alpha\bar{u}$$

as

$$|\frac{\lfloor n\alpha\rfloor}{n} - \alpha| = |\frac{\lfloor n\alpha\rfloor - n\alpha}{n}| \le \frac{1}{n} \to 0,$$

and

$$\frac{1}{n}\int_{\lfloor n\alpha\rfloor}^{n\alpha} \tilde{u}(y)\,\mathrm{d}y \le \frac{1}{n}\int_{\lfloor n\alpha\rfloor}^{\lfloor n\alpha\rfloor+1} |\tilde{u}(y)|\,\mathrm{d}y = \frac{1}{n}\|u\|_1 \to 0.$$

□

Corollary 42. *("weak continuity is too strong") Let $g\colon \mathbb{R} \to \mathbb{R}$ be measurable with $|g(u)| \le C(1+|u|^p)$ for some $p \in [1,\infty)$. Consider the functional*

$$I\colon L^p(0,1) \to \mathbb{R}, \quad I(u) = \int_0^1 g(u(x))\,\mathrm{d}x.$$

Then the following are equivalent:

1. *I is sequentially weakly continuous on $L^p(0,1)$, and,*

2. *I is* affine, *and there exists $b \in \mathbb{R}$ such that $I(u) = I(0) + b\bar{u}$.*

Proof. 2. $\Rightarrow$ 1. follows directly from the definition of weak convergence.
1. $\Rightarrow$ 2. in two steps:

Step 1: Consider arbitrary $u \in L^p(0,1)$ with periodic extension $\tilde{u}\colon \mathbb{R} \to \mathbb{R}$ and $u_n(x) := \tilde{u}(nx)$ for $x \in (0,1)$. By Example 41, $u_n \overset{L^p}{\rightharpoonup} \bar{u}$. Thus, by weak continuity,

$$I(u_n) = \int_0^1 g(u_n(x))\,\mathrm{d}x \to I(\bar{u}) = g(\bar{u}).$$

On the other hand, $I(u_n) \overset{y=nx}{=} \frac{1}{n}\int_0^n g(\tilde{u}(y))\,\mathrm{d}y = I(u)$, so for all $u \in L^p(0,1)$,

$$I(u) = g(\bar{u}).$$

Step 2: g is convex and concave and hence, g is affine: Let $\lambda \in (0,1)$, $\alpha, \beta \in \mathbb{R}$, and

$$v(x) := \begin{cases} \alpha, & x \in (0,\lambda), \\ \beta, & x \in (\lambda, 1). \end{cases}$$

By Step 1,

$$g(\lambda\alpha + (1-\lambda)\beta) = g(\bar{v}) = I(v) = \lambda g(\alpha) + (1-\lambda)g(\beta),$$

so g is concave and convex. It follows that there exist $a, b \in \mathbb{R}$ such that for all $u \in \mathbb{R}$, $g(u) = a + bu$. (Exercise!) This implies that for all $u \in L^p(0,1)$,

$$I(u) = a + b\int_0^1 u(x)\,\mathrm{d}x = I(0) + b\bar{u}.$$

In particular, I is affine linear.

□

3.2.3 Weak derivatives

We say that a function $u \in L^p(\Omega)$, $p \in [1,\infty]$ has *weak derivative* $D^\alpha u \in L^p(\Omega)$, α a multiindex, if there exists $w \in L^p(\Omega)$ such that

$$\int_\Omega w(x)\varphi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega u(x)D^\alpha\varphi(x)\,\mathrm{d}x$$

for all $\varphi \in C_c^\infty(\Omega)$ with (classical) derivative $D^\alpha\varphi$. Then we set $w = D^\alpha u$ and note that w is unique in $L^p(\Omega)$.
Note that if u is classically differentiable, then its classical and weak derivatives coincide.
Note that if u is classically differentiable almost everywhere, this does not imply its weak differentiability (Exercise).

Example 43. Let $\Omega = (-1,1)$ and

$$u(x) = \begin{cases} 0, & -1 < x \leq 0, \\ x^\delta, & 0 < x < 1, \end{cases}$$

so that $u \in L^p(\Omega)$ if and only if $\delta > -\frac{1}{p}$. Does $u' \in L^p(\Omega)$ exist?

- Consider $\varphi \in C_c^\infty((-1,0)) \subset C_c^\infty(\Omega)$, then

$$\int_{-1}^{1} w\varphi = \int_{-1}^{0} w\varphi \overset{!}{=} -\int_{-1}^{0} 0\varphi' = 0,$$

  so $w|_{(-1,0)} = 0$ in $L^p((-1,0))$.

- Let $\varphi \in C_c^\infty((0,1)) \subset C_c^\infty(\Omega)$ such that $\operatorname{supp}\varphi \subset [\varepsilon, 1)$. Then

$$\int_{0}^{1} w\varphi = \int_{\varepsilon}^{1} w\varphi \overset{!}{=} -\int_{\varepsilon}^{1} x^\delta \varphi'(x)\,\mathrm{d}x = \int_{\varepsilon}^{1} \delta x^{\delta-1}\varphi(x)\,\mathrm{d}x,$$

  so $w|_{(0,1)} = \delta x^{\delta-1}$ and $w \in L^p(0,1) \Leftrightarrow \delta > 1 - 1/p$.

- Assume $\delta > 1 - \frac{1}{p}$ and set

$$w(x) = \begin{cases} 0, & -1 < x < 0, \\ \delta x^{\delta-1}, & 0 < x < -1, \\ \frac{\pi^2}{17}, & x = 0. \end{cases}$$

  Then we show that $w = u' \in L^p(-1,1)$: For all $\varphi \in C_c^\infty(-1,1)$,

$$\int_{-1}^{1} w\varphi = \delta \int_{0}^{1} x^{\delta-1}\varphi(x)\,\mathrm{d}x.$$

  At the same time,

$$\begin{aligned} -\int_{-1}^{1} u\varphi' &= \int_{0}^{1} \delta x^{\delta-1}\varphi + x^\delta\varphi(x)|_{x=0} + x^\delta\varphi(x)|_{x=1} \\ &= \int_{-1}^{1} w\varphi, \text{ if } \delta > 0. \end{aligned}$$

- Note that u is classically differentiable if and only if $\delta > 1$.

3.2.4 Sobolev spaces

For $p \in [1,\infty]$ and a domain $\Omega \subset \mathbb{R}^n$, the *Sobolev space* of order $k \in \mathbb{N}$ is

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k, D^\alpha u \in L^p(\Omega)\},$$

with norm

$$\|u\|_{k,p} = \|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_p^p \right)^{1/p}.$$

Some facts:

- $W^{k,p}(\Omega)$ are *Banach spaces.*
- If $p \in (1,\infty)$, then $W^{k,p}(\Omega)$ are *reflexive.*
- If $p = 2$, then $W^{k,2}(\Omega) =: H^k(\Omega)$ are *Hilbert spaces.*
- If $p \in [1,\infty)$, then $\{u \in C^\infty(\Omega) : \|u\|_{k,p} < \infty\}$ is *dense* in $W^{k,p}(\Omega)$. In particular,

  $$W^{k,p}(\Omega) = H^{k,p}(\Omega) = \text{completion of } C^k(\Omega) \text{ w.r.t. } W^{k,p}\text{-norm.}$$

  Note that in general, $C^\infty(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$ only if $\partial\Omega$ is sufficiently regular (for example, in the case discussed in the paragraph on boundary regularity below) .

For proofs of these facts and Theorems 46, 47 and 48, we refer to [Adams]. It is a comprehensive and well-written monograph on the topic of Sobolev Spaces that also contains some history, sharp results and counterexamples, and details on the requirements of the geometry of Ω that extensively generalize what we are using here. You can find compact introductions to the topic in [Alt] and in [Evans]. The first one is close to (but not identical to) what is in these notes, but, for example, defines Sobolev spaces via density. The second one provides an additional point-of-view, also inspired by applications to PDEs as well as variational problems.

Lemma 44. *(Weak convergence in $W^{k,p}$) Let $p \in [1,\infty)$, $u_n, u \in W^{k,p}(\Omega)$. Then the following are equivalent:*

- *$u_n \rightharpoonup u$ in $W^{k,p}(\Omega)$, and,*
- *$\forall \alpha \in \mathbb{N}^n$ with $|\alpha| \le k$, $D^\alpha u_n \rightharpoonup D^\alpha u$ in $L^p(\Omega)$.*

Proof. (only $k = 1$) Embed $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)^{n+1}$ via

$$J\colon \begin{cases} W^{1,p}(\Omega) & \to L^p(\Omega)^{n+1} \\ u & \mapsto (u, \partial_1 u, \dots, \partial_n u). \end{cases}$$

Then J is injective and norm-preserving. Moreover, $J(W^{1,p}(\Omega))$ is a closed subspace of $L^p(\Omega)^{n+1}$ (Exercise!). Hence, J is an isomorphism of Banach spaces (Exercise!). This shows that weak convergence in $W^{1,p}(\Omega)$ must be the same as weak convergence in $L^p(\Omega)^{n+1}$. $\square$

Example 45. Let $\Omega = (0, 2\pi)$, $u_n(x) = \frac{1}{n}\sin(nx)$, $u_n'(x) = \cos(nx)$. Then

$$\|u_n\|_2^2 = \int_0^{2\pi} \frac{1}{n^2}\sin^2(nx)\,\mathrm{d}x = \frac{\pi}{n^2} \to 0,$$
$$\|u_n'\|_2^2 = \int_0^{2\pi} \cos(nx)^2\,\mathrm{d}x = \pi,$$

so $\|u_n\|_{1,2}^2 = \pi(1 + \frac{1}{n^2}) \le 2\pi$. Abstractly, since $H^1(0,2\pi)$ is reflexive, the bounded sequence $(u_n)_n$ has a weakly convergent subsequence. However, we know more: We have $u_n \xrightarrow{L^2} 0$ (by norm convergence) and by Example 41, $u_n' \overset{L^2}{\rightharpoonup} \overline{\cos} = 0$, hence, by Lemma 44, $u_n \overset{H^1}{\rightharpoonup} 0$.

