

6 Viscoelastic fluid models

Polymer solutions and other fluids are called viscoelastic, because they exhibit two types of behaviour. On the one hand, they are viscous and flow like a “usual” Newtonian fluid. On the other hand, they can store or release energy, like an elastic solid. This can e.g. be seen very nicely in the die swell effect, cf. [18, pages 2 and 5]. This property is attributed to the polymers, which can change their structure to store energy and to release it again at a later time, when the flow changes. In modelling, this can be interpreted as the constitutive equations having to have some memory of the flow. In particular, the stress tensor at a given time t should not only depend on $D(v(t))$, but on earlier instances $D(v(s))$, $s < t$ also. Two ideas of how to implement memory are given in the next section.

The approach to viscoelastic fluid models is somehow less rigorous than defining generalized Newtonian fluids. In the latter case, one could judge by a few general principles and some experimental facts, that (3.2) is really a good generalization of the Newtonian relation (1.6), under the assumption that the stress should not depend on time or space explicitly. Deriving and evaluating viscoelastic models is even much more involved, because the few restrictions that apply a priori do not really seem to limit the possibilities. Roughly speaking, the approach seems to be the following: Choose an integral or a differential model for including memory (see the next section), then tamper, so that it satisfies material frame indifference, see Section 6.2. Then, additional terms and material parameters can be included, to fit the fluid at hand. Additional criteria on the models are, which predictions does the model make for simple flow, compared to experiments, and whether the corresponding system of equations is well-posed or yields otherwise good results.¹

Again, I do not have a sufficient overview to present a coherent picture of the literature on analysis of viscoelastic fluids. There is an overview in [18]. In Chapter 7, some results are cited or discussed. The tools in analysis are in general very different from the ones used for generalized Newtonian flow, but there is a subsection on the paper by Fernandez-Cara, Guillen and Ortega [8], where maximal regularity estimates for the Stokes problem, cf. Theorem 1 in Section 1.3.4, are used together with a Schauder fixed point argument.

6.1 Linear models: how to include memory

There are basically two main ideas of how to include memory, attributed to Boltzmann [4] and Maxwell [13]. The following summary is based on [18, Chapter 2].

¹This is a simplifying and sloppy explanation. There are very precise ways of deriving (aspects of) the models. In [20, p. 25-34], some ideas like the concept of *simple fluids* and general principles in their modelling and the modelling of memory or fading memory are explained, see also [15] and [22].

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Boltzmann's model is an integral equation for S , assuming that it depends linearly on $D(v)$, but not linearly on $D(v(t, x))$,

$$S(t, x) = 2 \int_{-\infty}^t G(t-s) D(v(t, x)) \, ds. \quad (6.1)$$

Here, G is called the *stress relaxation modulus* and it should be positive and monotone. Its derivative $-G'$ is called the *memory function*. The Newtonian model can be recovered if G is a multiple of the delta function. The *viscosity* of a Boltzmann fluid is the integral

$$\eta = \int_0^{\infty} G(s) \, ds, \quad (6.2)$$

see [18, p. 14].

On the other hand, Maxwell's model says that T is given by a differential equation

$$\partial_t S + \lambda S = 2\mu D(v). \quad (6.3)$$

This ODE can be integrated to give an equation of the form (6.1) for S , where the stress relaxation modulus is $G(s) = \mu \exp(-\lambda s)$. In the linear case, Maxwell is thus a special instance of Boltzmann and the relation of S to $D(v)$ is also quasi-linear.

6.1.1 The Weissenberg number

The constant $\frac{1}{\lambda}$, derived from λ in (6.3) is called the *relaxation time*, and it roughly measures for how long a fluid will remember. It is important to see how this number relates to the time scale of the flow. This is encoded in the *Weissenberg number*

$$\text{We} = \frac{\lambda}{T}, \quad (6.4)$$

where $T = \frac{L}{U}$ is a characteristic time scale, given by an appropriate scale for length L and for the velocity, U , cf. the scaling for the Reynolds number in Section 1.33. The bigger the Weissenberg number, the more the fluid will behave like an elastic solid, the smaller it gets, the more it will be like viscous Newtonian flow.

