Solving Differential Equations in Terms of Bessel Functions

Masterarbeit bei
Professor Dr. Wolfram Koepf

am Fachbereich Mathematik der
Universität Kassel

vorgelegt von

Ruben Debeerst, geboren am 26. November 1982 in Lingen (Ems)

Kassel, Oktober 2007

Ich versichere hiermit, dass ich diese Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

RUBEN DEBEERST Kassel, Oktober 2007

Contents

List of Notations

This is a list of symbols with the first page on which they occur.

Introduction

Ordinary differential equations have always been of interest since they occur in many applications. Yet, there is no general algorithm solving every equation. There have been developed various methods for different classes of differential equations and functions. Apart from those there are methods using symmetry properties, the computation of integrating factors and there certainly are many more.

A special class of ordinary differential equations is the class of linear differential equations $Ly = 0$, for a linear differential operator

$$L = \sum_{i=0}^{n} a_i \partial^i$$

with coefficients in some differential field K, e.g. $K = \mathbb{Q}(x)$ and $\partial = \frac{d}{dx}$. The algebraic properties of those operators and their solutions spaces are studied very well, e.g. in [22].

Solutions that correspond to an order one right factor can always be found by Beke's algorithm but also with algorithms described in [8] and [18]. Each of those factors corresponds to a hyperexponential solution. This is a solution of the form $\exp(\int r)$ for a rational function r. In general, one can also factor L into factors of lower degree [23].

From this point on, one will have to consider special functions, which are functions defined by a differential operator. The question of solving an equating in terms of a special function is equivalent to the question whether two differential operators can be transformed into each other by certain transformations. We will consider a change of variables $x \to f$, exp-products $y \to \exp(\int r)y$ and gauge transformations $y \to r_0 y + r_1 y'$. The parameters f, r, r_0 and r_1 are rational functions.

There are several algorithms solving special cases of the transformation mentioned above. If one just allows exp-products and gauge transformations, the problem is also called the equivalence of differential operators, which is solved in [3]. The problem for a given rational function f is described in [5] and [25]. And finally, the algorithm in [7] is restricted to exp-products and Möbius transformations $f = \frac{\alpha x^k+\beta}{\gamma x^k+\delta}$ in the change of variables.

The approach we develop in this thesis will be restricted to Bessel functions but there will be no restrictions on the rational parameters. We will solve whether an operator can be obtained from the Bessel operator by a change of variables, and exp-product and a gauge transformation.

The idea for this algorithm is by Mark van Hoeij, teaching at Florida State University in Tallahassee, FL, USA. During my visit in Tallahassee from August to December 2006 we implemented the whole algorithm in Maple.

This thesis will explain all the main ideas and illustrate the algorithm by examples. However, there are still some more details in the algorithm which could not be described here because it would go beyond the scope of a master thesis. But these details are all speed-ups or Maple addicted aspects which are not necessary for the algorithm to work.

After introducing some preliminaries we will describe the transformations that we use in Chapter 2. We will introduce the exponent-difference which allows us to solve the change of variables apart from the other two transformations. Chapter 3 will describe the change of variables in the Bessel case and will also handle the constant parameter ν of the Bessel function. Furthermore, we will handle the algorithm case by case and give examples to each of the cases. We finally also show how we can apply the same algorithm to solve differential equations in terms of Whittaker functions.

The Maple source for the examples in this thesis can be downloaded from my website[1] or on the enclosed CD. There you will also find the Maple implementation of the algorithm.

[1] `http://www.mathematik.uni-kassel.de/~debeerst/master/`

1

Preliminaries

We will first introduce some facts about differential equations, their corresponding differential operators and singularities which are well known to many readers. After an overview over hypergeometric functions we will deal with formal solutions and generalized exponents. For the proofs of the statements given in this chapter we will refer to other sources.

1.1 Differential Operators

Definition 1.1 *Let K be a field. A derivation on K is a linear map $D : K \to K$ such that all $a, b \in K$ satisfy the product rule*

$$D(ab) = aD(b) + bD(a).$$

A field K with a derivation D is called differential field.

In our context we will consider functions in terms of the variable x with the "normal" derivation $\partial := \frac{d}{dx}$. If k is a field, then $K = k(x)$ with ∂ is a differential field. Another derivation that is often used is $\delta := x\partial$.

Definition 1.2 *Let K be a differential field with derivation ∂, then*

$$L = \sum_{i=0}^{n} a_i \partial^i, a_i \in K \tag{1.1}$$

is called differential operator. The coefficient $a_n \neq 0$ and n is the degree of L, denoted by $\deg(L)$. The leading coefficient of L refers to the coefficient a_n.

The set of differential operators with coefficients in K, denoted by $K[\partial]$, is a ring. The addition is canonical, i.e. $a\partial^i + b\partial^i = (a+b)\partial^i$, and the multiplication is defined by $\partial a = a\partial + a'$.

Remarks 1.3

1. In general there exists $a \in K$ with $a' \neq 0$ and $K[\partial]$ is not commutative.

2. The ring $K[\partial]$ is an euclidean ring. For two operators $L_1, L_2 \in K[\partial], L_2 \neq 0$ there are unique operators $Q, R \in K[\partial]$ such that $L_1 = QL_2 + R$ and $\deg R < \deg L_2$. This operation is called *left division*. If $R = 0$, then Q is called *left divisor* of L_1. Similarly there exists a *right division* on $K[\partial]$.

An euclidean ring is also a principal ideal ring. Hence, we can define the *least common left multiple* $\mathrm{LCLM}(L_1, L_2)$ as the unique monic generator of $K[\partial]L_1 \cap K[\partial]L_2$ and the *greatest common right divisor* $\mathrm{GCRD}(L_1, L_2)$ as the unique monic generator of $K[\partial]L_1 + K[\partial]L_2$. When working with right ideals we can similarly define the *least common right multiple* and the *greatest common left divisor*.

Note that every differential operator L corresponds to a homogeneous differential equation $Ly = 0$ and vice versa. We will always assume that $L \neq 0$.

Example 1.4
In Maple we can compute the corresponding equation with the following commands:

```
> with(DEtools):
> L:=D^2-x:
> eq:=diffop2de(L,[D,x],y(x));
```

$$eq := -xy(x) + \frac{d^2}{dx^2}y(x)$$

The second parameter `[D,x]` introduces the variable x and the variable `D` used for the derivation ∂. We can define these variables globally:

```
> _Envdiffopdomain:=[D,x]:
```

Then this parameter can always be omitted, e.g. when computing the corresponding operator:

```
> de2diffop(eq,y(x));
```

$$\partial^2 - x$$

From now on we will always assume that the `DEtools` package is loaded and that the differential domain `[D,x]` $= [\partial, x]$ is defined.