Boundary regularity: In the following, unless stated otherwise, we fix Ω to be a *bounded domain with Lipschitz (graph) boundary*, in short, *bounded Lipschitz domain,* see e.g. [Alt, A 6.2]:
A bit more precisely, this means that there is a finite number of open sets $U^1, \dots, U^l \subset \mathbb{R}^n$ that cover $\partial\Omega$ in such a way that the sets $\partial\Omega \cap U^j$, $j = 1, \dots, l$, are each given as a part of the graph of a Lipschitz continuous function $g^j : \mathbb{R}^{n-1} \to \mathbb{R}$, and $\Omega \cap U^j$ is located on one side of this graph.

This assumption is stronger than what is needed for most of the following Theorems. They may also hold for less regular and/or unbounded domains/unbounded boundaries. However, the assumptions needed for sharper statements quickly become technically involved. In particular, there may be an interaction between the types of unboundedness and the types of non-regularity that are admitted. For a comprehensive overview and treatment, including important counterexamples, we refer to [Adams]. For the functionals we study in this course, the assumption of bounded Lipschitz domains seems an acceptable compromise between achieving maximal generality and avoiding too many technicalities.
In particular, due to the local Lipschitz graph property, for bounded Lipschitz domains Ω, it is straightforward to define the integrals, L^p-norms and spaces $L^p(\partial\Omega)$ for functions $f : \partial\Omega \to \mathbb{R}$ via coordinate transform to $\mathbb{R}^{n-1}$. For example, if $\operatorname{supp}(f) \subset U^j$, then we can set

$$\int_{\partial\Omega} f(x)\,\mathrm{d}x' = \int_{\mathbb{R}^{n-1}} f\Big(\sum_{i=1}^{n-1} y_i e^i + g^j(y) e^n\Big) \sqrt{1 + |\nabla g^j(y)|^2}\,\mathrm{d}y,$$

which is well-defined due to Lipschitzianity of g^j and, e.g. Rademacher's Theorem. For details, see e.g. [Alt, A 6.5.1 and A 6.5.2].
Moreover, if Ω is a bounded Lipschitz domain, the *unit outer normal vector field*

$$\nu : \partial\Omega \to \mathbb{R}^n, \quad \nu \in L^\infty(\partial\Omega; \mathbb{R}^n), \tag{3.3}$$

is well-defined, i.e. it is independent of the choice of a local representation of $\partial\Omega$, cf. [Alt, A 6.5.3].

Theorem 46. *(Sobolev and Hölder embeddings) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $k > \tilde{k} \in \mathbb{N}_0$. Then*

1. *If $p, \tilde{p} \in [1, \infty)$ satisfy*
$$k - \frac{n}{p} \geq \tilde{k} - \frac{n}{\tilde{p}},$$
then $W^{k,p}(\Omega) \hookrightarrow W^{\tilde{k},\tilde{p}}(\Omega)$, i.e. there is a constant $C(\Omega, k, \tilde{k}, p, \tilde{p}) > 0$ such that for all $u \in W^{k,p}(\Omega)$, also $u \in W^{\tilde{k},\tilde{p}}(\Omega)$ with
$$\|u\|_{\tilde{k},\tilde{p}} \leq C \|u\|_{k,p}.$$

2. *If*
$$k - \frac{n}{p} \geq \tilde{k} + \gamma,$$
with $p \in [1, \infty)$ and $\gamma \in (0, 1)$, then
$$W^{k,p}(\Omega) \hookrightarrow C^{\tilde{k},\gamma}(\Omega),$$

with $C^{\tilde{k},\gamma}(\Omega)$ *the* Hölder space *given by*

$$C^{0,\gamma}(\Omega) := \{u \in C(\Omega) : \|u\|_{C^{0,\gamma}} < \infty\},$$
$$\|u\|_{C^{0,\gamma}} := \|u\|_\infty + \sup_{x,y\in\Omega, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma},$$

or, for $\tilde{k} \neq 0$,

$$C^{\tilde{k},\gamma}(\Omega) := \{u \in C^{\tilde{k}}(\Omega) : \|u\|_{C^{\tilde{k},\gamma}} < \infty\},$$
$$\|u\|_{C^{\tilde{k},\gamma}} := \|u\|_{C^{\tilde{k}}} + \sum_{|\alpha|=\tilde{k}} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma}.$$

The number $k - \frac{n}{p}$ *is called* Sobolev number *of* $W^{k,p}(\Omega)$.

The proof of this Theorem is elaborate. Here, we just sketch two heuristics:

1. How can we trade weak differentiability for larger integrability exponents (and how does the spatial dimension come in)?

   We look at one specific example: Let $n = 2, k = 1, p = 1$ and assume that $u \in W^{1,1}(\Omega)$ with $u = 0$ on $\partial\Omega$. Then we can extend u by zero to $u \in W^{1,1}(\mathbb{R}^2)$. We see that $u \in L^2(\Omega)$ by the following estimates:
   We have
   $$u(x_1, x_2) = \int_{-\infty}^{x_1} \partial_1 u(y_1, x_2)\,\mathrm{d}y_1 = \int_{-\infty}^{x_2} \partial_2 u(x_1, y_2)\,\mathrm{d}y_2,$$
   hence
   $$|u(x)| \leq \int_{\mathbb{R}} |\nabla u(x_j, y_i)|\,\mathrm{d}y_i, \qquad \text{für } i = 1,2, j \neq i. \tag{3.4}$$
   Consequently,
   $$|u(x)|^2 \leq \left(\int_{\mathbb{R}} |\nabla u(x_1, y_2)|\,\mathrm{d}y_2\right)\left(\int_{\mathbb{R}} |\nabla u(y_1, x_2)|\,\mathrm{d}y_1\right).$$
   Integrating with respect to x_1 and x_2 gives the estimate we were looking for,
   $$\|u\|_2^2 \leq \left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(x_1, y_2)|\,\mathrm{d}y_2\mathrm{d}x_1\right)\left(\int_{\mathbb{R}}\int_{\mathbb{R}} |\nabla u(y_1, x_2)|\,\mathrm{d}y_1\mathrm{d}x_2\right) \leq \|u\|_{1,1}^2.$$

2. How can we trade weak differentiability for Hölder regularity?

   We just look at the case $n = 1, k = 1$. Then
   $$|u(x) - u(y)| = |\int_x^y u'(s)\,\mathrm{d}s| \overset{\text{Hölder}}{\leq} \|1\|_{L^{p'}((x,y))}\|u'\|_p = |x-y|^{1-1/p}\|u'\|_p,$$
   with $0 \leq \gamma = 1 - 1/p \leq 1$ the Hölder exponent and $\|u'\|_p$ the Hölder norm.

Also worth thinking about: there is clearly no 'partial' vs 'total' weak differentiability – the definition is 'partial', but 'global' (what does this mean?). The Embedding Theorem shows that sufficient (for example, second-order, large p) Sobolev regularity implies (total) classical differentiability.

The following compactness theorem will also be crucial for our analysis of (DP).

Theorem 47. *(Rellich Compactness Theorem) With the same assumptions as in Theorem 46, if $\gneqq$ holds for the conditions on the Sobolev number in 1. or 2., then the corresponding embeddings are* compact.

In particular, note that this means that if a sequence, e.g. an infimizing sequence, is (a priori) bounded in $W^{k,p}(\Omega)$, it has a strongly convergent subsequence in $W^{\tilde{k},\tilde{p}}(\Omega)$, with $k, p, \tilde{k}, \tilde{p}$ subject to the assumptions of Rellich's Theorem.
Since we are interested in functionals of type (3.1), defined via volume and surface integral densities for functions $u \in W^{1,p}(\Omega)$, we need to understand and quantify the properties of u on $\partial\Omega$. If $u \in W^{1,p}(\Omega)$ is insufficient for $u \in C(\overline{\Omega})$, we need to think carefully about the definition of the integral $\int_{\partial\Omega} g(u(x))\,\mathrm{d}x'$. The essential tool is the following trace operator that uniquely determines $u|_{\partial\Omega}$.

Theorem 48. *(Trace operator) Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. For $p \in [1, \infty]$, define*

$$p^* = \begin{cases} \infty, & \text{if } p > n, \\ \text{arbitrary, but } < \infty, & \text{if } p = n, \\ \frac{p(n-1)}{n-p} \geq p, & \text{if } p < n. \end{cases}$$

Then there is a bounded linear operator

$$\gamma : W^{1,p}(\Omega) \to L^{p^*}(\partial\Omega)$$
$$u \mapsto \gamma(u) = u|_{\partial\Omega}$$

that is the extension of the restriction operator

$$\tilde{\gamma} \colon C^1(\overline{\Omega}) \to C(\partial\Omega)$$
$$u \mapsto u|_{\partial\Omega}.$$

For $u \in W^{1,p}(\Omega)$, we call $u|_{\partial\Omega} \in L^{p^*}(\Omega)$ the trace *of u on $\partial\Omega$.*

We see immediately that if $p > n$, then $W^{1,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ by the embedding theorem and the trace of u is just its continuous extension. Otherwise, more work is needed to obtain the right estimate, see e.g. [Adams, Thm 5.22] that can then be combined with a density argument.