6.2 Non-linear models, examples

In the following, we focus more on the Maxwell approach, because it is more prominent in the mathematical literature. We have seen that it is a special case of Boltzmann. It is also more difficult to find "correct" integral models. For example, the K-BKZ model is a widely used nonlinear generalization of (6.1), but it also does not fit experiments in some respects, cf. [18, p. 17] and the references therein.

The problem with the linear models from the last section is that they violate material frame indifference, in particular, the operator $\partial_t S$ in the Maxwell model (6.3) is not *objective*, cf Chapter 3. To overcome this problem, it is replaced by an objective derivative, e.g. the *upper convected derivative*

$$S^\nabla := \frac{DS}{Dt} - (\nabla v)S - S(\nabla v)^T,$$

the *lower convected derivative*

$$S^\Delta := \frac{DS}{Dt} + S(\nabla v) + (\nabla v)^T S,$$

or the *co-rotational derivative*

$$S^\circ := \frac{1}{2}(S^\nabla + S^\Delta),$$

cf. [20, p. 35], or a linear superposition of any of them. Note that $\frac{DS}{Dt}$ denotes the material derivative of S , cf. Section 1.2.2, given by $\frac{DS}{Dt} = \partial_t S + (u \cdot \nabla)S$.

An example of this is the non-linear *Johnson-Segalman model*, which superposes S^∇ , S^Δ and a constant viscosity, Newtonian stress part $2\eta_2 D(v)$. We define

$$S_\zeta^\diamond := (1 - \frac{\zeta}{2})S^\nabla + \frac{\zeta}{2}S^\Delta.$$

The full stress is then given by the equation

$$S + \lambda S_\zeta^\diamond = 2(\eta_1 + \eta_2)(D(v) + \lambda \frac{\eta_2}{\eta_1 + \eta_2} (D(v))_\zeta^\diamond), \quad (6.5)$$

cf. [20, p. 35]. It really looks more complicated than it is! The four parameters $\frac{1}{\lambda}$, for relaxation time, ζ , for favouring upper or lower convected derivative and η_1, η_2 for “viscosities” and balancing Newtonian and viscoelastic contributions can now be chosen to model a given fluid.

Special cases of the Johnson-Segalman model where $\eta_2 = 0$ and $\zeta = 0$ or $\zeta = 2$ are the *lower convected Maxwell model*

$$S + \lambda S^\Delta = 2\eta D(v)$$

and the *upper convected Maxwell model*

$$S + \lambda S^\nabla = 2\eta D(v). \quad (6.6)$$

The reason for adding a Newtonian, constant viscosity stress part in (6.5) is that convected Maxwell models overpredict stresses if the deformation $D(v)$ is large. One more special case of the Johnson-Segalman model ($\zeta = 2$) which also includes this correction is the *Oldroyd-B model*,

$$S + \lambda S^\nabla = 2(\eta_1 + \eta_2)(D(v) + \lambda \frac{\eta_2}{\eta_1 + \eta_2} (D(v))^\nabla)$$

Instead of the superposition with $2\eta_2 D(v)$, the upper convected Maxwell model (6.6) can also be corrected by adding objective non-linear terms like κT^2 for the *Giesekus model*, or $\kappa(\text{tr } T)T$ for the *Phan-Thien-Tanner (PTT) model*, or the function

$$\mu_0(\text{tr } S)D(v) + \mu_1(\text{tr } SD(v))\text{Id} + \mu_2 D^2(v) + \mu_3(\text{tr}^2(D(v)))\text{Id}$$

for the *Oldroyd 8-constants model*, where the μ_i are constants, cf. the paper by Oldroyd, [17].

6.2.1 About simple shear flow and normal stress differences

For much more background on this short subsection, we refer to [18, Chapter 3] and [20, Section 7].

In order to evaluate, understand or construct non-Newtonian fluid models, it is good to look at simple flows, where the velocity of the fluid is known, so that the stresses predicted by the model can be calculated directly. Measurements mostly refer to simple flows.

