# Introduction

These notes should serve as an introduction to the *Internet seminar on non-Newtonian fluids* held at TU Darmstadt in the summer term of 2010. It is a short collection of notions from modelling and analyis of (non-) Newtonian fluid flow with some emphasis on the first part. The main purpose is to raise mathematical questions regarding the analysis of non-Newtonian fluids to be discussed in the seminar meeting and during its preparation. Furthermore, it is an opportunity to list and relate to selected literature on the subject.

Each part of the notes has a particular topic. The first part is on basics from continuum mechanics and the modelling of fluids and it contains a motivational chapter. The subsequent parts will be in particular on generalized Newtonian fluids (May 4), Oldroyd-B type fluids (May 18) and modelling and problems for dilute polymer solutions (June 1). The focus in all parts is on stating the inital and boundary value problems and on existence and uniqueness results in three space dimensions. A detailed table of contents will be added later and each part will contain references, but they will also be included in one bibliography later on. The literature on the subject is extensive and my overview is very limited. Suggestions are very welcome! Books on modelling and mathematical analysis of non-Newtonian fluids are for example [1], [4] and [7].

Understanding the analysis of non-Newtonian fluids as well as organizing my first seminar seem to me to be two very difficult tasks. I would be very happy to receive your comments and corrections and I look forward to our discussion!

# 1 Basics from continuum mechanics and the Navier-Stokes equations

We start out with some basics from continuum mechanics and repeat some basic steps in the modelling of the Navier-Stokes equations. Even though non-Newtonian fluids are those that show flow behaviour which cannot be described by these equations, they are still fluids and their governing equations inherit characterisitic features from the Navier-Stokes equations, e.g the presence of a pressure term linked to the "divergence-free" condition and the non-linear term $(u \cdot \nabla)u$, which shows that the fluid velocity is transported by itself. Roughly speaking, the difference will "only" appear in the constitutive equations for the stress tensor of the fluid, but of course, this will also be the term of highest order in the velocity $u$.

The following remarks on modelling are brief and simplifying. For a detailed background and explanations, we refer e.g. to the books [1] and [10]. Moreover, I got great help for preparing this chapter from the lecture of Professor Bothe on *Continuum Mechanical Modeling of Flows* and from Professor Robertson's *Lecture Notes on Non-Newtonian Fluids*[1].

## 1.1 Continuum hypothesis

A basic assumption in continuum mechanics is that the behaviour of the material at hand can be described by quantities which are piecewise continuous functions of time and space. This continuity assumption makes the process and the quantities involved macroscopic. This is an idealization of the real situation which makes sense if the scales of the process are sufficiently big.

For non-Newtonian fluids, it is assumed that their flow dynamics is influenced by the structure of the molecules they contain. This makes modelling very difficult. In many models, this influence is implemented "within" the continuum mechanical framework as an extra continuous stress on the fluid. In the fourth part of the notes on dilute polymer solutions, we will see how different scales are included in one model. There will be a continuum mechanical part of the problem as well as a contribution from molecular forces, Brownian motion. Moreover, the continuum mechanical Oldroyd-B model will arise as a special case of a coupled dilute polymer model.

[1] I found them on: http://numerik.iwr.uni-heidelberg.de/Oberwolfach-Seminar/Robertson_NonNewtonianNotes.pdf. Professor Robertson teaches at the University of Pittsburgh. Please also look at [8]. Both texts also contain very good references on the topic.

## 1.2 Balance equations

Consider some extensive quantity $\Phi$ which has value $\Phi(t, V)$ at time $t$ on a volume $V$. The term *extensive quantity* means that $\Phi$ can be assigned a value for every volume $V$ and that it is additive with respect to $V$, i.e. $\Phi(t, \cup_{i\in\mathbb{N}} V_i) = \sum_{i\in\mathbb{N}} \Phi(t, V_i)$ for given volumina $V_i$. For example, $\Phi$ can be a mass $m$, a momentum $mv$ or an energy $\frac{1}{2}mv^2$. By the continuum hypothesis, a pointwise limit and a density $\phi$ can be assigned to $\Phi$, i.e.

$$\phi(t, x) = \lim_{R\to 0_+} \frac{\Phi(t, B_R(x))}{|B_R(x)|},$$

where $B_R(x)$ denotes the ball centered at $x$ with radius $R$.