Definition 1.5 *By the solutions of L we mean the solutions of the homogeneous linear differential equation $Ly = 0$. They are denoted by $V(L)$.*

When talking about differential equations, the term *order* is commonly used for the degree of the corresponding operator.

Considering the solutions of $Ly = 0$ for $L \in K[\partial]$ a constant factor in K does not change the solution space $V(L)$. Thus, we can always work with monic operators.

A linear differential equation is commonly solved by transforming it into a matrix equation of order one.

Theorem 1.6 *The set $V(L)$ is a vector space of dimension* $\deg(L)$.

Proof. [15, Chapter 5]. □

We will later give an algebraic definition of $V(L)$.

Definition 1.7 *A set of* $\deg(L)$ *linearly independent solutions of L is called fundamental system of L.*

1.2 Singular Points

Let $y(x)$ be a function with values in $\mathbb{C}$.

Definition 1.8 (Solutions and singular points) *A function $y(x)$ is called*

(i) regular at $p \in \mathbb{C}$ if there exists a neighborhood U of p such that $y(x)$ is continuous on U,

(ii) regular at ∞ if $y(\frac{1}{x})$ is regular at 0.

A point p is called singular or a singularity if it is not regular.

Two equivalent terms *holomorphic* points and *analytic* points often occur with regular points.

A *holomorphic* point is a point, where y is differentiable in a open set around the point, an *analytic* point p is a point, where the function y can be represented as a *power series*

$$y(x) = \sum_{i=0}^{\infty} a_i (x-p)^i, \quad a_i \in \mathbb{C}.$$

The three terms arose in different contexts and therefore they are all still being used.

An important fact of the one-dimensional case is that singularities are always isolated. At those isolated singularities we need a Laurent series to represent $y(x)$ at a point p.

Theorem 1.9 *Let $y(x)$ be holomorphic around p such that p itself may be a singularity. Then $y(x)$ can be written as a Laurent series*

$$y(x) = \sum_{i=-\infty}^{\infty} a_i (x-a)^i, \quad a_i \in \mathbb{C}. \tag{1.2}$$

Proof. [9, Theorem 10.6.2]. □

The next definition will give a classification of singularities.

Definition 1.10 *Let $y(x)$ be of the form (1.2) and let p be a singularity of $y(x)$. Then p is called*

(i) a removable singularity if $a_i = 0$ for $i < 0$,
(ii) a pole if there exists $N \in \mathbb{N}$ such that $a_i = 0$ for $i \leq -N$, or
(iii) an essential singularity otherwise.

From this definition we see that an analytic point is almost equivalent to a regular point. Therefore, one often introduces *ordinary points* which include regular points and removable singularities. Then ordinary points and analytic points are equivalent. However, we will only use the fact that every regular point is analytic in the majority of cases and we will not confuse with another term.

For the rest of this section we define the fields $k = \mathbb{C}$ and $K = k(x) = \mathbb{C}(x)$. Furthermore, let $L \in K[\partial]$ be a differential operator of the form (1.1).

Definition 1.11 (Operators and singular points) *A point p is called a singular point of the operator $L \in K[\partial]$ if p is a zero of the leading coefficient of L or if p is a pole of one of the other coefficients. All other points are called regular.*

If $y(x)$ is a solution of the differential operator L, every singularity of $y(x)$ must be a singularity of L. But the converse is not true, i.e. a singularity of L can be a regular point of $y(x)$. At all regular points of L we can find a fundamental system of power series solutions.

In our context a regular or singular point will usually refer to a differential operator.

Definition 1.12 *Let $L = \sum_{i=0}^{n} a_i \partial^i \in K[\partial]$ with $a_n = 1$. A singularity p of L is called*

(i) apparent singularity if all solutions of L are regular at p,
(ii) regular singular if $(x-p)^i a_{n-i}$ is regular at p for $1 \leq i \leq n$, and
(iii) irregular singular otherwise.

If L has apparent singularities we can always remove these singularities from L, i.e. we can find another operator $L' \in K[\partial]$ with $V(L) \subset V(L')$ which has no apparent singularities (see appendix of [1]). However, the degree of L' can be higher than the degree of L.

Now let L be a differential operator of degree $\deg(L) = 2$. We can then find the following solutions.

Theorem 1.13 *Let* $L = \partial^2 + p(x)\partial + q(x) \in K[\partial]$.

(i) If L is regular at p, there exists a unique solution $y(x) = \sum_{i=0}^{\infty} a_i(x-p)^i$ *of L satisfying the initial conditions* $y(p) = c_0$ *and* $y'(p) = c_1$*, where* c_0 *and* c_1 *are arbitrary constants.*

(ii) If L is regular singular at p, there exist the two linearly independent solutions

$$y_1(x) = (x-p)^{e_1} \sum_{i=0}^{\infty} a_i(x-p)^i \tag{1.3}$$

$$\text{and} \quad y_2(x) = (x-p)^{e_2} \sum_{i=0}^{\infty} b_i(x-p)^i + c y_1(x) \ln(x-p), \tag{1.4}$$

where $e_1, e_2, a_i, b_i, c \in k$ *are constants and* $c = 0$ *if* $e_1 - e_2 \notin \mathbb{Z}$.

(iii) If p is an irregular singularity, two linearly independent solutions are

$$y_1(x) = (x-p)^{e_1} \sum_{i=-\infty}^{\infty} a_i(x-p)^i \tag{1.5}$$

$$\text{and} \quad y_2(x) = (x-p)^{e_2} \sum_{i=-\infty}^{\infty} b_i(x-p)^i + c y_1(x) \ln(x-p), \tag{1.6}$$

with constants $e_1, e_2, a_i, b_i, c \in k$ *and* $c = 0$ *if* $e_1 - e_2 \notin \mathbb{Z}$.

Proof. [24, Chapters 2.1-2.4]. □

Remarks 1.14

1. The constants e_1 and e_2 are called *exponents*, and in the regular singular case they can be determined solving the *indicial equation*

$$\lambda(\lambda - 1) + p_0\lambda + q_0 = 0$$

which can be obtained by taking the constant coefficients p_0 and q_0 from the power series expansion of $(x-p)p(x)$ and $(x-p)^2 q(x)$ at the point p, respectively.