For example, we consider simple shear flow, where the velocity “points” in x -direction, depends only on the y -component and is constant in time, $v = (V(y), 0, 0)$. The velocity gradient is

$$\nabla v = \begin{pmatrix} 0 & V'(y) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $V'(y) = \dot{\gamma}$ is also the shear rate, cf. Section 3.1. By isotropy, in simple shear flow, the stress tensor has to be of the form

$$S = \begin{pmatrix} S_{11}(\dot{\gamma}) & S_{12}(\dot{\gamma}) & 0 \\ S_{12}(\dot{\gamma}) & S_{22}(\dot{\gamma}) & 0 \\ 0 & 0 & S_{33}(\dot{\gamma}) \end{pmatrix},$$

cf. [18, p. 23] for the derivation of this relation. Arbitrary constants can be subtracted from the diagonal of S to go into the pressure term, so only the differences of the diagonal entries characterize the flow. There are thus only three functions to consider, the *viscosity*

$$\eta(\dot{\gamma}) = \frac{S_{12}(\dot{\gamma})}{\dot{\gamma}},$$

and the *first* and *second normal stress differences*

$$N_1 = S_{11} - S_{22} \quad \text{and} \quad N_2 = S_{22} - S_{33}.$$

These functions are called *viscometric functions* and they determine the simple shear flow, or more generally, every simple *viscometric* flow of fluids. In Newtonian fluids, η is constant and $N_1 = N_2 = 0$. In generalized Newtonian fluids, it also holds that $N_1 = N_2 = 0$ and for this reason it is assumed that the typical viscoelastic effects cannot be explained by this model.

For the Johnson-Segalman model (6.5), it can be calculated that

$$\begin{aligned} \eta(\dot{\gamma}) &= \dot{\gamma} \left(\frac{\eta_1}{1 + 2\zeta\lambda^2(1 - \zeta/2)\dot{\gamma}^2} + \eta_2 \right), \\ N_1(\dot{\gamma}) &= \frac{2\eta_1\lambda\dot{\gamma}^2}{1 + 2\zeta\lambda^2(1 - \zeta/2)\dot{\gamma}^2}, \\ N_2(\dot{\gamma}) &= -\frac{\zeta\eta_1\lambda\dot{\gamma}^2}{1 + 2\zeta\lambda^2(1 - \zeta/2)\dot{\gamma}^2}, \end{aligned}$$

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see [20, p. 40]. At this reference, there is also a table for viscometric functions of different viscoelastic fluid models. For the calculations of these functions for several models, we refer to [18].

There is an interesting point concerning the viscosity. Before, I wondered how shear-thinning can be read off the viscoelastic models, when it is so clear for the generalized Newtonian models. Here, it can for example be seen that Johnson-Segalman is (strongly) shear-thinning for $\zeta \in (0, 2)$, but not in a power-law type sense. For Oldroyd-B, $\zeta = 0$, so that the viscosity is constant. Also for the linear model, in (6.2) above, the viscosity of a linear Maxwell fluid was defined as the constant $\int_0^\infty \mu \exp(-\lambda s) ds = \frac{\mu}{\lambda}$.

Remark 1. In the Johnson-Segalman model (6.5), the Newtonian “correction” can be replaced by a generalized Newtonian one, i.e. $\eta_2 = \eta_2(|D(v)|_2^2)$. This would not destroy objectivity and introduce different possibilities of shear-thinning and maybe -thickening. There is also analysis literature on *generalized Oldroyd fluids*, cf. e.g. [2], [7], [1].

7 Literature in analysis

7.1 The paper by Fernandez-Cara/Guillen/Ortega

In this section we look at a paper by Fernandez-Cara, Guillen and Ortega [8] where local-in-time existence of Oldroyd-B type fluid flow is shown. More generally, a Johnson-Segalman model is considered: The model below including the constants a , λ_1, λ_2 , η and $\alpha = 1 - \frac{\lambda_1}{\lambda_2}$ is of the form (6.5) if we put $\zeta = 1 - a$, $\lambda = \lambda_1$, $\eta_1 = \eta\alpha$ and $\eta_2 = \frac{\lambda_1}{\lambda_2}\eta$. Thus, there is a Newtonian stress part $\tau_N := 2(1 - \alpha)\eta D(v) = 2\eta_2$ and a Maxwell-type stress given by the transport equation