Heuristics for the right estimate in the case $n = 2, p = p* = 1$, locally, assuming that $U^j \subset \operatorname{span}(e^1)$ is flat. As in (3.4), we have

$$|u(x_1, 0)| \leq \int_{\mathbb{R}} |\partial_2 u(x_1, y_2)|\,\mathrm{d}y_2,$$

for every $(x_1, 0) \in U^j$, so

$$\|u(\cdot, 0)\|_{L^1(\mathbb{R})} \leq \|\nabla u\|_{L^1(\mathbb{R}^2)} \leq \|u\|_{1,1}.$$

Now for general bounded $\Omega \subset \mathbb{R}^n$ and $k \geq 1$, $p \in [1, \infty)$, recall the spaces

$$W_0^{k,p}(\Omega) = \overline{C_c^\infty(\Omega)}^{\|\cdot\|_{k,p}} \subset W^{k,p}(\Omega),$$

given by closure of the set of $C_c^\infty(\Omega)$-functions in $W^{k,p}(\Omega)$. Without proof, we note that if Ω is a bounded Lipschitz domain, then they relate to the usual Sobolev space in a sensible way, i.e.

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) \colon u|_{\partial\Omega} = 0\}.$$

Theorem 49. *(Poincaré inequality) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $p \in (1,\infty)$. Then*

1. *For $\delta = \operatorname{diam}(\Omega) = \sup\{|x-y| : x,y \in \Omega\} < \infty$, for all $u \in W_0^{1,p}(\Omega)$ (defined above), we have*
$$\|u\|_p \leq \delta \|\nabla u\|_p. \tag{3.5}$$

2. *Let Ω be a bounded Lipschitz domain. Assume that $V \subset W^{1,p}(\Omega)$ is a closed subspace with the property*
$$u \in V \text{ and } \|\nabla u\|_{L^p(\Omega)} = 0 \Rightarrow u = 0.$$
Then there exists a constant $C_p > 0$, called Poincaré constant, *such that*
$$\|u\|_{1,p} \leq C_p \|\nabla u\|_p \tag{3.6}$$
for all $u \in V$.

Proof. For these two versions of "Poincaré", it is convenient to use two very different strategies:

1. By density, it suffices to prove (3.5) for $u \in C_c^\infty(\Omega)$. Extend u to $\mathbb{R}^n$ by 0 (and still call this function u). For $x \in \Omega$ and e^1 the first unit vector, we have $x + \delta e^1 \notin \Omega$, so
$$u(x) = u(x+\delta e^1) - \int_0^\delta \partial_1 u(x+se^1)\,\mathrm{d}s = -\int_0^\delta \partial_1 u(x+se^1)\,\mathrm{d}s,$$
and thus,
$$\begin{aligned}|u(x)|^p &= \left|\int_0^\delta \partial_1 u(x+se^1)\,\mathrm{d}s\right|^p \\ &\overset{H\ddot{o}lder}{\leq} \left|\int_0^\delta 1\,\mathrm{d}s\right|^{p/p'} \left|\int_0^\delta |\partial_1 u(x+se^1)|^p\,\mathrm{d}s\right|^1 \\ &\leq \delta^{p/p'} \int_0^\delta |\nabla u(x+se^1)|^p\,\mathrm{d}s.\end{aligned}$$
Hence, by Fubini,
$$\begin{aligned}\int_{x\in\Omega} |u(x)|^p\,\mathrm{d}x &\leq \delta^{p/p'} \int_{x\in\Omega}\int_0^\delta |\nabla u(x+se^1)|^p\,\mathrm{d}s\,\mathrm{d}x \\ &\overset{y=x+se^1}{=} \delta^{p/p'} \int_0^\delta \int_{y\in\Omega+se^1} |\nabla u(y)|^p\,\mathrm{d}y\,\mathrm{d}s \\ &\leq \delta^{p/p'+1} \int_{\mathbb{R}^n} |\nabla u(y)|^p\,\mathrm{d}y \\ &= \delta^p \|u\|_p^p.\end{aligned}$$

2. By contradiction: Assume that for all $n \in \mathbb{N}$, there exists $u_n \in V$ such that
$$\|u_n\|_{1,p} \geq n \|\nabla u_n\|_p.$$

Now consider $v_n = \frac{u_n}{\|u_n\|_{1,p}}$ with $\|v_n\|_{1,p} = 1$ and

$$\|\nabla v_n\|_p = \frac{\|\nabla u_n\|_p}{\|u_n\|_{1,p}} \leq \frac{1}{n}. \tag{3.7}$$

Since $W^{1,p}(\Omega)$ is reflexive and $(v_n)_n$ is bounded, there exists a weakly converging subsequence $(v_{n_k})_k$ of $(v_n)_n$ with limit

$$v_{n_k} \overset{W^{1,p}}{\rightharpoonup} v_*.$$

Since $V \subset W^{1,p}(\Omega)$ is a weakly closed subspace (is is linear! − apply Mazur), we obtain $v_* \in V$. By (3.7), we conclude $\nabla v_{n_k} \to 0$ in $L^p(\Omega)$. Using $\nabla v_{n_k} \rightharpoonup \nabla v_*$ (Lemma 44!), this means that $\nabla v_* = 0$, so $v_* = 0$ by assumption. At the same time, by Rellich's Compactness Theorem, there is a subsequence of $(v_{n_k})_k$ with $v_{n_{k'}} \to v_* = 0$ strongly in $L^p(\Omega)$ (here, some boundary regularity is used). This contradicts $\|v_n\|_{1,p} = 1$.

□

Remark 50. What are choices for V in Poincaré's Theorem?

- Let $\Gamma \neq \emptyset$ be a part of the boundary $\Gamma \subset \partial\Omega$ of a bounded Lipschitz domain Ω. Choose $V = X_0 = \{u \in W^{1,p}(\Omega) : u|_\Gamma = 0\}$. Then if $u \in V$ and $\nabla u = 0$, $u = \text{const}$. Depending on Γ, e.g. if it is open in $\partial\Omega$, this implies $u = 0$. Weaker assumptions on Ω and Γ may suffice.
- Another typical choice: $V = \{u \in W^{1,p}(\Omega) : \bar{u} = \frac{1}{|\Omega|}\int_\Omega u = 0\}$. In particular, Theorem 49 implies that for all $u \in W^{1,p}(\Omega)$,

$$\|u - \bar{u}\|_{1,p} \leq C_p \|\nabla u\|_p.$$

More useful facts that close out this section:

- *Gauss' Theorem* holds, i.e. in Theorem 6, it is sufficient to ask that Ω is bounded Lipschitz and $u \in W^{1,1}(\Omega)$, to get

$$\int_\Omega \partial_j u(x)\,\mathrm{d}x = \int_{\partial\Omega} u|_{\partial\Omega}(x)\nu_j(x)\,\mathrm{d}x'$$

  with $\nu \in L^\infty(\partial\Omega;\mathbb{R}^n)$ the outer normal vector field as in (3.3) and $u|_{\partial\Omega} \in L^1(\partial\Omega)$ by the Trace Theorem.
- *Product rule:* if $u \in W^{1,p}(\Omega)$, $v \in W^{1,p'}(\Omega)$, then $uv \in W^{1,1}(\Omega)$ and $(uv)' = u'v + uv'$ (just use "Hölder").
- *Chain rule:* (take care!) If $f\colon \mathbb{R} \to \mathbb{R}$ is uniformly Lipschitz and $u \in W^{1,p}(\Omega)$, then $f \circ u \in W^{1,p}(\Omega)$ and $(f \circ u)' = (f' \circ u)u'$. For $p \in [1,\infty)$, the mapping $T_f\colon W^{1,p}(\Omega) \ni u \mapsto f \circ u \in W^{1,p}(\Omega)$ is bounded.

3.3 Properties of $I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x$ on $W^{1,p}(\Omega)$

The aim of this section is to provide necessary and sufficient conditions on p, Ω, f and g for the functionals

$$
\begin{aligned}
I\colon W^{1,p}(\Omega) \supseteq X &\to \mathbb{R}_\infty \\
u &\mapsto \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x + \int_{\partial\Omega} g(x, u(x))\,\mathrm{d}x
\end{aligned}
$$

of the form we looked at in Chapter 2, to allow us to solve (DP). The strategy is to apply Tonelli's Theorem 32, so, in particular, we want

- X to be reflexive (choose $p \in (1, \infty)$ and $X = W^{1,p}(\Omega)$ or a suitable subspace, see Examples),
- I to be well-defined (Proposition 52),
- I to be coercive (Theorem 54), and
- I to be weakly lower semi-continuous (see Subsection 3.3.1).

For simplicity (and clarity), for the moment, assume that $g = 0$. Criteria for general g are discussed in the Exercises.

We start from the following definition.

Definition 51. (Carathéodory function) Let $\Omega \subset \mathbb{R}^n$ be open and $N \in \mathbb{N}$. Then $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}_\infty$ is called a *Carathéodory function* if

1. for a.e. $x \in \Omega$, $f(x, \cdot)\colon \mathbb{R}^N \to \mathbb{R}_\infty$ is continuous, and,
2. for all $a \in \mathbb{R}^N$, $f(\cdot, a)\colon \Omega \to \mathbb{R}_\infty$ is measurable.

The point of this definition is the following result on I being well-defined if f is Carathéodory and satisfies a suitable growth condition.

Proposition 52. *Let $f\colon \Omega \times \mathbb{R}^N \to \mathbb{R}_\infty$ be a Carathéodory function. Then*

1. *If $u\colon \Omega \to \mathbb{R}^N$ is measurable, then*

$$
\begin{aligned}
h\colon \Omega &\to \mathbb{R}_\infty \\
x &\mapsto f(x, u(x)),
\end{aligned}
$$

is measurable.