2. When working with a differential operator L one will hardly work with $k = \mathbb{C}$. One would rather define k to be the smallest field extension of $\mathbb{Q}$ such that $L \in k(x)[\partial]$ and move on to bigger fields when it is required.

1.3 Hypergeometric Series

This section should give a short overview about hypergeometric series and hypergeometric functions, especially for differential equations of order two which we are most interested in. For details we refer to [24] and a lot of identities can also be found in [19].

Definition 1.15 *A generalized hypergeometric series ${}_pF_q$ is defined by*

$$ {}_pF_q\left(\begin{array}{c} \alpha_1, \alpha_2, \ldots \alpha_p \\ \beta_1, \beta_2, \ldots \beta_q \end{array} \middle| \, x \right) := \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k k!} x^k, \tag{1.7} $$

where $(\lambda)_k$ denotes the Pochammer symbol

$$ (\lambda)_k := \lambda \cdot (\lambda+1) \cdots (\lambda+k-1). $$

The function is also denoted as ${}_pF_q\left(\alpha_1, \alpha_2, \ldots \alpha_p; \beta_1, \beta_2, \ldots \beta_q; x\right)$.

Example 1.16
Many special functions can be written as a generalized hypergeometric series. Some well-known series are the exponential and trigonometric series

$$ \begin{aligned} \exp(x) &= \sum_{k=0}^{\infty} \frac{x^k}{k!} = {}_0F_0(x), \\ \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = {}_0F_1\left(\begin{array}{c} - \\ \frac{1}{2} \end{array} \middle| \, \frac{-x^2}{4} \right), \\ \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = x {}_0F_1\left(\begin{array}{c} - \\ \frac{3}{2} \end{array} \middle| \, \frac{-x^2}{4} \right) \end{aligned} $$

Furthermore, if one of the upper parameters is a negative integer, the series breaks into a polynomial. But we won't consider that case.

Theorem 1.17 *The generalized hypergeometric series ${}_pF_q$ defined in (1.7) satisfies the differential equation*

$$ \delta(\delta+\beta_1-1)\cdots(\delta+\beta_q+1)y(x) = x(\delta+\alpha_1)\cdots(\delta+\alpha_p)y(x) \tag{1.8} $$

where $\delta = x\frac{d}{dx}$.

Proof. This can easily be seen if we plug the series ${}_pF_q$ into (1.8) and equate coefficients. □

Remarks 1.18
1. For $p \leq q$ the series ${}_pF_q$ is convergent for all z. For $p > q+1$ the radius of convergence is zero, and for $p = q+1$ the series converges for $|z| < 1$.
2. For $p \leq q+1$ the series and its analytic continuation is called a *hypergeometric function*.
3. There are identities connecting several hypergeometric functions. A lot of these formulas can be found in [19] but they can contain typing errors. An algorithmic approach to check these identities is presented in [17].

1.3.1 Hypergeometric Differential Equation

We will now consider the more special case where the differential equation has order two. From Theorem 1.17 we know that we can have at most three parameters $\alpha = \alpha_1, \beta = \alpha_2$ and $\gamma = \beta_1$, which turns (1.8) into

$$x(1-x)\partial^2 y(x) + (\gamma - (\alpha+\beta+1)x)\partial y(x) - \alpha\beta y(x) = 0. \tag{1.9}$$

This equation is called the *hypergeometric differential equation*. It has regular singular points at $0, 1$ and ∞, and the solutions can all be expressed by ${}_2F_1$ functions which are also called *Gauss' hypergeometric functions*.

An important equation that appears in this context is the *Riemann P-equation*

$$y(x) = P\left\{\begin{matrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1 & , x \\ \alpha_2 & \beta_2 & \gamma_2 & \end{matrix}\right\}$$

which represents a solution of (1.9). The constants a, b and c are the three regular singularities and $\alpha_{1,2}, \beta_{1,2}$ and $\gamma_{1,2}$ are their corresponding exponents.

We also receive a second order differential equation in the cases ${}_1F_1$, ${}_2F_0$ and ${}_0F_1$. In those cases the equations we obtain have a regular singularity at 0 and an irregular singularity at ∞. These equations are also called *confluent hypergeometric equation* because they can be obtained from (1.9) by the confluence of two of its singularities. This creates an irregular singularity. The equation in the ${}_1F_1$-case is

$$x\frac{d^2}{dx^2}F(x) + (\gamma - x)\frac{d}{dx}F(x) - \alpha F(x) = 0 \tag{1.10}$$

and is called the *Kummer equation*. The solution $M(\alpha,\gamma,x) := {}_1F_1(\alpha;\gamma;x)$ is called *Kummer function*. A second independent solution is defined as

$$U(\alpha,\gamma,x) := \frac{\pi}{\sin(\pi\gamma)}\left(\frac{M(\alpha,\gamma,x)}{\Gamma(1+\alpha-\gamma)\Gamma(\gamma)} - x^{1-\gamma}\frac{M(1+\alpha-\gamma,2-\gamma,x)}{\Gamma(\alpha)\Gamma(2-\gamma)}\right),$$

where $\Gamma(x)$ denotes the *Gamma function*

$$\Gamma(x) := \int_0^\infty t^{x-1}\exp(-t)\,dt.$$

It is called the second Kummer function and satisfies the relation

$$U(\alpha,\gamma,z) = x^{-\alpha}{}_2F_0\left(\begin{matrix}\alpha, 1+\alpha-\gamma \\ -\end{matrix}\,\middle|\, -\frac{1}{x}\right).$$

The solutions in the case ${}_0F_1$ can also be expressed in terms of Kummer functions with the *Kummer formula*:

$$\exp\left(-\frac{x}{2}\right){}_1F_1\left(\begin{array}{c}\alpha\\2\alpha\end{array}\middle|\,x\right)={}_0F_1\left(\begin{array}{c}-\\\frac{1}{2}+\alpha\end{array}\middle|\,\frac{x^2}{16}\right).$$

Concluding, we can separate the hypergeometric functions that satisfy a second order differential equation, into two classes. The ${}_2F_1$ functions with three regular singularities and the Kummer functions with one regular and one irregular singularity.

The ${}_0F_1$-functions are a special case of the Kummer functions. They are close to the Bessel functions, which we will deal with in the following section.