$$\text{We}(\partial_t \tau + (v \cdot \nabla) \tau + \frac{1}{2}(a+1)\tau^\nabla + \frac{1}{2}(1-a)\tau^\Delta) + \tau = 2\alpha D(v), \quad (7.1)$$

where $\text{We} = \frac{\lambda_1}{T}$ is the corresponding Weissenberg number. In the following, we write $g_\alpha(\nabla v, \tau) := (a+1)\tau^\nabla + \frac{1}{2}(1-a)\tau^\Delta$. Together, τ_N and τ give the full stress tensor $S = \tau_N + \tau$ of the fluid. The equations governing the fluid flow are

$$\left\{ \begin{array}{ll} \text{Re}(\frac{\partial v}{\partial t} + (u \cdot \nabla)u) - (1-\alpha)\Delta v + \nabla q = \text{div } \tau + f, & \text{in } \mathbb{R}_+ \times \Omega, \\ \text{div } v = 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ v|_{\partial\Omega} = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ v|_{t=0} = v_0, & \text{on } \Omega, \end{array} \right. \quad (7.2)$$

combined with (7.1) and an initial condition $\tau(0) = \tau_0$.

The following is the main result in [8], on the existence of local-in-time strong solutions.

Theorem 2. *Let Ω be a bounded domain in \mathbb{R}^3 with boundary of class $C^{2,1}$. Let $1 < p < \infty$ and $3 < q < \infty$ and $T > 0$. If $v_0 \in Z_{p,q}$ (cf. Theorem 1 in 1.3.4), $\tau_0 \in W^{1,q}(\Omega)$, $f \in L^p(0, T; L^q(\Omega))$, then there exists a maximal time $T_* \in (0, T]$ and a unique strong solution of (7.1),(7.2) satisfying*

$$\begin{aligned} v &\in X_{p,q}^{T_*}(\Omega), \\ q &\in L^p(0, T; \widehat{W}^{1,q}(\Omega)), \\ \tau &\in W^{1,p}(0, T_*; L^q(\Omega)) \cap C(0, T_*; W^{1,q}(\Omega)) =: V_{p,q}^{T_*}(\Omega). \end{aligned}$$

The idea for the proof is to solve the coupled problem by a fixed point argument. The estimates on the Stokes problem from Theorem 1 in Section 1.3.4 are crucial, as they give the solution of the linear part of (7.2) for fixed τ on the right hand side. In addition, estimates on the transport equation (7.1) for fixed v are needed.

Lemma 3. *If Ω, p, q, T, τ_0 are as in Theorem 2 and $v \in L^p(0, T; D(A_q))$, where $D(A_q) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L_\sigma^q(\Omega)$ is the domain of the Stokes operator, cf. Section 1.3.4, then there exists a function $\tau \in V_{p,q}^T(\Omega)$ which solves (7.1) and satisfies the estimates*

$$\begin{aligned} \|\tau\|_{L^\infty(0,T;W^{1,q}(\Omega))} + \frac{4\alpha}{CWe} &\leq (\|\tau_0\|_{W^{1,q}(\Omega)} + \frac{4\alpha}{CWe}) \exp(C_2\|v\|_{L^1(0,T;W^{2,q}(\Omega))}) =: \Lambda, \\ \|\tau'\|_{L^p(0,T;L^q(\Omega))} &\leq 2^{1-\frac{1}{s}} C\Lambda (\|v\|_{L^p(0,T;W^{1,q}(\Omega))} + \frac{T^{1/s}}{CWe}). \end{aligned}$$

Here, we do not prove the lemma, but only look at some parts of the proof. In particular, the following a priori estimate is used:

$$\frac{1}{q} \frac{d}{dt} (We\|\tau\|_{W^{1,p}(\Omega)}^q) + \|\tau\|_{W^{1,q}(\Omega)}^q \leq 4\alpha C\|v\|_{W^{2,q}(\Omega)} \|\tau\|_{W^{1,q}(\Omega)}^q + CWe\|u\|_{W^{2,q}(\Omega)} \|\tau\|_{W^{1,q}(\Omega)}^{q-1}. \quad (7.3)$$