2. *For $p \in [1, \infty)$, if there are $a \in L^1(\Omega)$, $b \in \mathbb{R}$ such that for all $u \in \mathbb{R}^N$, a.e. $x \in \Omega$,*

$$f(x, u) \geq a(x) + b|u|^p, \tag{3.8}$$

then the functional

$$I\colon L^p(\Omega; \mathbb{R}^N) \to \mathbb{R}_\infty, \quad I(u) = \int_\Omega f(x, u(x))\,\mathrm{d}x,$$

is well-defined.

Proof.

1. Recall that measurable functions can be approximated by step functions [Klenke, Satz 1.96] and that pointwise limits of measurable functions are measurable [Klenke, Satz 1.92]. We first show 1. in the case that u is a step function,

$$u(x) = \sum_{i=1}^{k} \alpha_i \chi_{A_i}(x)$$

where $\alpha_i \in \mathbb{R}_\infty$ and $A_i \subset \Omega$ are disjoint measurable sets with $\cup_{i=1}^k A_i = \Omega$. Then for every $a \in \mathbb{R}$, the set

$$\begin{aligned}\{x \in \Omega : h(x) \leq a\} &= \cup_{i=1}^k \{x \in A_i : f(x, u(x)) \leq a\} \\ &= \cup_{i=1}^k (A_i \cap f^{-1}((-\infty, a); \alpha_i)\end{aligned}$$

is measurable, as it is the union of measurable sets, as the functions $f(\cdot, \alpha_i)$ are measurable by the Carathéodory assumption. Now we approximate general measurable u by step functions u_n and use the continuity of $f(x, \cdot)$ to get that for a.e. $x \in \Omega$,

$$h(x) = f(x, u(x)) = \lim_{n\to\infty} f(x, u_n(x)).$$

Hence h is measurable as the limit of measurable functions.

2. What is needed for I to be well-defined? By 1., it remains to see that the integral $\int_\Omega f(x, u(x))\,\mathrm{d}x$ does not return the value $-\infty$, i.e. $(f(\cdot, u(\cdot)))_-$ should be integrable. But this follows directly from (3.8).

□

Example 53. Let $f(x, a) = \kappa(x)|a|^p$ with $\kappa \in L^\infty(\Omega)$ and $p \in [1, \infty)$. Then

$$I(u) = \int_{\mathbb{R}} f(x, \nabla u(x))\,\mathrm{d}x$$

is well-defined on $W^{1,p}(\Omega)$.

Theorem 54. *(Coercivity) Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz. Assume that*

$$f \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times n} \to \mathbb{R}_\infty$$

is a Carathéodory function ($N = m + n \cdot m$) that satisfies

$$f(x, u, A) \geq c|A|^p - \delta(x)|u|^r - h(x),$$

for a.e. $x \in \Omega$, all $u \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m\times n}$. Here, $p \in (1, \infty)$, $c > 0$ is a constant, $r \in [1, p)$, $\delta \in L^{\frac{p}{p-r}}(\Omega)$ and $h \in L^1(\Omega)$.
Then there exist constants $C > 0$ and $\beta \in \mathbb{R}$ such that

$$I(u) = \int_\Omega f(x, u(x), \nabla u(x))\,\mathrm{d}x \geq C\|u\|_{1,p}^p - \beta$$

for all $u \in W_0^{1,p}(\Omega)$. Hence, I is coercive *on $X = u_0 + W_0^{1,p}(\Omega)$ for any $u_0 \in W^{1,p}(\Omega)$.*

Proof. Let $u \in W_0^{1,p}(\Omega)$. By assumption,

$$I(u) \geq c\|\nabla u\|_p^p - \int_\Omega \delta(x)|u(x)|^r \,\mathrm{d}r - \|h\|_1.$$

By Poincaré's Inequality, $c\|\nabla u\|_p^p \geq \frac{c}{C_P}\|u\|_{1,p}^p$. Together with Hölder's Inequality with $\left(\frac{p}{r}\right) > 1$, $\left(\frac{p}{r}\right)' = \frac{p}{p-r}$, this gives

$$I(u) \geq \frac{c}{C_p}\|u\|_{1,p}^p - \|\delta\|_{\frac{p}{p-r}}\|u\|_p^r - \|h\|_1.$$

Note that this can be shown for any $u \in V$ where V is a subspace of $W^{1,p}(\Omega)$ such that the Poincaré Inequality holds. Note that since $p > r \geq 1$, there exists an $s_0 > 0$ such that $\frac{c}{2C_P}s^p \geq \|\delta\|_{\frac{p}{p-r}}s^r$ for all $s \geq s_0$. Set $\tilde{\beta} = \|\delta\|_{\frac{p}{p-r}}s_0^r$, then

$$\frac{c}{2C_P}s^p + \tilde{\beta} - \|\delta\|_{\frac{p}{p-r}}s^r \geq 0$$

for all $s \geq 0$. Then we have shown

$$I(u) \geq \frac{c}{2C_P}\|u\|_{1,p}^p + \tilde{\beta} - \|h\|_1 =: C\|u\|_{1,p}^p - \beta.$$

□

Example 55. (Dirichlet problem) Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz and

$$I(u) = \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x.$$

Then I is coercive on $W_0^{1,p}(\Omega)$ if and only if $p \leq 2$.

Proof. There are two cases to be considered:

$p \leq 2$ Then there exists $\beta \in \mathbb{R}$ such that for all $s \geq 0$,

$$s^2 \geq s^p - \beta.$$

It follows that

$$I(u) \geq \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x - \beta|\Omega| \overset{\text{Poincaré}}{\geq} \frac{1}{C_p}\|u\|_{1,p}^p - \beta|\Omega|,$$

so I is coercive on $W_0^{1,p}(\Omega)$.

$p > 2$ Let $u_0 \in W_0^{1,2}(\Omega) \setminus W_0^{1,p}(\Omega)$ and choose $(u_n)_n \subset C_c^\infty(\Omega)$ such that $u_n \to u_0$ strongly in $W^{1,2}(\Omega)$. Then

$$I(u_n) = \|\nabla u_n\|_2^2 \to I(u_0) < \infty,$$

but

$$\|u_n\|_{1,p} \to \infty,$$

so I is not coercive on $W_0^{1,p}(\Omega)$.

□

Example 56. (Is $r < p$ necessary?) Let $p \in (1, \infty)$, $\lambda \in \mathbb{R}$ and

$$I_\lambda(u) = \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x + \lambda \int_\Omega |u(x)|^p \,\mathrm{d}x.$$

Let C_P be the *optimal* constant in the Poincaré inequality (3.6) on $W_0^{1,p}(\Omega)$. Then I_λ is coercive on $W_0^{1,p}(\Omega)$ if and only if $\lambda \gneqq -\frac{1}{C_P^p - 1} =: -\tilde{c}$. This shows that we cannot just omit the condition $r < p$ in Theorem 54.

Proof. Note that from (3.6), we get that $\tilde{c}$ is the optimal constant in the inequality $\|\nabla u\|_p^p \geq \tilde{c}\|u\|_p^p$ for all $u \in W_0^{1,p}(\Omega)$.

$\lambda > -\tilde{c}$: Then I_λ is coercive (Exercise!)

$\lambda \leq -\tilde{c}$: We show that I_λ is not coercive on $W_0^{1,p}(\Omega)$. Let $\tilde{u} \in W_0^{1,p}(\Omega) \setminus \{0\}$ with $\|\nabla \tilde{u}\|_p^p = \tilde{c}\|\tilde{u}\|_p^p$. (We prove later that the optimal constant $\tilde{c} > 0$ and the minimizer of constants $\tilde{u}$ exist.) Let $\alpha \in \mathbb{R}$. Clearly,

$$\|\alpha \tilde{u}\|_{1,p} \to \infty, \quad \text{as } |\alpha| \to \infty.$$

On the other hand,

$$\begin{aligned} I_\lambda(\alpha\tilde{u}) &= |\alpha|^p \left(\int_\Omega |\nabla \tilde{u}(x)|^p \,\mathrm{d}x + \lambda \int_\Omega |\tilde{u}(x)|^p \,\mathrm{d}x \right) \\ &= |\alpha|^p (\tilde{c} + \lambda) \|\tilde{u}\|_p^p. \end{aligned}$$

Since $\tilde{c} + \lambda \leq 0$, this shows that $I_\lambda(\alpha\tilde{u}) \leq 0$ for all $\alpha \in \mathbb{R}$, so I_λ is not coercive.

□

3.3.1 Quasiconvexity and weak lower semicontinuity of I

In this Subsection, we show that, roughly,

1. if I is wlsc in $W^{1,p}$, then f is qc in the gradient component,
2. if f is qc in the gradient component and satisfies a growth condition, then I is wlsc in $W^{1,p}$.

We start out from the necessity of quasiconvexity and show only the simpler case of no x- and u-dependence in f.

Theorem 57. *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and let $f \colon \mathbb{R}^{m \times n} \to \mathbb{R}$ be continuous and*

$$I(u) := \int_\Omega f(\nabla u(x)) \,\mathrm{d}x.$$

Then for $p \in [1, \infty)$, if I is wlsc on $W^{1,p}(\Omega)$, then f is quasiconvex.