1.3.2 Bessel Functions

Definition 1.19 *The solutions $V(L_{B1})$ of the operator*

$$L_{B1} := x^2\partial^2 + x\partial + (x^2 - \nu^2) \tag{1.11}$$

with the constant parameter $\nu \in \mathbb{C}$ are called Bessel functions. The two linearly independent solutions

$$J_\nu(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu} \tag{1.12}$$

$$\text{and}\quad Y_\nu(x) := \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{1.13}$$

generate $V(L_{B1})$ and they are called Bessel functions of first and second kind respectively.

Similarly the solutions of

$$L_{B2} := x^2\partial^2 + x\partial - (x^2 + \nu^2) \tag{1.14}$$

are called the modified Bessel functions of first and second kind and they are generated by

$$I_\nu(x) := \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k} \tag{1.15}$$

$$= \exp\left(\frac{-\nu\pi i}{2}\right) J_\nu\left(\exp\left(\frac{\pi i}{2}\right)x\right)$$

$$\text{and}\quad K_\nu(x) := \frac{\pi(I_{-\nu}(x) - I_\nu(x))}{2\sin(\nu\pi)} \tag{1.16}$$

Remarks 1.20

1. If $2\nu \notin \mathbb{Z}$, then $J_\nu(x)$ and $J_{-\nu}(x)$ also generate $V(L_{B1})$. This is proven in [24, 7.2].

2. In terms of hypergeometric functions, the Bessel functions of the first kind are given by

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} {}_0F_1\left(\begin{array}{c} - \\ \nu+1 \end{array} \middle| -\frac{1}{4}x^2\right) \tag{1.17}$$

$$\text{and} \quad I_\nu(x) = \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} {}_0F_1\left(\begin{array}{c} - \\ \nu+1 \end{array} \middle| \frac{1}{4}x^2\right). \tag{1.18}$$

Lemma 1.21 *Replacing x by ix where $i = \sqrt{-1}$ reduces L_{B2} to L_{B1} and vice versa.*

Proof. That this is true for Bessel functions of the first kind can easily be seen when comparing the hypergeometric representations (1.17) and (1.18). There we get $I_\nu(x) = i^\nu J_\nu(ix)$, so there only remains a constant coefficient i^ν. That the statement for the corresponding operators is also true can be seen as follows.

Let $y(x)$ be a solution of L_{B2} and consider $g = g(x) = y(ix)$, then

$$g'(x) = i\,\frac{d}{dx}y(x)\bigg|_{x=ix}$$

$$\text{and} \quad g''(x) = -\frac{d^2}{dx^2}y(x)\bigg|_{x=ix} = \frac{1}{(ix)^2}\left(ix\,\frac{d}{dx}y(x)\bigg|_{x=ix} - ((ix)^2+\nu^2)y(ix)\right).$$

A general differential operator for $g(x)$ is $L = \partial^2 + a_1\partial + a_0$. Using the equations for g' and g'' we can transform $Lg = 0$ into

$$\left(\frac{1}{ix} + ia_1\right)\frac{d}{dx}y(x)\bigg|_{x=ix} + \left(\frac{-1}{x^2}(x^2-\nu^2) + a + 0\right)y(ix) = 0.$$

If we equate coefficients we obtain $a_1 = 1/x$ and $a_0 = (x^2-\nu^2)/x^2$. Then $L = 1/x^2 L_{B1}$ and $g(x) \in V(L) = V(L_{B1})$. So $g(x)$ is a solution of L_{B1}.

The reverse works similarly. □

Since L_{B1} and L_{B2} are so closely connected we just need to consider one of the two in later issues. From now on L_B will refer to the modified Bessel operator L_{B2}. The modified Bessel operator is easier to handle in some cases, as we will see in Example 1.32.

There are various recurrence equations and relationships for the Bessel function and its derivative.

Lemma 1.22 *The Bessel functions satisfy*

$$J_{\nu+1}(x) = \frac{2\nu}{x}J_\nu(x) - J_{\nu-1}(x), \qquad J'_\nu(x) = \frac{\nu}{x}J_\nu(x) - J_{\nu+1}(x) \tag{1.19}$$

and similarly the modified Bessel functions satisfy

$$I_{\nu+1}(x) = I_{\nu-1}(x) - \frac{2\nu}{x}I_\nu(x), \qquad I'_\nu(x) = \frac{\nu}{x}I_\nu(x) + I_{\nu+1}(x). \tag{1.20}$$

Moreover, $Y_\nu(x)$ satisfies the same equations as $J_\nu(x)$ and $(-1)^\nu K_\nu(x)$ satisfies the same equations as $I_\nu(x)$.

Proof. [2, Equations 9.1.27 and 9.6.26]. □

The following result will be important to find solutions of differential operators.

Corollary 1.23 *Consider*

$$S := \mathbb{C}(x)B_\nu + \mathbb{C}(x)B'_\nu \tag{1.21}$$

where $B'_\nu = \frac{d}{dx}B_\nu$ and B_ν is a linear combination of either J_ν and Y_ν or I_ν and $(-1)^\nu K_\nu$. The space S is invariant under the substitution $\nu \to \nu+1$.

Proof. It follows from the last lemma that this is true for each Bessel function on its own. A linear combination of J_ν and Y_ν does no harm since they satisfy the same equations. The same holds for a linear combination of I_ν and $(-1)^\nu K_\nu$. □

1.4 Formal Solutions and Generalized Exponents

In this section we will study differential operators with power series coefficients in $K = \mathbb{C}((x))$. In this context the derivation that is usually used is $\delta = x\frac{d}{dx}$.

Definition 1.24 *A universal extension U of K is a minimal (simple) differential ring in which every operator $L \in K[\partial]$ has precisely* $\deg(L)$ $\mathbb{C}$*-linear independent solutions.*

Theorem 1.25 *The universal extension U of K is unique and has the form*

$$U = \overline{K}[\{x^a\}_{a\in M}, \{e(q)\}_{q\in\mathfrak{Q}}, l], \tag{1.22}$$

where $M \subset \mathbb{C}$ is such that $M \oplus \mathbb{Q} = \mathbb{C}$ and $\mathfrak{Q} := \cup_{m\geq 1} x^{-1/m}\mathbb{C}[[x^{-1/m}]]$. Here $\overline{K}$ denotes an algebraic closure of K and the following rules hold:

(i) *The only relations between the symbols are* $x^0 = 1, x^{a+b} = x^a x^b, e(0) = 1$ *and* $e(q_1 + q_2) = e(q_1)e(q_2)$.

(ii) *The differentiation is given by* $\delta x^a = ax^a, \delta e(q) = qe(q)$ *and* $\delta l = 1$.