The *balance* of $\Phi$ over a volume $V$ is an equation for the rate of change of $\Phi$ on $V$ and it is given by

$$\frac{d}{dt}\Phi(t, V) = -\int_{\partial V} j \cdot n \, \mathrm{d}\sigma + \int_V f \, \mathrm{dV},$$

where $j$ is the flux of $\Phi$ through the boundary $\partial V$ and where $f$ gives the rate of change of $\Phi$ attributed to sources and sinks within $V$. By the Divergence theorem, this gives the transport equation

$$\frac{\partial \phi}{\partial t} + \operatorname{div} j = f$$

in terms of the density $\phi$. The rate of change of $\Phi$ on $V$ is thus decomposed into exactly the two components $j$ and $f$. The question remains how $j$ and $f$ are to be modelled by constitutive equations. In the next two sections, we turn to the specific balances of mass and momentum.

### 1.2.1 Balance of mass and incompressibility

We can assume that there are no sources or sinks of mass in a material, i.e. $f = 0$. The rate of change of mass $m$ on a volume $V$ is given by the rate with which mass is being transported out of $V$ through the boundary $\partial V$. The flux $j$ of mass is proportional to the density $\rho$ of the material and its velocity $v$ at the boundary, so that the balance is

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho \, \mathrm{dV} = -\int_{\partial V} \rho v \cdot n \, d\sigma.$$

This gives the *continuity equation*

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0 \tag{1.1}$$

in $\rho$.

In these notes, we will always assume that the Newtonian or non-Newtonian fluid under consideration is incompressible, so $\frac{\partial \rho}{\partial t} = 0$, and that $\rho$ is moreover constant, so that (1.1) reduces to

$$\operatorname{div} v = 0. \tag{1.2}$$

### 1.2.2 Material derivative and balance of momentum

If we assume constant mass $m$, a formulation of the balance of momentum is Newton's second law $m\frac{dv}{dt} = F$ or

$$\frac{d}{dt}\int_{V(t)} \rho v \,\mathrm{d}V = F_{V(t)}.$$

The term $F_{V(t)} = F_b + F_c$ includes two types of forces, $F_b$ and $F_c$, which may act on the volume $V(t)$.

*Body forces* $F_b$ are external forces. They arise from an external field like gravity or an electromagnetic field and can be written in terms of a density $f$ on the material, i.e. $F_b = \int_{V(t)} \rho f \,\mathrm{dV}$.

*Contact forces* $F_c$ act on the boundary surface $\partial V(t)$ from the surrounding continuum itself. They depend on the position of the surface given by its outer normal $n$, $F_c = F_c(n)$. *Cauchy's Theorem* states that this dependence can only be linear and that we can write $F_c(t, x) = T(t, x)n(x)$ for some matrix or tensor $T$, which is called the stress tensor. It is not easy to give a precise proof of this theorem, even though it seems somehow reasonable. For a survey and new proof we refer to [2].

In conclusion, the balance of momentum in integral form is given by

$$\frac{d}{dt}\int_{V(t)} \rho v \,\mathrm{d}V = \int_{\partial V(t)} Tn \,\mathrm{d}\sigma + \int_{V(t)} \rho f \,\mathrm{d}V.$$

We can calculate the derivative of the integral on the left hand side (see also the *Reynolds Transport Theorem*),

$$\frac{d}{dt}\int_{V(t)} \rho v \,\mathrm{d}V \quad = \quad \int_{V(t)} \rho \frac{Dv}{Dt} \,\mathrm{d}V$$

where $\frac{Dv}{Dt}$ denotes the *material derivative* of $v$,

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + (v \cdot \nabla)v.$$

The corresponding transport equation is also called the *Cauchy equation*. It is given by

$$\rho \frac{Dv}{Dt} = \operatorname{div} T + \rho f, \tag{1.3}$$

where $(\operatorname{div} T)_i = \sum_{j=1}^n \partial_j T_{ij}$ denotes the divergence of the $i$th row of $T$.