Proof. We show that for $[0,1]^n = D \subset \mathbb{R}^n$ the unit cube, for every $\xi \in \mathbb{R}^{m\times n}$ and for every $w \in W_0^{1,p}(D;\mathbb{R}^m)$,

$$\frac{1}{|D|}\int_D f(\xi + \nabla w(y))\,\mathrm{d}y \geq f(\xi). \tag{3.9}$$

(Recall Definition 17 of quasiconvexity of f). Let $w \in W_0^{1,p}(D;\mathbb{R}^m)$ be extended periodically to $\mathbb{R}^n$, i.e.

$$w(x+z) = w(x), \quad \text{for all } x \in D \text{ and } z \in \mathbb{Z}^n.$$

Consider the sequence

$$w_k(x) := \frac{1}{k} w(kx) \in W_0^{1,p}(D;\mathbb{R}^m).$$

Then $w_k \rightharpoonup 0$ in $W_0^{1,p}(D;\mathbb{R}^m)$ by Lemma 44 and Example 41 and an exercise: the result of Example 41 modified to hold on cubes D (instead of intervals $(0,1)$). Now define $u_\xi(x) := \xi x \in W^{1,p}(\Omega;\mathbb{R}^m)$ (Ω is bounded!) and

$$u_k(x) := \begin{cases} u_\xi(x), & x \in \Omega \setminus D, \\ u_\xi(x) + w_k(x), & x \in D. \end{cases}$$

Then $u_k \rightharpoonup u_\xi$ in $W^{1,p}(D:\mathbb{R}^m)$ and

$$\begin{aligned} I(u_k) = \int_\Omega f(\nabla u_k(x))\,\mathrm{d}x &= \int_{\Omega\setminus D} f(\xi)\,\mathrm{d}x + \int_D f(\xi + \nabla w_k(kx))\,\mathrm{d}x \\ &= |\Omega \setminus D| f(\xi) + \frac{1}{k^n}\int_{kD} f(\xi + \nabla w(y))\,\mathrm{d}y \\ &= |\Omega \setminus D| f(\xi) + \int_D f(\xi + \nabla w(y))\,\mathrm{d}y. \end{aligned}$$

Taking the limit and using wlsc'ity of f gives (3.9) via

$$I(u_\xi) = |\Omega| f(\xi) \leq \liminf_k I(u_k) = |\Omega \setminus D| f(\xi) + \int_D f(\xi + \nabla w(y))\,\mathrm{d}y.$$

□

Corollary 58. *If $n = 1$ or $m = 1$ in Theorem 57, then f is necessarily convex.*

Remark 59. Theorem 57 holds more generally for f that also depend on x and u [Dac08, Lemma 3.18] and it can be shown also for unbounded Ω [Dac08, Theorem 3.13].

We now want to prove something like a converse statement to Theorem 57. We first prove a simple preliminary results on continuity of separately convex functions that satisfy a growth condition:

Definition 60. (Separate Convexity – should have been included in Chapter 2!) A function $f\colon \mathbb{R}^N \to \mathbb{R}_\infty$ is *separately convex* or *convex in each variable,* if for every $x = (x_1,\dots,x_N)$, for every $i = 1,\dots,N$, the functions

$$\begin{aligned} f_{y_i}\colon \mathbb{R} &\to \mathbb{R}_\infty \\ y_i &\mapsto f(x_1,\dots,x_{i-1},y_i,x_{i+1},\dots,x_N) \end{aligned}$$

are convex.

Remark 61. Clearly, a rank-1-convex function is separately convex. The converse is not true, consider, for example, the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = x_1 x_2$. It is separately convex but not convex. This example also shows how separate convexity is not objective in its dependence of the choice of coordinates.

Lemma 62. *Let $f\colon \mathbb{R}^N \to \mathbb{R}$ be separately convex and such that for some $\alpha \geq 0$, $p \geq 1$, for every $x \in \mathbb{R}^N$,*

$$|f(x)| \leq \alpha(1 + |x|^p). \tag{3.10}$$

Then there exists $\beta \geq 0$ such that for every $x, y \in \mathbb{R}^N$,

$$|f(x) - f(y)| \leq \beta(1 + |x| + |y|)^{p-1}|x - y|.$$

Proof. In three steps:

Step 1: If a function $g\colon \mathbb{R} \to \mathbb{R}$ is convex, then for every $\lambda > \mu > 0$ and every $t \in \mathbb{R}$,

$$\frac{g(t \pm \mu) - g(t)}{\mu} \leq \frac{g(t \pm \lambda) - g(t)}{\lambda}. \tag{3.11}$$

(Picture!) This is proved by

$$\begin{aligned} g(t \pm \mu) &= g\Big(\frac{\mu}{\lambda}(t \pm \lambda) + (1 - \frac{\mu}{\lambda})t\Big) \\ &\leq \frac{\mu}{\lambda} g(t \pm \lambda) + (1 - \frac{\mu}{\lambda}) g(t), \end{aligned}$$

using convexity of g.

Step 2: Now fix $x^1 = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$ and define

$$g(t) := f(t, x^1)$$

for $t \in \mathbb{R}$. Then we show that there is $\beta_1 \geq 0$ such that

$$|g(x_1) - g(y_1)| \leq \beta_1(1 + |x| + |y|)^{p-1}|x_1 - y_1|). \tag{3.12}$$

Wlog, let $x_1 < y_1$. In (3.11), set

$$\lambda := 1 + |x| + |y|, \quad \mu := y_1 - x_1,$$

then

$$\begin{aligned} g(y_1) - g(x_1) &= g(x_1 + (y_1 - x_1)) - g(x_1) \\ &\leq (y_1 - x_1)\frac{g(x_1 + 1 + |x| + |y|) - g(x_1)}{1 + |x| + |y|} \end{aligned}$$

and

$$\begin{aligned} g(x_1) - g(y_1) &= g(y_1 - (y_1 - x_1)) - g(y_1) \\ &\leq (y_1 - x_1)\frac{g(y_1 - (1 + |x| + |y|)) - g(y_1)}{1 + |x| + |y|}. \end{aligned}$$

Using (3.10), this proves (3.12).

Step 3: With

$$\begin{aligned} f(x) - f(y) = (f(x) - f(y_1, x_2, \dots, x_N)) + \\ \dots (f(y_1, \dots, y_{i-1}, x_i, \dots, x_N) - f(y_1, \dots, y_i, x_{i+1}, \dots, x_N)) \dots \\ + (f(y_1, \dots, y_{N-1}, x_N) - f(y)), \end{aligned}$$

the result follows from Step 2.

□

Theorem 63. *(Quasiconvexity and growth condition imply wlsc'ity) Let $p \in (1, \infty)$ and $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be quasiconvex, additionally satisfying the growth condition*

$$0 \le f(\xi) \le \alpha(1 + |\xi|^p) \tag{3.13}$$

for every $\xi \in \mathbb{R}^{m\times n}$ and some $\alpha \ge 0$. Let $\Omega \subset \mathbb{R}^n$ be open and bounded and

$$I(u) := \int_\Omega f(\nabla u(x))\,\mathrm{d}x.$$

Then I is wlsc in $W^{1,p}(\Omega; \mathbb{R}^m)$.

Proof. In two steps:

1. show wlsc'ity only for $u_n \rightharpoonup u$ with $u(x) = Ax + b$ affine.
2. show wlsc'ity for general u using approximation by piecewise affine functions and Step 1.

Note that a consequence of condition (3.13) is that f is quasiconvex in A if and only if $\int_D f(A + \nabla w(y))\,\mathrm{d}y \ge |D| f(A)$ holds for all $w \in W_0^{1,p}(D; \mathbb{R}^m)$ (instead of: only for all $w \in PC_0^1(D; \mathbb{R}^m)$).

Step 1: Let $(u_n)_n \subset W^{1,p}(\Omega)$ with $u_n \rightharpoonup u$ and $u(x) = Ax + b$ for some $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. If we knew $u_n|_{\partial\Omega} = u|_{\partial\Omega}$, then $u_n = u + w_n$ where $w_n \in W_0^{1,p}(\Omega)$. Then from quasiconvexity, we would get

$$I(u_n) = \int_\Omega f(A + \nabla w_n(x))\,\mathrm{d}x \ge |\Omega| f(A) = I(u) \tag{3.14}$$

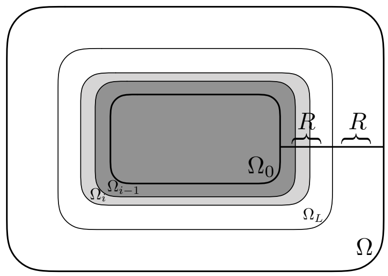
and we would be done. However, in general, $u_n|_{\partial\Omega} \ne u|_{\partial\Omega}$ so we need additional arguments:
Consider

- a set $\Omega_0 \subset \Omega$ such that $R := \frac{1}{2}\mathrm{dist}(\Omega, \Omega_0) > 0$,
- arbitrary $L \in \mathbb{N} \setminus \{0\}$,
- for integers $1 \le i \le L$, the sets

$$\Omega_i := \{x \in \Omega : \mathrm{dist}(x, \Omega_0) \lneqq \frac{i}{L} R\},$$

- cut-off functions $\varphi_i \in C_0^1(\Omega_i)$, with $0 \le \varphi_i \le 1$, $\varphi_i|_{\Omega_{i-1}} = 1$, and $|\nabla \varphi_i| \le \frac{2(L+1)}{R}$,

as in the Figure below:



.