From this estimate, the estimates in the lemma roughly follow from a Gronwall argument. The essential thing about this estimate is that it does not include the second space derivatives of τ , even though $\partial_t(\nabla\tau)$ is estimated. This follows from the fact that the transport of τ is done by a vector v with zero boundary conditions. It can be seen from the following calculations: We take the gradient of equation (7.1), multiply by $|\nabla\tau|^{q-2}\nabla\tau$ and integrate by parts. This gives

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} (We\|\nabla\tau\|_{L^q(\Omega)}^q) + \|\nabla\tau\|_{L^q(\Omega)}^q &= 2\alpha(\nabla D(v), |\tau|^{q-2}\tau) - We(\nabla g_\alpha(\nabla v, \tau), |\nabla\tau|^{q-2}\nabla\tau) \\ &\quad - We(\nabla((u \cdot \nabla)\tau), |\nabla\tau|^{q-2}\nabla\tau). \end{aligned}$$

Except for the last term on the right hand side, this fits into (7.3). We calculate this term, using that $\partial_k|\nabla\tau|^q = q \sum_{i,j,k} (\partial_k\partial_l\tau_{ij})(\partial_l\tau_{ij})|\nabla\tau|^{q-2}$,

$$\begin{aligned} \sum_{i,j,k,l} \int_{\Omega} \partial_l(u_k\partial_k\tau_{ij})(\partial_l\tau_{ij})|\nabla\tau|^{q-2} &= \sum_{i,j,k,l} \int_{\Omega} (\partial_l u_k)(\partial_k\tau_{ij})(\partial_l\tau_{ij})|\nabla\tau|^{q-2} \\ &\quad + \sum_k \frac{1}{q} \int_{\Omega} u_k(\partial_k|\nabla\tau|^q), \end{aligned}$$

where $\sum_k \frac{1}{q} \int_{\Omega} u_k(\partial_k|\nabla\tau|^q) = -\int_{\Omega} \operatorname{div} u |\nabla\tau|^q + \int_{\partial\Omega} u \cdot n |\nabla\tau|^q = 0$, so also this part of the equation can be estimated as in (7.3).

For the fixed point argument, a map Φ is constructed, taking $(\bar{v}, \bar{\tau}) \in X_{p,p}^T \times V_{p,q}^T$ to a solution v of (7.2) with right hand side $\operatorname{div} \bar{\tau}$ and $(\bar{v} \cdot \nabla)\bar{v}$ (so the left hand side becomes a Stokes problem) and to a solution τ of (7.1) with transport coefficient \bar{v} , right hand side $D(\bar{v})$ and including $g_\alpha(\nabla\bar{v}, \tau)$. By Theorem 1 in Section 1.3.4 and Lemma 3 it is shown that there exist $0 < R$ and $0 < T_* < T$, such that Φ continuously maps a ball $B_R^{T_*} \subset X_{p,q}^T \times V_{p,q}^T$ into itself. By Schauder's fixed point theorem, Φ has at least one fixed point, which is a solution of the Oldroyd problem. A different argument is used to show uniqueness of the solution, cf. [8, page 10].

7.2 A different approach by Renardy

In the analysis literature, many more papers are concerned with the Oldroyd-B or Segalman-Johnson model than with pure Maxwell-type models, which do not include a Newtonian linear “correction” of the stress. It is because maybe these are the better models, but also they could be easier to consider mathematically, because the Stokes part is well-understood.

An interpretation of the fixed point argument in the last section would be that the solution is just a perturbation of Newtonian flow, as its maximal time of existence depends on the parameter α , which expresses the strength of the Maxwellian part. I do not know wheter this is true or how to make this point precise. Maybe we can discuss about this in the seminar.

In a paper by Renardy [19], a local existence theorem for strong solutions for very general Maxwell-type models is shown. In this case, the equations in the fluid velocity really are hypberbolic. There is a very concise presentation of this paper in Renardy’s book, [18, p. 35-37]. Here, there is only a short sketch of his sketch, omitting details on regularity and many arguments.