A principal aspect of the remainder of the notes is to determine the relation between $T$ and $v$ by constitutive equations for particular types of fluids, which will determine the model. The next brief section is concerned with the basic linear constitutive equations for Newtonian fluids, which yield the Navier-Stokes equations.

## 1.3 Constitutive equations for Newtonian fluids

In all fluid models we consider,

$$T = S - p\mathrm{Id} \tag{1.4}$$

will be devided into a *deviatoric part* $S$ and a *spherical part* $p\mathrm{Id}$, where $p$ is called the *pressure*. If the fluid were compressible, additional constitutive relations of $p$ to density or temperature would be needed. In incompressible fluids, it is the mechanical pressure which arises as part of the solution, due to the incompressibility constraint. In other words, the stress includes a part which does not result in a motion which satisfies the solenoidal condition.

We only give some ideas of how to derive $S$.

1. Relative rotation of the fluid should not affect its internal stress, i.e. the stress acting on a control volume from the surrounding medium. In the case of relative rotation, there is no internal exchange of momentum. This property is called *material frame indifference*. Therefore, $S$ only depends on the symmetric part of the gradient of $v$, the *deformation tensor* $D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T)$. It does not depend on the rotation $W(v) = \frac{1}{2}(\nabla v - (\nabla v)^T)$ and it is invariant under rigid motions in the sense that $T(QDQ^T) = QT(D)Q^T$ for every orthogonal matrix $Q$.

2. From the balance of angular momentum, it can be seen that $T$ and thus $S$ should be symmetric, like $D$.

3. It is assumed that the fluid is *homogeneous*, so that its basic properties do not depend on time or position, i.e. $S$ does not depend explicitly on $t$ or $x$.

The above assumptions 1,2,3 reduce the possible relations of $S$ and $v$ to that of a *Stokes fluid*,

$$S = \alpha\mathrm{Id} + \beta D(v) + \gamma D^2(v), \tag{1.5}$$

where $\alpha, \beta, \gamma$ only depend on the first three invariants of $D(v)$, $\mathrm{tr}(D), D : D$ and $\det(D)$, see e.g. [8, Section 3.1] or [6]. Note that the incompressibility condition implies that $\mathrm{tr}(D(v)) = \mathrm{div}\, v = 0$.

For Newtonian fluids, it is assumed that this relation is linear. In (1.5), this means that $\gamma = 0$ and that $\beta$ is constant, whereas $\alpha$ can be absorbed into the pressure, so that

$$S = \eta D(v), \tag{1.6}$$

for some constant $\eta \geq 0$. The coefficient $\eta$ determining this linear dependence is called the *viscosity* of the fluid. We can think of $\eta$ as a "diffusion coefficient" for momentum, roughly measuring how strongly fluid particles interact with each other.

In the non-Newtonian models we consider, the incompressibility condition and the decomposition of $T$ as in (1.4) will be preserved, as well as assumption 1. The model (1.5) also applies to generalized Newtonian fluids (cf. the second part of the notes). For the other models in the notes, 2 and 3 may be false.

### 1.3.1 Dirichlet boundary conditions

If we consider fluid flow in some domain $\Omega$, boundary conditions have to be prescribed on $\partial\Omega$. We always use Dirichlet boundary conditions, i.e.

$$v|_{\partial\Omega} = 0, \tag{1.7}$$

if the fluid domain $\Omega$ is not moving. This condition is well-established for Newtonian fluids in contact with solid walls, but it may be in dispute for non-Newtonian fluids, cf. e.g. [5]. The mathematical literature quoted in the notes mainly adopts this condition, for all types of non-Newtonian fluids considered.

Dirichlet boundary conditions are based on two assumptions: There is no outflow or flow through the wall, i.e. $v \cdot n|_{\partial\Omega} = 0$, and secondly, there is a strong exchange of momentum between fluid and wall, i.e. the fluid sticks to the wall and no *slip* occurs.