Now let

$$v_{i,n} := u + \varphi_i(u_n - u),$$

then $I(v_{i,n}) \ge I(u)$ as in (3.14). Moreover, (omitting the argument "(x)" in the following),

$$\begin{aligned} I(u) &\le I(v_{i,n}) \\ &= \int_{\Omega\setminus\Omega_i} f(\nabla u)\,\mathrm{d}x + \int_{\Omega_i\setminus\Omega_{i-1}} f(\nabla v_{i,n})\,\mathrm{d}x + \int_{\Omega_{i-1}} f(\nabla u_n)\,\mathrm{d}x \\ &\overset{f\ge 0}{\le} \int_{\Omega\setminus\Omega_0} f(\nabla u)\,\mathrm{d}x + \int_{\Omega_i\setminus\Omega_{i-1}} f(\nabla v_{i,n})\,\mathrm{d}x + \int_{\Omega} f(\nabla u_n)\,\mathrm{d}x. \end{aligned}$$

Since

$$\nabla v_{i,n} = (1-\varphi_i)\nabla u + \varphi_i \nabla u_n + \nabla\varphi_i(u_n - u),$$

by (3.13),

$$\begin{aligned} &\int_{\Omega_i\setminus\Omega_{i-1}} f(\nabla v_{i,n})\,\mathrm{d}x \\ &\le \int_{\Omega_i\setminus\Omega_{i-1}} C(1 + |\nabla v_{i,n}|^p)\,\mathrm{d}x \\ &\overset{\text{Young}}{\le} \int_{\Omega_i\setminus\Omega_{i-1}} C\left(1 + |\nabla u|^p + |\nabla u_n|^p + \left(\frac{2(L+1)}{R}\right)^p |u_n - u|^p\right)\mathrm{d}x. \end{aligned}$$

Putting the estimates above together, summing over i and dividing by L, we get

$$\begin{aligned} &\int_{\Omega_0} f(\nabla u)\,\mathrm{d}x \\ &\le \frac{C}{L}\int_{\Omega}\left(1 + |\nabla u|^p + |\nabla u_n|^p + \left(\frac{2(L+1)}{R}\right)^p |u_n - u|^p\right)\mathrm{d}x \\ &\quad + \int_{\Omega} f(\nabla u_n)\,\mathrm{d}x. \end{aligned}$$

Now recall that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$, so $\int_\Omega |\nabla u_n(x)|^p \,\mathrm{dx}$ is bounded and by Rellich, $u_n \to u$ strongly in $L^p(\Omega)$, so taking n large,

$$\left(\frac{2(L+1)}{R}\right)^p \int_\Omega |u_n - u|^p(x)\,\mathrm{d}x$$

is bounded independently of L. Taking the liminf and using wlsc'ity of the norm, we obtain

$$\int_{\Omega_0} f(\nabla u)\,\mathrm{d}x \le \frac{C}{L} + \liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\,\mathrm{d}x.$$

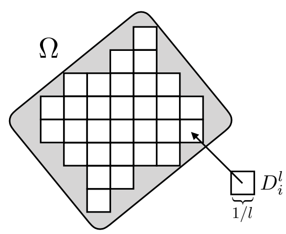
As $\Omega_0 \subset \Omega$ and $L \in \mathbb{N}$ were arbitrary, conclude

$$\int_\Omega f(\nabla u)\,\mathrm{d}x \le \liminf_{n\to\infty} \int_\Omega f(\nabla u_n)\,\mathrm{d}x.$$

Step 2: Now we look at general $(u_n)_n \subset W^{1,p}(\Omega)$ with $u_n \rightharpoonup u \in W^{1,p}(\Omega)$ not necessarily affine. We approximate Ω by $S(l)$ cubes D_s^l of size $\frac{1}{l}$, $l \in \mathbb{N}$. In particular, for given $\delta > 0$, l can be made so large and the cubes can be placed so that

$$H^l = \cup_{i=1}^{S(l)} D_i^l \subset \Omega \quad \text{with} \quad |\Omega \setminus H^l| < \delta, \tag{3.15}$$

see Figure below:



Set $\xi_i^l := \frac{1}{|D_i^l|}\int_{D_i^l} \nabla u(x)\,\mathrm{d}x$. Then for $\varepsilon > 0$ we can choose $l \in \mathbb{N}$ such that

$$\sum_{i=1}^{l} \int_{D_i^l} |\nabla u - \xi_i^l|^p \,\mathrm{d}x < \varepsilon. \tag{3.16}$$

(To see this, approximate $\nabla u \in L^p(\Omega;\mathbb{R}^{m\times n})$ by $w_\varepsilon \subset C(\Omega;\mathbb{R}^{m\times n})$ such that $\|\nabla u - w_\varepsilon\|_p^p < \varepsilon$. Since w_ε is continuous, it follows directly that for sufficiently small l,

$$\sum_{i=1}^{l} \int_{D_i^l} |w(x) - \frac{1}{|D_i^l|}\int_{D_i^l} w(y)\,\mathrm{d}y|^p \,\mathrm{d}x < \varepsilon.$$

It is then straightforward to show (3.16).)

Now look at

$$\begin{aligned}
I(u_n) - I(u) &= \int_\Omega f(\nabla u_n) - f(\nabla u) \, \mathrm{d}x \\
&= \int_{\Omega \setminus H^l} f(\nabla u_n) - f(\nabla u) \, \mathrm{d}x \\
&\quad + \int_{H^l} f(\nabla u + \nabla(u_n - u)) - f(\nabla u) \, \mathrm{d}x \\
&= J_1^{n,l} + J_2^{n,l}.
\end{aligned}$$

From (3.13), using (3.15), we get

$$J_1^{n,l} \geq -C \int_{\Omega \setminus H^l} (1 + |\nabla u(x)|^p) \, \mathrm{d}x \overset{l \to \infty}{\to} 0.$$

To estimate $J_2^{n,l}$, let ξ^l be the piecewise constant function with $\xi^l(x) = \xi_i^l$ if $x \in D_i^l$ and set

$$\begin{aligned}
J_2^{n,l} = &\int_{H^l} f(\xi^l + \nabla(u_n - u) \, \mathrm{d}x \\
&+ \int_{H^l} f(\nabla u + \nabla(u_n - u)) - f(\xi^l + \nabla(u_n - u)) \, \mathrm{d}x \\
&+ \int_{H^l} -f(\nabla u) \, \mathrm{d}x. \\
=: &\, J_{21}^{n,l} + J_{22}^{n,l} + J_{23}^l.
\end{aligned}$$

From Step 1, for all D_i^l, we obtain

$$\liminf_{n \to \infty} \int_{D_i^l} f(\xi_i^l + \nabla(u_n - u)) \, \mathrm{d}x \geq \int_{D_I^l} f(\xi_i^l) \, \mathrm{d}x,$$

so we have

$$\liminf_{n \to \infty} J_{21}^{n,l} + J_{23}^l \geq \int_{H^l} f(\xi^l(x)) - f(\nabla u(x)) \, \mathrm{d}x =: J_3^l.$$

Now by Lemma 62, there exists $\beta, C \geq 0$ such that

$$\begin{aligned}
|J_3^l| &\leq \int_{H^l} |f(\xi^l(x)) - f(\nabla u(x)) \, \mathrm{d}x \\
&\leq \beta \int_{H^l} (1 + |\xi^l(x)| + |\nabla u(x)|)^{p-1} |\xi^l(x) - \nabla u(x)| \, \mathrm{d}x \\
&\overset{\text{Hölder}}{\leq} \beta \|(1 + |\xi^l| + |\nabla u|)^{p-1}\|_{L^{p'}(H^l)} \|\xi^l - \nabla u\|_{L^p(H^l)} \\
&= \beta \|1 + |\xi^l| + |\nabla u|\|_{L^p(H^l)}^{p-1} \|\xi^l - \nabla u\|_{L^p(H^l)} \\
&\leq C \|\xi^l - \nabla u\|_{L^p(H^l)},
\end{aligned}$$

where we used that $\|\xi^l\|_p$ is bounded by $\|\nabla u\|_p$ considering (3.16). In the same way, for large n,

$$\begin{aligned}|J_{22}^{n,l}| &\leq \beta' \int_{H^l} (1 + |\xi^l| + |\nabla u| + |\nabla u_n|)^{p-1} |\xi^l - \nabla u| \,\mathrm{d}x \\ &\leq C\|\xi^l - \nabla u\|_{L^p(H^l)}.\end{aligned}$$

Putting everything together, we obtain

$$\begin{aligned}\liminf_{n\to\infty} I(u_n) - I(u) &\geq \liminf_{n\to\infty} J_1^{n,l} + J_2^{n,l} \\ &\geq \liminf_{n\to\infty} J_1^{n,l} + J_{22}^{n,l} + J_3^l \\ &\geq -C \int_{\Omega\setminus H^l} (1 + |\nabla u(x)|^p) \,\mathrm{d}x - C\|\xi^l - \nabla u\|_{L^p(H^l)} \\ &\overset{l\to\infty}{\to} 0,\end{aligned}$$

and have proved the claim.

□

Remark 64. The growth condition (3.13) can be weakened, but it cannot be omitted, as the example $f(\nabla u) = \det(\nabla u)$ shows. If $m = n = 2$, then f is quasiconvex, but clearly, (3.13) is not satisfied, as the determinant could be negative. Moreover, I is not wlsc in $W^{1,2}(\Omega)$ [Dac08, Example 8.6].

Remark 65. Theorem 63 holds more generally for f that also depend on x and u [Dac08, Theorem 8.8].

Remark 66. In spite of the results of this Subsection, for a given non-convex function it remains difficult to decide whether it is quasiconvex and/or weakly lower semicontinuous. Moreover, the assumption of quasiconvexity and the growth condition (3.13) may be too restrictive. The need for functionals

$$f\colon \Omega \times \mathbb{R}^{m\times n} \to [0,\infty]$$

that are Carathéodory, non-convex, but with more straightforward structure than quasiconvex functions, and such that $I\colon W^{1,p}(\Omega) \to \mathbb{R}_\infty$ is wlsc for suitable p leads to the concept of *polyconvexity,* see e.g. [Dac08, Chapter 5].