Proof. To give a complete proof we would have to introduce to many details about differential rings. This is why we sketch the idea here only and refer to [22, Chapter 3.2] for more details. □

In order to get an intuitive idea of the structure above, we will use the following statement from [22]:

To get a differential ring where all equations $y' = Ay$ with a matrix A over K have a fundamental system it is sufficient that all equations

$$\delta y = ay, \quad a \in \overline{K} \tag{1.23}$$

and $\delta y = 1$ have a solution.

Hence, to give a structure of U, we need to classify order one differential equations of the form (1.23) and define a solution of each of them in U. Doing this, we need to take care about equations that have the same solution. The function $\exp(\int \frac{a}{x}dx)$ is a solution of (1.23). So two equations $\delta y = ay$ and $\delta y = by$ with $a, b \in \bar{k}$ have the same solutions if $\exp(\int \frac{a}{x}dx) = c\exp(\int \frac{b}{x}dx)$ for some constant $c \in \overline{K}$. This is exactly the case, if $b = a + \frac{\delta f}{f}$ for some $f \in \overline{K}, f \neq 0$.

Thus, if $I := \{\frac{\delta f}{f} \mid f \in \overline{K}, f \neq 0\}$, then $\overline{K}/I$ gives us a classification of all the equations $\delta y = ay$ with different solutions.

We know that the algebraic closure consists of series with fractional exponents, i.e. $\overline{K} = \overline{\mathbb{C}((x))} = \cup_{n\in\mathbb{N}}\mathbb{C}((x^{1/n}))$. The degree of the lowest monomial in $f \in \overline{K}$ and δf are the same. Thus, taking the quotient will give a series starting with a rational constant and we get the representation

$$I = \left\{ c + \sum_{k=1}^{\infty} c_k z^{k/n} \;\middle|\; c \in \mathbb{Q}, c_k \in \mathbb{C}, n \in \mathbb{N} \right\}.$$

Then, of course, the elements in $\overline{K}/I$ are series in $\overline{K}$ which have a non-rational constant term. In Theorem 1.25 this was denoted by $M \oplus \mathfrak{Q}$.

We conclude that it is sufficient if the universal extension U has solutions of (1.23) for all $a \in M$ and all $a \in \mathfrak{Q}$. These are the elements $\{x^a\}_{a\in M}$ and $\{e(q)\}_{q\in\mathfrak{Q}}$ respectively and finally, the symbol l is the solution to $\delta y = 1$.

The interpretation of the symbols in the universal extension is the following:

1. x^a is the function $\exp(a\ln(x))$,

2. l is the function $\ln(x)$, and

3. $e(q)$ is the function $\exp(\int \frac{q}{x}\,dx)$.

Note that this construction at the point $x = 0$ can also be performed at other points $x = p$ by replacing x with the *local parameter* t_p which is $t_p := x - p$ for a point $p \in \mathbb{C}$ and $t_p = \frac{1}{x}$ for $p = \infty$.

Remark 1.26 (Logarithmic solutions)
A solution whose formal representation in the universal extension U involves $l = \ln(x)$ is called a *logarithmic solution*.

An important fact about logarithmic solutions is that we get other solutions of the differential equation if we replace $\ln(x)$ by $\ln(x) + c$, where c is some constant. This is due to the fact that the derivation of $\ln(x) + c$ does not depend on c. So if we have a solution of the form $f(x)\ln(x)$ and replace $\ln(x)$ by $\ln(x) + 1$, we get another solution $f(x) + f(x)\ln(x)$. Since the difference will also be a solution we get the non-logarithmic solution $f(x)$.

If the degree of the operator is two, the highest power of a logarithm that can appear is one. Assume we have $\ln(x)^2$ in a solution, then sending $\ln(x)$ to $\ln(x)+1$ successively will give a solution involving $\ln(x)^2 + 2\ln(x) + 1$ and one involving $\ln(x)^2 + 4\ln(x) + 4$. These solutions are independent, so the dimension of the solution space $V(L)$ is at least 3. This can only happen if $\deg(L) \geq 3$.

A more detailed structure of the universal extension is given by the following lemma.

Lemma 1.27 *The universal extension U of K is a $K[\delta]$-module which can be written as a direct sum of $K[\delta]$-modules:*

$$U = \bigoplus_{q \in \mathcal{Q}} e(q)\overline{K}[\{x^a\}_{a \in \mathbb{C}/\mathbb{Q}}, l] \tag{1.24}$$

$$= \bigoplus_{q \in \mathcal{Q}} \bigoplus_{a \in \mathbb{C}/\left(\frac{1}{r_q}\mathbb{Z}\right)} e(q)x^a\mathbb{C}((x^{1/r_q}))[l], \tag{1.25}$$

where, in the latter equation, r_q is the ramification index of q, i.e. the smallest number such that $q \in \mathbb{C}[[x^{-1/r_q}]]$.

Proof. Equation (1.24) is proven in [22, Chapter 3.2] and for equation (1.25) we refer to [23, Chapter 2.8]. □

Definition 1.28 *Let $L \in K[\partial]$ and let p be a point with local parameter t_p. An element $e \in \mathbb{C}[[t_p^{-1/r}]], r \in \mathbb{N}$ is called a generalized exponent of L at the point p if* *there exists a formal solution of the form*

$$y(x) = \exp\left(\int \frac{e}{t_p}\,dt_p\right) S, \quad S \in \mathbb{C}((t_p^{1/r}))[\ln(t_p)], \tag{1.26}$$

where the constant term of the Puiseux series S, i.e. the coefficient of $t_p^0 \ln(t_p)^0$*, is non-zero. For a given solution this representation is unique.*

The set of generalized exponents at a point p is denoted by $\mathrm{gexp}(L, p)$.

Similarly, we call e a generalized exponent of the solution y at the point p if $y = y(x)$ *has the representation (1.26) for some* $S \in \mathbb{C}((t_p^{1/r}))[\ln(t_p)]$.

The series s in (1.26) is also called a *Puiseux series* with coefficients in $\mathbb{C}$.

Example 1.29

Generalized exponents can be computed in Maple with the command `gen_exp`, which belongs to the package `DEtools`. An explanation of the algorithm is given in [23].

The input is an operator L, a variable t to express the generalized exponent and a point at which we want to compute the generalized exponent. The output is a list of pairs $[g, eq]$ which each represent a generalized exponent at the given point. In this pair the equation eq describes the variable t which is used to express the generalized exponent g.