### 1.3.2 Navier-Stokes equations

Putting (1.2), (1.3), (1.4), (1.6) and (1.7) together yields the Navier-Stokes equations

$$\left\{\begin{array}{rcll} \rho(\frac{\partial v}{\partial t} + (v\cdot\nabla)v) - \eta\Delta v + \nabla p &=& f, & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} v &=& 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ v|_{\partial\Omega} &=& 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ v|_{t=0} &=& a, & \text{on } \Omega, \end{array}\right. \tag{1.8}$$

for the flow of a Newtonian fluid in a domain $\Omega$, where $a$ is a suitable initial value for $v$. Note that we used $\operatorname{div}(\eta D(v)) = \eta\Delta v$, which follows from the incompressibility condition.

### 1.3.3 Reynolds number and the Stokes equations

We scale the NSE to a characteristic length $L$ and velocity $U$ to get a formulation in dimensionless form. The constants $L, U$ can be chosen arbitrarily, but the idea is to use numbers which fit the situation under consideration.

They yield a time scale $T = \frac{L}{U}$ and new arguments $\tau = \frac{t}{T}$ and $y = \frac{x}{L}$. We set $u(\tau, y) = \frac{v(\tau T, Ly)}{U}$ and $q(\tau, y) = \rho U^2 p(\tau T, Ly)$. We plug the derivatives of $v, p$ expressed in terms of $u, q$ into (1.8) and multiply the first line by $\frac{L}{\rho U^2}$ to get

$$\left\{\begin{array}{rcll} \frac{\partial u}{\partial t} + (u\cdot\nabla)u - \frac{1}{\mathrm{Re}}\Delta u + \nabla q &=& f, & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u &=& 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{\partial\Omega} &=& 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u|_{t=0} &=& a, & \text{on } \Omega, \end{array}\right. \tag{1.9}$$

as the new form of the equations, where the constant

$$\mathrm{Re} = \frac{\rho L U}{\eta} > 0$$

is called the *Reynolds number.* The characteristics of the flow are thus reduced to this one parameter. If Re is very small, $\frac{1}{\mathrm{Re}}$ is very large so that the influence of the non-linear term $(u \cdot \nabla)u$ may be neglected with respect to $\frac{1}{\mathrm{Re}}\Delta u$. This yields the idealization of (1.9) by the linear *Stokes equations,*

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + \nabla q &= f, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u &= 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{\partial\Omega} &= 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u|_{t=0} &= a, \quad \text{on } \Omega. \end{cases} \tag{1.10}$$

### 1.3.4 Strong solutions for the Stokes equations

Very many things can be said about the analysis of the NSEs, but this would somehow extend the seminar... A this point, we only cite one result on the existence of *strong solutions* for the Stokes equations and the corresponding estimate, which gives us a background for comparison in the later parts and which will be used again. The proof is due to Solonnikov [9].

**Theorem 1.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a domain of class $C^2$ with compact boundary and $1 < p, q < \infty$, $0 < T < T_0$, $f \in L^p(0,T; L^q_\sigma(\Omega))$ and $a \in Z_{p,q} := (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$. Then there exists a unique solution*

$$\begin{aligned} u &\in L^p(0,T; W^{2,q}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^q_\sigma(\Omega)) \cap W^{1,p}(0,T; L^q_\sigma(\Omega)) =: X^T_{p,q,\sigma}, \\ p &\in L^p(0,T; \widehat{W}^{1,q}(\Omega)) =: Y^T_{p,q}, \end{aligned}$$

*to the inhomogeneous Stokes problem* (1.10) *in* $(0,T) \times \Omega$ *and there exists a constant* $C > 0$ *independent of* $T, a$ *and* $f$, *such that*

$$\|u\|_{X^T_{p,q,\sigma}} + \|p\|_{Y^T_{p,q}} \leq C(\|f\|_{p,q} + \|a\|_{Z_{p,q}}). \tag{1.11}$$

The function spaces and notation we use is standard. In particular, $p$ may be the pressure or the integrability exponent. The space $L^q_\sigma(\Omega) := \overline{C^\infty_{c,\sigma}(\Omega)}^{\|\cdot\|_q}$ contains the solenoidal vector fields, where $C^\infty_{c,\sigma}(\Omega)$ denotes the space of divergence-free $C^\infty$-functions with compact support. The notation $\widehat{W}^{m,q}(\Omega)$ is used for the homogeneous Sobolev spaces of equivalence classes of functions in $D^{m,q} := \{f \in L^1_{\mathrm{loc}}(\Omega) : \partial^\alpha f \in L^q(\Omega), |\alpha| = m\}$ with respect to the polynomials of degree $m-1$. For $0 < \theta < 1$ and $1 \leq p \leq \infty$ we denote the real interpolation spaces of $X, Y$ by $(X,Y)_{\theta,p}$ if $X, Y$ form an interpoation couple.

