Exercise 1: (Chebyshev polynomials)

The Chebyshev polynomials $T_n(x)$ ($n \in \mathbb{N}_0$) have a series of properties. Two general formulas are given by

$$T_n(x) = \cos(n \arccos(x)) \tag{1}$$

or

$$T_n(x) = \frac{1}{2}\Big(\big(x + \sqrt{x^2-1}\big)^n + \big(x - \sqrt{x^2-1}\big)^n\Big). \tag{2}$$

The $T_n(x)$ are polynomials with integer coefficients which satisfy the recurrence equation

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_n(x) = 2\,x\,T_{n-1}(x) - T_{n-2}(x), \tag{3}$$

from which one can see that only integer coefficients appear. Another identity of those polynomials is the equation

$$2T_n(x)T_m(x) = T_{n+m}(x) + T_{n-m}(x) \tag{4}$$

für $n, m \in \mathbb{N}_0$ und $n \geq m$.

(a) Program a function `T1`, which calculates T_n via (1). (Hint: Use `TrigExpand`).

(b) Program a function `T2`, which calculates T_n via (2). Which simplifications are necessary to get a representation as polynomial with integer coefficients?

(c) Program a function `T3`, which calculates T_n via (3) without remember effect.

(d) Program a function `T4`, which calculates T_n via (3) with remember effect.

(e) Program a function `T5`, which calculates T_n via (3) iteratively with a loop.

(f) Program a function `T6`, which calculates T_n via (3) using the function `Nest`.

(g) Program a function `T7`, which calculates T_n using (4) and the Divide-and-Conquer strategy (don't use `Expand`!).

(h) Generate with each of those functions and the internal function `ChebyshevT` the list $\{T_1(x), \ldots, T_{100}(x)\}$. What is interesting about the calculation times?

(i) With which procedure `T1`–`T7` is it possible, to calculate the polynomial $T_{1000000}$ (without showing the output) and to evaluate it at $x = 1$? Can you explain?

(12 points)