Let's take the Kummer operator:

```
> LK:=x*D^2+(nu-x)*D-mu:
```

```
> gen_exp(LK,t,x=infinity);
```

$$[[\mu, t = \frac{1}{x}], [-\frac{1}{t} - \mu + \nu, t = \frac{1}{x}]]$$

The equation $t = \frac{1}{x}$ indicates that t is the local parameter. The algorithm computes two generalized exponents μ and $-\frac{1}{t_\infty} - \mu + \nu$. They each belong to a local solution.

A second example is the following operator:

```
> L:=D^2-x:
```

```
> gen_exp(L,t,x=infinity);
```

$$[[\frac{1}{t^3} + \frac{1}{4}, t^2 = \frac{1}{x}]]$$

Here t is a square root of the local parameter. The pair in the output now represents two generalized exponents: $(\sqrt{t_\infty})^{-3} + \frac{1}{4}$ and $(-\sqrt{t_\infty})^{-3} + \frac{1}{4}$.

In both cases the point $x = \infty$ is an irregular singularity, since at least on of the generalized exponent is not a constant.

If generalized exponents are equal modulo $\frac{1}{r}\mathbb{Z}$, where r is the ramification index, the pairs are combined in the output of `gen_exp`. At a regular point this is always the case:

```
> gen_exp(L,t,x=1);
```

$$[[0,1,t=x-1]]$$

Since the decompositions in Lemma 1.27 are direct sums one can derive the following statement for the solution space $V(L)$.

Theorem 1.30 *Let $L \in K[\partial]$, $d = \deg(L)$ and let $r \in \mathbb{N}$. Suppose that the ramification indices of the generalized exponents divide r. Then there exists a basis $y_1, \ldots, y_n$ of $V(L)$ which satisfies the condition*

$$y_i = \exp\left(\int \frac{e_i}{x} dx\right) S_i \text{ for some } S_i \in \mathbb{C}((x^{1/r}))[\ln(x)] \tag{1.27}$$

where $e_1, \ldots, e_n \in \mathbb{C}[[x^{-1/r}]]$ are generalized exponents and the constant term of S_i is non-zero.

Proof. [23, Chapter 4.3.3, Theorem 5]. □

Remarks 1.31

1. Again, replacing x by t_p for some point p, results in a fundamental system of formal solutions at the point p.

2. For a given generalized exponent there is a unique solution of the form (1.26) if we require the constant term of the series in to be one.

3. The set of generalized exponents is unique modulo $\frac{1}{r}\mathbb{Z}$, where r is the ramification index (see [23]).

Furthermore, with the use of generalized exponents we also get such a representation in the irregular singular case, i.e. we do not need a Laurent series for the representation as we had in Theorem 1.13 (iii).

Example 1.32

1. Let's compute the generalized exponents for the Bessel functions. The differential operator we consider is

$$L = x^2\partial^2 + x\partial + (x^2 - \nu^2),$$

which has singularities at 0 and ∞. Therefore the generalized exponents at any point p other than 0 and ∞ are 0 and 1. For $p = 0$ we compute with Maple:

```
> L:=x^2*D^2+x*D+(x^2-nu^2):
```

```
> gen_exp(L,t,x=0);
```

$$[[\nu, t = x], [-\nu, t = x]]$$

Since ν is a constant the point $p = 0$ is a regular singularity. Furthermore, we can express the fundamental system of solutions at $p = 0$ in the form $x^\nu S_1$ and $x^{-\nu} S_2$, where $S_1, S_2 \in \mathbb{C}[[x]][\ln(x)]$.

For the point $p = \infty$ we do the same computation:

```
> gen_exp(L,t,x=infinity);
```

$$[[\frac{RootOf\left(1 + \_Z^2\right)}{t} + \frac{1}{2}, t = x^{-1}]]$$

The *RootOf* in the output indicates that algebraic extensions are involved. The `gen_exp` command will always pick the field of the input as the base field. In our case L is defined over $\mathbb{Q}(x)$, so the field of constants is $\mathbb{Q}$. If we do a computation at the point $p \neq \infty$ the `gen_exp` command will take $\mathbb{Q}(p)$ as the field of constants. In order to distinguish between the two generalized exponents we have to consider the algebraic extension of $\mathbb{Q}$ obtained by the irreducible polynomial $Z^2 + 1 \in \mathbb{Q}[Z]$. Taking $\mathbb{Q}(i)$ as the field of constants we get a fundamental system of solutions at $p = \infty$:

$$\exp\left(\frac{i}{t} + \frac{1}{2}\right) S_1 \quad \text{and} \quad \exp\left(-\frac{i}{t} + \frac{1}{2}\right) S_2,$$

where $t = \frac{1}{x}$ and $S_1, S_2 \in \mathbb{C}[[t]][\ln(t)]$.

2. For the modified Bessel functions the situation is similar. Let L be the modified Bessel operator. The singular points are at 0 and ∞. Again, at $p = 0$ the generalized exponents are ν and $-\nu$:

```
> L:=x^2*D^2+x*D-(x^2+nu^2):
> gen_exp(L,t,x=0);
```

$$[[\nu, t = x], [-\nu, t = x]]$$

At the point $p = \infty$ we compute the generalized exponents:

```
> gen_exp(L,t,x=infinity);
```

$$[[\frac{1}{t} + \frac{1}{2}, t = \frac{1}{x}], [-\frac{1}{t} + \frac{1}{2}, t = \frac{1}{x}]]$$

They correspond to the solutions

$$\exp\left(\frac{1}{t} + \frac{1}{2}\right) S_1 \quad \text{and} \quad \exp\left(-\frac{1}{t} + \frac{1}{2}\right) S_2,$$

with $t = \frac{1}{x}$ and $S_1, S_2 \in \mathbb{C}[[t]][\ln(t)]$. We see that the modified Bessel operator is easier to handle since the generalized exponents do not create new algebraic extensions.

The Puiseux series S_1, S_2 of these examples are not important for us. The important thing is that the generalized exponents can be expressed with the local parameter in all cases. We never need a root of it. Hence, the ramification index is $r = 1$. So the series S_1 and S_2 will not contain fractional exponents.