# 2 A short motivation for considering non-Newtonian fluids

There are many difficulties in modelling non-Newtonian flows - for example, measurements may be elaborate and imprecise and there is a great diversity of non-Newtonian fluid behaviour. There is definitely not one model to fit them all. For one fluid, different models may apply for different scales or it may exhibit a threshold kind of behaviour, as for example Bingham fluids, which behave more like solids and start flowing only after a particular amount of stress is applied.

Mathematical results on the models are very new, maybe mostly less than 30 years old, and there are not yet very many. In his book published in 2000, Renardy moreover wrote that they can be considered as results obtained only in the cases of *small perturbations* in some sense, cf. [7, p. 33].

In the notes, we will see some of the difficulties like the non-linear character of the equations or the problem of coupled parabolic and hyperbolic parts (cf. in particular the third part of the notes). The models are mainly concerned with polymer solutions, which show elastic behaviour due to the spring-like (cf. the fourth part of the notes) character of the polymers as well as viscous flow. These fluids are also called viscoelastic.

There are many tools for evaluating models. Basic assumptions can be disputed or ignored, as for example the principle of material frame indifference. Experiments and measurements help especially for special types of flow like simple flow in a pipe. In these cases, one can often find a solution to the model by simple calculations. Computational Rheology and testing models via simulations is also a very important field of research. But also, mathematical results on the well-posedness of the corresponding initial-value problems or on existence and stability of steady flows evaluate the models.

## 2.1 Examples

Even though it is difficult, in applications, it is important to understand non-Newtonian fluids as for example oils, plastics, paints, ... which appear in many industrial processes. Other materials like glaciers, foams, sand, etc. also exhibit a non-Newtonian flow.

Many examples of non-Newtonian fluids and their good or bad properties are known from household applications, like toothpaste, which becomes less viscous ($\eta$ decreases) as more stress is applied ($D(v)$ increases) or ketchup, which does not flow (stays in the bottle) until it has been sufficiently squeezed and stressed (then, it will be much less viscous than expected and flood the plate). These are very standard examples. Nice descriptions and explanations can be found on the internet[1].

Examples of Newtonian fluids are water and, in some cases, air. In contrast to their behaviour, typical non-Newtonian effects of viscoelastic, polymer fluids are for example rod-climbing, die swell and stable jets, cf. e.g. [7, Section 1.2] and the references therein or [1, Section 2.3]. They arise from *normal stresses* in the fluid, in particular, stresses which do not satisfy (1.5). When the fluid flows, the polymer molecules tend to align with the flow direction. This generates a tension force in this direction. If these effects can be predicted, fluids can be tampered with by adding polymer molecules to control their flow and improve their properties.

Another typical example of a non-Newtonian fluid is blood. The modelling and analysis of blood flow in the body is also a very active field of research. It includes not only a complicated fluid, but also flows in unusual domains and fluid-structure interaction problems. There will probably be not more on blood in the notes, but there are references in the literature, especially for Oldroyd-B type fluids. For example, the new volume [3] surveys related mathematical problems and results.

## 2.2 Related Questions

- It would be great if typical non-Newtonian effects (see 2.1 above) could be shown (pictures or actual experiments!) on the seminar day. Maybe somebody wants to do this as part of the presentation...? In any case, I strongly recommend looking at the pictures in the literature!
- I think it is interesting to see what kind of predictions a model makes for simple flow in a pipe (for example: Eulerian flow vs. Stokes flow). I only know some results for the fluid models we consider and I am not sure whether the question can make sense for the dilute polymer bead-spring models, but we could discuss this if somebody is interested. There is more literature available.
- If anything regarding the notes is unclear, if there are mistakes or if there are questions, please contact me!

[1] I hope it's OK to say this: I tried wikipedia and google+video+non-Newtonian fluid to get nice results

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