3.4 Examples

3.4.1 p-Laplace

Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz, $p \in (1,\infty)$. Consider the functional

$$I(u) = \int_\Omega \frac{1}{p}|\nabla u(x)|^p \,\mathrm{d}x - \int_\Omega h(x)u(x)\,\mathrm{d}x$$

in $W_0^{1,p}(\Omega)$. Then for every $h \in L^{p'}(\Omega)$, there exists a unique minimizer of I in $W_0^{1,p}(\Omega)$.

Proof and discusssion in four steps:

1. I is well-defined and coercive in $W^{1,p}(\Omega)$

   Consider $f\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, $f(x,u,A) := \frac{1}{p}|A|^p - h(x)u$. Clearly, f is a Carathéodory function. We can estimate f from below,

   $$f(x,u,A) \geq \frac{1}{p}|A|^p - |h(x)||u| \geq \frac{1}{p}(|A|^p - |u|^p) - \frac{1}{p'}|h(x)|^{p'}.$$

   In the second estimate, if $h \in L^{p'}(\Omega)$, then $|h|^{p'} \in L^1(\Omega)$, so the condition in Proposition 52 is satisfied and thus I is well-defined. The first estimate shows that the growth condition in Theorem 54 is satisfied with $r = 1$ if $h \in L^{p'=p/p-1}(\Omega)$. Hence, I is coercive on $W_0^{1,p}(\Omega)$.

2. I is wlsc

   The function $f_1\colon \mathbb{R}^n \to [0,\infty)$, $A \mapsto \frac{1}{p}|A|^p$ is convex and satisfies the growth condition in Theorem 63, so for a weakly convergent sequence $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, using $h \in L^{p'}(\Omega) \subset (W_0^{1,p}(\Omega))'$, we have

   $$\begin{aligned}\liminf_n I(u_n) &\geq \int_\Omega f_1(\nabla u(x))\,\mathrm{d}x - \lim_n \int_\Omega h(x)u_n(x)\,\mathrm{d}x \\ &= \int_\Omega f_1(\nabla u(x))\,\mathrm{d}x - \int_\Omega h(x)u(x)\,\mathrm{d}x = I(u),\end{aligned}$$

   so I is wlsc in $W_0^{1,p}(\Omega)$.

   By Tonelli's Theorem, 1. and 2. give the existence of a minimizer of I in $W_0^{1,p}(\Omega)$.

3. Uniqueness of the minimizer follows from strict convexity: Exercise!

4. Why p-Laplace?

   The Euler-Lagrange equations for I are

   $$\begin{cases}\operatorname{div}(|\nabla u|^{p-2}\nabla u(x)) + h(x) = 0, & x \in \Omega,\\ u(x) = 0, & x \in \partial\Omega.\end{cases}$$

   They generalize Laplace's Equation (the case $p = 2$). By the direct method, we have shown the existence of a unique (weak) solution.

3.4.2 Optimal Poincaré constant

Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz, $p \in (1,\infty)$. Then there exist $\tilde{c} > 0$ and $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that for all $v \in W_0^{1,p}(\Omega)$,

$$\|\nabla v\|_p^p \geq \tilde{c}\|v\|_p^p, \tag{3.17}$$

and

$$\|\nabla u\|_p^p = \tilde{c}\|u\|_p^p. \tag{3.18}$$

We prove this in three steps:

1. Reformulation as a minimization problem with constraint
   If (3.17) is true for all $v \in W_0^{1,p}(\Omega)$ with $\|v\|_p = 1$, then it is true for all $u \in W_0^{1,p}(\Omega)$ (if (3.17) is true for some $u \in W_0^{1,p}(\Omega)$, it must hold for $v := \frac{u}{\|u\|_p}$.) Thus, we minimize
   $$I(u) := \int_\Omega |\nabla u(x)|^p \,\mathrm{d}x = \|\nabla u\|_p^p$$
   in the set $M := \{v \in W_0^{1,p}(\Omega) : \|v\|_p = 1\}$.

2. Application of the Direct Method
   Let $(u_n)_n \subset M$ be an infimizing sequence for I.
   - Since I is coercive (Poincaré's Inequality), $(u_n)_n$ is bounded.
   - Since $W_0^{1,p}(\Omega)$ is reflexive, $(u_n)_n$ is weakly convergent.
   - If M is weakly sequentially closed, then the weak limit $u_n \rightharpoonup u$ is again in M.
   - Since I is wlsc on M (wlsc on $W_0^{1,p}(\Omega)$ by convexity and Theorem 63), then $\liminf_n I(u_n) \geq I(u)$, so $u \in M$ is a minimizer.

   It remains to show that M is weakly sequentially closed. Since M is not convex, Mazur's Lemma does not apply. Instead we use Rellich's Theorem: Let $(u_n)_n \subset M$ such that $u_n \rightharpoonup u_0 \in W^{1,p}(\Omega)$. Then $u_0 \in W_0^{1,p}(\Omega)$ since $W_0^{1,p}(\Omega)$ is a linear and thus weakly sequentially closed subspace of $W^{1,p}(\Omega)$. By Rellich's Theorem, there is a subsequence $u_{n_k} \xrightarrow{k} u_0$ that converges strongly in $L^p(\Omega)$, so
   $$\|u_0\|_p = \lim_k \|u_{n_k}\|_p = 1$$
   and hence $u_0 \in M$.

3. Conclusion
   By 2., there is an $u \in M$ such that $\|\nabla u\|_p^p \leq \|\nabla v\|_p^p$ for all $v \in M$. Define $\tilde{c} := I(u)$, then
   $$I(v) = \|\nabla v\|_p^p \geq \tilde{c}\|v\|_p^p$$
   for all $v \in M$ and hence for all $v \in W_0^{1,p}(\Omega)$ by 1. In addition,
   $$\|\nabla u\|_p^p = \tilde{c} \cdot 1 = \tilde{c}\|u\|_p^p,$$
   so (3.17) and (3.18) are proved. It remains to check that $\tilde{c} \neq 0$. Assume $\tilde{c} = 0$, then $\nabla u = 0$ in Ω, so $u = \text{const.}$ on each connected component of Ω. Since $u \in W_0^{1,p}(\Omega)$, this implies $u = 0$. But this is in contradiction to $\|u\|_p = 1$.

3.4.3 Phase separation (Cahn-Hilliard energy)

Let $\Omega \subset \mathbb{R}^3$ be bounded Lipschitz and $\kappa, \varepsilon > 0$. Define
$$I(u) := \int_\Omega \frac{\varepsilon}{2}|\nabla u(x)|^2 + \frac{\kappa}{\varepsilon}(1 - u(x)^2)^2 \,\mathrm{d}x \tag{3.19}$$
on the set
$$V := \{u \in W^{1,2}(\Omega) : \int_\Omega u(x)\,\mathrm{d}x = 0\}$$
– the double-well potential has returned! Then, I has a minimizer in the space V.

1. Proof

   Since the Poincaré Inequality holds in the reflexive Banach space V and for all $u \in \mathbb{R}$, $(1-u^2)^2 \geq 0$, I is well-defined and coercive. If we show that I is wlsc in V, then by Tonelli's Theorem, we have a minimzer $u \in V$. To show that I is wlsc, it remains to show that

   $$\begin{array}{rlll} I_2: & W^{1,2}(\Omega) & \to & \mathbb{R} \\ & u & \mapsto & \int_\Omega (1-u(x)^2)^2 \,\mathrm{d}x \end{array}$$

   is wlsc in V. This follows from the following proposition using that

   $$g\colon \mathbb{R} \supset u \mapsto (1-u^2)^2 \in \mathbb{R}$$

   is continuous with $|g(u)| \leq C(1+|u|^4)$ for some $C > 0$ and that the embedding $W^{1,2}(\Omega) \overset{c}{\hookrightarrow} L^r(\Omega)$ is compact for $r = 4$.

Proposition 67. *(Non-convex functionals of lower order) Let $\Omega \subset \mathbb{R}^n$ be bounded Lipschitz, $p \in (1,\infty)$ and $r \in [1,\infty)$ such that the embedding $W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^r(\Omega)$ is compact, i.e. $1 - \frac{n}{p} > -\frac{n}{r}$. Moreover, let $g\colon \mathbb{R}^m \to \mathbb{R}$ be continuous with $|g(u)| \leq C(1+|u|^r)$ for all $u \in \mathbb{R}^m$. Then the functional*

$$\begin{array}{rlll} I_g: & W^{1,p}(\Omega) & \to & \mathbb{R} \\ & u & \mapsto & \int_\Omega g(u(x)) \,\mathrm{d}x \end{array}$$

is weakly sequentially continuous.