3. Formal solutions can be computed with the command `formal_sol` in Maple. Let L be the modified Bessel operator with $\nu = 0$. Then we get the following local solutions at $x = 0$:

```
> L:=subs(nu=0,L):
> formal_sol(L,t,x=0);
```

$$[[\ln(t) + \left(-\frac{1}{4} + 1/4\ln(t)\right)t^2 + \left(\frac{1}{64}\ln(t) - \frac{3}{128}\right)t^4 + O\left(t^6\right),$$
$$1 + \frac{1}{4}t^2 + \frac{1}{64}t^4 + O\left(t^6\right), t = x]]$$

If we just want to know whether the operator L has logarithmic solutions at $x = 0$, we can also use the command:

```
> formal_sol(L,`has logarithm?`,x=0);
```

$$true$$

Here, `formal_sol` will make sure that enough terms of the Puiseux series are computed such that we know whether a logarithm appears in the formal solution.

Remark 1.33
Let us finally summarize what we learn from this section for operators of degree two:

- At every point p there are two generalized exponents e_1 and e_2 such that the solution space is generated by two solutions of the form (1.26).
- If e_1 and e_2 are both non-negative integers the local solutions are power series and p is either a regular point or an apparent singularity.
- If p is a non-apparent singularity and $e_1, e_2 \in \mathbb{C}$ are both constants, p is regular singular. If $e_1, e_2 \notin \mathbb{C}$, p is irregular singular.
- Each generalized exponents e is unique modulo $\frac{1}{r_e}\mathbb{Z}$, where r_e is the ramification index of the generalized exponent e.
- If $e_1 \neq e_2$ modulo $\frac{1}{r_{e_1}}\mathbb{Z}$, the generalized exponents belong to different submodules of the universal extension. It follows from Remark 1.26 that there are no logarithmic local solution at the point p.

- If $e_1 = e_2$ modulo $\frac{1}{r_{e_1}}\mathbb{Z}$, there *can* be logarithmic solutions.

Especially for the Bessel operator we know:

- The ramification index is always 1.
- At $p = \infty$ the two generalized exponents belong to different submodules and there are no logarithmic solutions.
- At $p = 0$ there can be logarithmic solutions only if $\nu = -\nu$ modulo $\mathbb{Z}$.

2

Transformations

From now on we will only work with differential operators of degree two.

2.1 Operators of Degree Two

Let k be a field and let $K = k(x)$ be the field of rational functions in x.

Definition 2.1 *A transformation between two differential operators $L_1, L_2 \in K[\partial]$ is a map from the solution space $V(L_1)$ onto the solution space $V(L_2)$.*

The transformation is invertible if there also exists a map from $V(L_2)$ onto $V(L_1)$.

For us the following transformations will be important.

Definition 2.2 *Let $L_1 \in K[\partial]$ be a differential operator of degree two.*

We define for $y = y(x) \in V(L_1)$ the following transformations:

(i) change of variables: $y(x) \to y(f), f \in K$,
(ii) exp-product: $y \to \exp(\int r)y, r \in K$, and
(iii) gauge transformation: $y \to r_0 y + r_1 y', r_0, r_1 \in K$.

For the resulting operator $L_2 \in K[\partial]$ we write $L_1 \xrightarrow{f}_C L_2$, $L_1 \xrightarrow{r}_E L_2$, and $L_1 \xrightarrow{r_0,r_1}_G L_2$, respectively. Furthermore, we write $L_1 \longrightarrow L_2$ if there exists a sequence of those transformations which transforms L_1 into L_2.

The rational functions f, r, r_0 and r_1 will be called parameters of the transformation and the function $\exp(\int r)$ in case (ii) is called a hyperexponential function.

Note that the parameters of a transformation do not uniquely define the operator L_2. Actually, the existence of such an operator for given parameters is not yet assured.

Theorem 2.3 *Let $L_1 \in K[\partial]$ be a differential operator of degree two. If the parameters of the transformations above are given, we can always find $L_2 \in K[\partial]$ with* $\deg(L_2) = 2$ *that satisfies the conditions.*

Proof. 1. Let $L_1 = a_2\partial^2 + a_1\partial + a_0$ be a differential operator with $a_2 \neq 0$ and let $y = y(x)$ be a solution of L_1, i.e. $a_2 y''(x) + a_1 y'(x) + a_0 y = 0$.

Let $f \in K$ and $z = y(f)$. Then

$$\begin{aligned} z &= (y(f)), \\ z' &= \frac{d}{dx} y(f) = \frac{d}{dx} y(x)\Big|_{x=f} f' \\ \text{and} \quad z'' &= \frac{d^2}{dx^2} y(f) = \frac{d^2}{dx^2} y(x)\Big|_{x=f} (f')^2 + \frac{d}{dx} y(x)\Big|_{x=f} f'' \\ &= -\left(\frac{a_1}{a_2}\frac{d}{dx} y(x) + \frac{a_0}{a_2} y(x)\right)\Big|_{x=f} (f')^2 + \frac{d}{dx} y(x)\Big|_{x=f} f''. \end{aligned}$$

We can rewrite the equation

$$z'' + b_1 z' + b_0 = 0 \tag{2.1}$$

in terms of $y(f)$ and $\frac{d}{dx} y(x)\big|_{x=f}$ using the equations above and get

$$y(f)\left(-\frac{a_0}{a_2}\Big|_{x=f} (f')^2 + b_0\right) + \left(\frac{d}{dx} y(x)\Big|_{x=f}\right)\left(-\frac{a_1}{a_2}\Big|_{x=f} (f')^2 + f'' + b_1 f'\right) = 0.$$

Equating coefficients finally yields

$$b_0 = \frac{a_0}{a_2}\Big|_{x=f} (f')^2 \quad \text{and} \quad b_1 = \frac{1}{f'}\left(\frac{a_1}{a_2}\Big|_{x=f} (f')^2 + f''\right). \tag{2.2}$$

Thus, we have found a differential equation for $z = y(f)$ which has order two.

Similarly we can prove the statement for exp-products and gauge transformations.