The equations are the following. For the fluid, there is the balance of momentum equation,

$$\rho(\partial_t v + (v \cdot \nabla)v) = \operatorname{div} S - \nabla q + f \quad (7.4)$$

together with the incompressibility condition, an initial condition and a homogeneous Dirichlet boundary condition. For S , the constitutive equation

$$(\partial_t + (v \cdot \nabla))S_{ij} = \sum_{k,l} A_{ijkl}(S)(\partial_l v_k) + g_{ij}(S) \quad (7.5)$$

is assumed, in particular generalizing the Maxwell models and their linear superpositions. The functions A_{ijkl} and g_{ij} are assumed to be very regular (C^4) and A_{ijkl} should be of the following form, due to the requirement of material frame indifference,

$$A_{ijkl}(S) = \frac{1}{2}(\delta_{ik}S_{lj} - \delta_{il}S_{kj} - S_{ik}\delta_{lj} + S_{il}\delta_{kj}) + B_{ijkl}(S),$$

where $B_{ijkl} = B_{klij} = B_{lkij} = B_{lkji}$ is symmetric. In addition, there is a strong ellipticity condition on A , in the form

$$\sum_{i,j,k,l} C_{ijkl}(S)\zeta_i\zeta_k\eta_j\eta_l \geq \kappa(S) > 0$$

for all $\zeta, \eta \in \mathbb{R}^3$, $|\zeta| = |\eta| = 1$, where $C_{ijkl} = A_{ijkl} - T_{il}\delta_{kj}$. It can be seen from the calculations for (7.6) below, why C has to be considered, instead of A .

Under these conditions, the existence of a unique local in time solution is shown. The proof is also based on a fixed point argument, but the iteration is different from the one in the last section, and it is only in the fluid velocity v . By applying

$$\partial_t + (v \cdot \nabla) + (\nabla v)^T$$

to (7.4) and using (7.5), the equation

$$\rho(\partial_t + (v \cdot \nabla))^2 v_i = -\partial_i p + \sum_{j,k,l} C_{ijkl}(S) \partial_j \partial_l v_k + h_i(v, \nabla v, \partial_t v, S, \nabla S, f, \nabla f, \partial_t f) \quad (7.6)$$

is obtained (for a modified pressure p). For given \bar{v} , the transport equation (7.5) is solved for \bar{S} . Plugging \bar{S} into (7.6) gives a new v , by a contractive map. The main difficulty is to solve (7.6). By strong ellipticity of C , it is hyperbolic, except for the pressure term and the incompressibility condition. Renardy explains that known techniques for semilinear hyperbolic equations apply, but “overcoming this technical difficulty requires a rather elaborate argument”, cf. [18, p. 37] (and cf. the Bothe/Prüss result, Section 4.1.1).

7.3 More literature

- In addition to the literature cited in the sections above, for well-posedness, existence of global solutions for small data and also of global weak solutions for large data, there are “classical” works by Guillopé and Saut, [9] and [10], and Lions and Masmoudi, [12] for global solutions. More recently, in [21], the problem is considered on unbounded domains, and there are more complete results in [6], [11] and [5]. Regarding strong solutions, it is important that for the hyperbolic Maxwellian models, development of shocks can be expected in finite time, so there are blow-up results if the Maxwellian part is large. The existence of global solutions for the Oldroyd problems rests on the Newtonian contribution, cf. e.g. [18, p.37].

For results on the steady flow problem, we refer e.g. to [16], [2] and [3].

In [14], it is shown that the Oldroyd flow converges to Newtonian flow for $We \rightarrow 0$.

I’m sorry, as always, the list is by far not complete, but maybe the references in the references will help!

7.4 Related question(s)

- Regarding the transport equation (7.1), I don’t really understand how to solve it in the classical setting, cf. [8, p. 26]. Maybe it is not a difficult argument, or at least standard theory, but it is of course crucial to understanding the overall result... Please let me know if you would like to present this on the seminar day!
- Is the solution in [8] really a perturbation of Newtonian flow? If yes, in what sense? The argument is made by Renardy, in [18, p.37].

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