Proof. Consider $u_n \overset{W^{1,p}}{\rightharpoonup} u$. Let $(u_{n_{k'}})_{k'}$ be an infimizing subsequence of $(u_{n_k})_k$ such that

$$\liminf_n I_g(u_n) = \lim_k I_g(u_{n_k}).$$

By Rellich's Theorem, there is a strongly converging subsequence $u_{n_{k'}} \overset{L^r(\Omega)}{\to} u$, so, in particular,

$$\int_\Omega (1 + |u_{n_{k'}}(x)|^r) \,\mathrm{d}x = |\Omega| + \|u_{n_{k'}}\|_r^r \overset{k'\to\infty}{\to} |\Omega| + \|u\|_r^r$$

and we can choose $(u_{n_{k'}})_{k'}$ such that it converges pointwise a.e. to u. We obtain

$$\begin{aligned} \liminf_n I_g(u_n) &= \lim_{k'} I_g(u_{n_{k'}}) + C(|\Omega| + \|u_{n_{k'}}\|_r^r) - C(|\Omega| + \|u_{n_{k'}}\|_r^r) \\ &\overset{\text{Fatou}}{\geq} \int_\Omega \lim_{k'} \left(g(u_{n_{k'}}(x)) + \frac{C}{|\Omega|}(|\Omega| + \|u_{n_{k'}}\|_r^r) \right) \mathrm{d}x - C(|\Omega| + \|u_{n_{k'}}\|_r^r) \\ &= \int_\Omega g(u(x)) \,\mathrm{d}x = I_g(u). \end{aligned}$$

Here, the condition $|g(u)| \leq C(1+|u|^r)$ ensures that Fatou is applicable as the integrand is positive. In the same way, we prove that

$$-\limsup_n I_g(u_n) = \liminf_n -I_g(u_n) \geq -I_g(u),$$

hence, $\lim_n I_g(u_n) = I_g(u)$. □

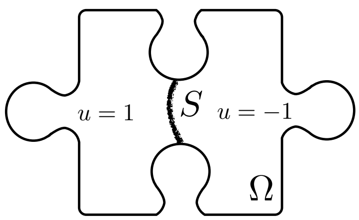
Remark 68. Note the importance of g acting in the lower order: Otherwise, g and I_g would be affine (Corollary 42) or g would have to be convex (Theorem 57).

2. Discussion – why "phase separation"?

   Consider u as the volume density of a given compound material, re-scaled, so that

$$\begin{aligned} u(x) = 1 &\simeq \text{material A at } x \text{ (e.g. zinc)}, \\ u(x) = -1 &\simeq \text{material B at } x \text{ (e.g. lead)}, \\ u(x) \in (-1,1) &\simeq \text{mixture of materials A and B at } x. \end{aligned}$$

   The assumption $u \in M$ just means that the total ratio of 50% zinc and 50% lead is fixed. Then I in (3.19) is an energy associated to the phases' wish to separate: the states $u(x) = 1$ and $u(x) = -1$ are preferable, due to the double-well potential. Mixed states are more costly, as well as changes of material composition ($\nabla u \neq 0$), due to the first integrand. The minimizer will thus be mostly constant with a minimal size of the separating layer S_ε (see Picture). S can be considered to approximate ($\varepsilon \to 0$) the free surface that separates two natural phases.



3.4.4 Nonlinear Elasticity

Recall the modelling and notation for elasticity from Section 2.7.3. Recall the condition

$$\det F_n > 0 \text{ with } \det F_n \overset{n\to\infty}{\to} 0 \Rightarrow W(x, F_n) \overset{n\to\infty}{\to} \infty \tag{3.20}$$

on the deformation gradient $F = \nabla\varphi$ that was introduced to avoid self-interprenetation. This condition implied that elastic energies would need to be non-convex.

For $n = 2$, set

$$W(A) = \begin{cases} \frac{1}{p}\|A\|^p + \frac{1}{\det A}, & \det A > 0, \\ +\infty, & \det A \leq 0, \end{cases}$$

as a model energy density for nonlinear elasticity that satisfies (3.20). Set

$$I(\varphi) = \int_\Omega W(\nabla\varphi(x))\,\mathrm{d}x$$

as the corresponding nonlinear elastic energy. What are properties of I? Do minimizers exist (in what space)?

As usual, we want to apply Tonelli's Theorem.

- For $p > 1$, I is well-defined in $W_0^{1,p}(\Omega)$ by Proposition 52 and coercive by Poincaré's Inequality,

$$I(\varphi) \geq \frac{1}{p}\|\nabla\varphi\|_p^p \geq C\|\varphi\|_{W^{1,p}(\Omega)}^p.$$

- To show wlsc'ity for $p > 2$, it remains to show wlsc'ity of

$$I_2(\varphi) = \int_\Omega f_2(\nabla\varphi(x))\,\mathrm{dx},$$

where

$$f_2(A) = \begin{cases} \frac{1}{\det A}, & \det A > 0, \\ +\infty, & \det A \leq 0. \end{cases}$$

We decompose I_2 on $W_0^{1,p}(\Omega)$ into the operator

$$B\colon W^{1,p}(\Omega) \ni \varphi \mapsto \det(\nabla\varphi) \in L^{p/2}(\Omega),$$

(this uses $p > 2$), and the functional

$$J_2\colon L^{p/2}(\Omega;\mathbb{R}) \ni \theta \mapsto J_2(\theta) = \int_\Omega g_2(\theta(x))\,\mathrm{d}x \in [0,\infty],$$

where

$$g_2(s) = \begin{cases} \frac{1}{s}, & s > 0, \\ +\infty, & s \leq 0. \end{cases}$$

Note that g_2 is positive, continuous and convex. Thus, wlsc'ity of J_2 holds by the following result.

Proposition 69. *(Convexity and "$+\infty$") Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Assume that*

$$f\colon \Omega \times \mathbb{R}^N \to [0,\infty]$$

is a Carathéodory function that satisfies

$$f(x,\cdot)\colon \mathbb{R}^N \to [0,\infty] \text{ is convex for a.e. } x \in \Omega.$$

Then for all $p \in (1,\infty)$, the functional

$$I(u) = \int_\Omega f(x,u(x))\,\mathrm{d}x$$

is wlsc in $L^p(\Omega)$.

Proof. Note that by assumption, I is well-defined. Since $f(x,\cdot)$ is convex, I is convex. Thus, by Theorem 34, it suffices to show that I is strongly lower semicontinuous. Hence, let $(u_n)_n \subset L^p(\Omega)$ such that $u_n \to u \in L^p(\Omega)$. Let $(u_{n_k})_k$ be a subsequence such that

$$\lim_k I(u_{n_k}) = \liminf_n I(u_n).$$

Let $(u_{n_{k'}})_{k'}$ be a further subsequence that converges pointwise a.e. to u. Since the $f(x,\cdot)$ are continuous and non-negative, by Fatou, we obtain

$$\begin{aligned}\liminf_n \int_\Omega f(x,u_n(x)) &= \liminf_{k'} \int_\Omega f(x,u_{n_{k'}}(x))\,\mathrm{d}x \\ &\geq \int_\Omega \lim_{k'} f(x,u_{n_{k'}}(x))\,\mathrm{d}x \\ &= \int_\Omega f(x,u(x))\,\mathrm{d}x.\end{aligned}$$

□

To prove wlsc'ity of I_2, it now suffices to show that the operator B is weakly sequentially continuous. So assume $\varphi_n \rightharpoonup \varphi$ weakly in $W_0^{1,p}(\Omega)$. We need to show that $\det(\nabla\varphi_n) \rightharpoonup \det(\nabla\varphi)$ weakly in $L^{p/2}(\Omega)$. By Lemma 39, it suffices to show

1. $\|\det(\nabla\varphi_n)\|_{p/2} \leq C$ for $C > 0$ independent of $n \in \mathbb{N}$ (this also shows that B is well-defined), and
2. $\int_\Omega \det(\nabla\varphi_n(x))\psi(x)\,\mathrm{d}x \to \int_\Omega \det(\nabla\varphi(x))\psi(x)\,\mathrm{d}x$ for all $\psi \in C_c^\infty(\Omega)$.

To show 1., note that for any $v \in W^{1,p}(\Omega)$,

$$\begin{aligned}\|\det(\nabla v)\|_{p/2}^{p/2} &\leq C\int_\Omega |\partial_1 v_1 \partial_2 v_2|^{p/2} + |\partial_1 v_2 \partial_2 v_1|^{p/2}\,\mathrm{d}x \\ &\leq C\int_\Omega |\partial_1 v_1|^p + |\partial_2 v_2|^p + |\partial_1 v_2|^p + |\partial_2 v_1|^p\,\mathrm{d}x \\ &= C\|\nabla v\|_p^p < +\infty.\end{aligned}$$

Since $(\varphi_n)_n$ is weakly convergent, it is bounded, and this proves the claim.
To show 2., apply Gauss' Theorem:

$$\begin{aligned}\int_\Omega \det(\nabla\varphi_n)\psi &= \int_\Omega (\partial_1\varphi_{n1}\partial_2\varphi_{n2} - \partial_2 u_{n1}\partial_1 u_{n2})\psi \\ &= \int_\Omega \operatorname{div}(\varphi_{n1}\begin{pmatrix}\partial_2\varphi_{n2} \\ -\partial_1\varphi_{n2}\end{pmatrix})\psi \\ &\overset{\text{Gauss}}{=} -\int_\Omega \varphi_{n1}\begin{pmatrix}\partial_2\varphi_{n2} \\ -\partial_1\varphi_{n2}\end{pmatrix}\cdot\begin{pmatrix}\partial_1\psi \\ \partial_2\psi\end{pmatrix} \\ &\to -\int_\Omega \varphi_1\begin{pmatrix}\partial_2\varphi_2 \\ -\partial_1\varphi_2\end{pmatrix}\cdot\begin{pmatrix}\partial_1\psi \\ \partial_2\psi\end{pmatrix} \\ &= \int_\Omega \det(\nabla\varphi)\psi.\end{aligned}$$

In conclusion, we have shown that the elastic energy defined above has a minimizer in $W_0^{1,p}(\Omega)$. Its density of the form $f(A) = g(A, \det A)$ with convex g is a special *polyconvex* function. The notion of polyconvexity is important in nonlinear elasticity theory. Roughly speaking,

$$f \text{ convex } \Rightarrow f \text{ polyconvex } \Rightarrow f \text{ quasiconvex} \Rightarrow \ldots,$$

and, at least in general,

$$f \text{ convex } \not\Leftarrow f \text{ polyconvex } \not\Leftarrow f \text{ quasiconvex } \ldots$$