2. Let $z = \exp(\int(r))y$ and $r \in K$. Then

$$\begin{aligned} z' &= \exp\left(\int r\right)\left(ry + y'\right) \\ \text{and} \quad z'' &= \exp\left(\int r\right)\left(r^2 y + 2ry' + r'y - a_1 y' - a_0 y\right). \end{aligned}$$

We rewrite the equation

$$z'' + b_1 z' + b_0 = 0 \tag{2.3}$$

in terms of y and y':

$$y\left(r'+r^2-a_0-b_1r+b_0\right)+y'\left(2r-a_1+b_1\right)=0.$$

Equation coefficients yields

$$b_1=-2r+a_1 \quad \text{and} \quad b_0=-r'-r^2+a_0+b_1r.$$

3. Let $z=r_0y+r_1y'$ and $r_0,r_1\in K$. The derivations are

$$\begin{aligned}
z' &= yr_0'+y'(r_0+r_1')+y''r_1\\
&= y(r_0'-a_0r_1)+y'(r_0+r_1'-a_1r_1)\\
\text{and}\quad z'' &= y(r_0''-2a_0r_1'-a_0r_0-a_0'r_1-a_0a_1r_1)\\
&\quad +y'(2r_0'-a_0r_1+r_1''-2a_1r_1'-a_1'r_1-a_1r_0+a_1^2r_1).
\end{aligned}$$

Again, we rewrite (2.1) and solve the equations for the coefficients. This yields

$$\begin{aligned}
b_0 =&-\big(-r_1a_0r_1''-3r_1a_0r_0'+r_1^2a_0a_1'-r_1a_0a_1r_0+r_0'r_1a_1^2-2r_0'r_1'a_1\\
&-r_0'r_1a_1'+r_0'r_1''-r_0'a_1r_0+2r_0'^2+a_0r_0^2-r_0''r_0-r_1a_0r_1'a_1\\
&+r_1a_0'r_0+3a_0r_0r_1'+a_0^2r_1^2-r_0''r_1'+2r_1'^2a_0+r_1a_0'r_1'+r_0''r_1a_1\\
&-r_1^2a_0'a_1\big)/\big(-r_0^2-r_0r_1'+r_0r_1a_1+r_1r_0'-r_1^2a_0\big)\\
\text{and}\quad b_1 =&(r_0r_1''+2r_0r_0'+r_0r_1a_1^2-2r_0r_1'a_1-r_0r_1a_1'-a_1r_0^2-a_0r_1^2a_1\\
&+r_1^2a_0'-r_1r_0''+2r_1r_1'a_0)/(-r_0^2-r_0r_1'+r_0r_1a_1+r_1r_0'-r_1^2a_0).
\end{aligned}$$

Concluding, if we apply an exp-product or a gauge transformation to a solutions y of a differential equation of order two, we will find a differential equation for the resulting function z. Furthermore the coefficients are determined by the formulas given above.

We can also avoid denominators in the formulas if we start with the equation $b_2z''+b_1z'+b_0=0$. This will then result in two equations with three variables and there will always be a non-trivial solution. □

Example 2.4
From the proof we can directly derive algorithms to compute the resulting operator. They are called `changeOfVar`, `expProduct` and `gauge` and take an operator L and the parameters, respectively, f or r or r_0,r_1.

We apply $x\to x^2$ to the modified Bessel operator L_B:

```
> LB:=x^2*D^2+x*D-(x^2+nu^2):
> L:=changeOfVars(LB,x^2);
```

$$x^2\partial^2+x\partial-4x^4-4\nu^2$$

Maple will still be able to find solutions:

```
> dsolve(diffop2de(L,y(x)),y(x));
```

$$y(x) = \_C1\, I_\nu\left(x^2\right) + \_C2\, K_\nu\left(x^2\right)$$

We apply a gauge transformation with parameters 0 and 1 to L_B with $\nu = 0$:

```
> L:=gauge(subs(nu=0,LB),0,1):
```

It follows from Lemma 1.22 that $I_0'(x) = I_1(x)$ and $K_0'(x) = K_1(x)$ so Maple will express the solutions by $I_1(x)$ and $K_1(x)$:

```
> dsolve(diffop2de(L,y(x)),y(x));
```

$$y(x) = \_C1\, I_1(x) + \_C2\, K_1(x)$$

An interesting question is always whether we can to a reverse operation. In this case:

```
> gauge(L,1/x,1);
```

$$x\partial^2 + \partial - x$$

The result differs from $L_B\big|_{\nu=0}$ by a factor $x \in K$. Since such a factor doesn't change the solution space those operators are considered to be equal. So we found a reverse operation.

The following lemma will prove that a reverse operation always exists for exp-products and gauge transformations.

Lemma 2.5 *The operations* $\longrightarrow_C, \longrightarrow_E$ *and* $\longrightarrow_G$ *are reflexive and transitive. The operations* $\longrightarrow_E$ *and* $\longrightarrow_G$ *are also symmetric.*

Proof. 1. We can derive the reflexivity by of $\longrightarrow_C, \longrightarrow_E$ and $\longrightarrow_G$ using the parameters $f = x, r = 0, r_0 = 1$ and $r_1 = 0$ in the definition of the transformations.

2. Let L_1, L_2 and L_3 be differential operators. If $L_1 \xrightarrow{f_1}_C L_2 \xrightarrow{f_2}_C L_3$ for $f_1, f_2 \in K$, then $L_1 \xrightarrow{f_3}_C L_3$ where $f_3 = f_1(f_2(x))$.

If $L_1 \xrightarrow{r_1}_E L_2 \xrightarrow{r_2}_E L_3$ for $r_1, r_2 \in K$, then $L_1 \xrightarrow{r_3}_E L_3$ where $r_3 = r_1 r_2$.

Let $L_1 = a_2\partial^2 + a_1\partial + a_0$. If $L_1 \xrightarrow{r_0,r_1}_G L_2 \xrightarrow{s_0,s_1}_G L_3$ for $r_0, r_1, s_0, s_1 \in K$, then a solution $y \in V(L_1)$ is mapped to

$$\begin{aligned}
& s_0(r_0 y + r_1 y') + s_1(r_0 y + r_1 y')' \\
= \; & s_0(r_0 y + r_1 y') + s_1(r_0 y' + r_0' y + r_1 y'' + r_1' y') \\
= \; & y\left(r_0 s_0 + r_0' s_1 - \frac{a_0}{a_2} r_1 s_1\right) + y'\left(r_1 s_0 + r_0 s_1 + r_1' s_1 - \frac{a_1}{a_2} r_1 s_1\right).
\end{aligned} \tag{2.4}$$

As a result, $L_1 \xrightarrow{t_0,t_1}_G L_3$ where t_0 and t_1 are the coefficients of y and y' in equation (2.4), respectively.

3. If $L_1 \xrightarrow{r}_E L_2$, then $L_2 \xrightarrow{-r}_E L_1$ since $\exp(\int r)\exp(-\int r) = 1$. Depending on the constant we choose in the integration of r we can get any other result $\exp(c)$ for a constant c. But since $\exp(c)V(L) = V(L)$ the solution space is not changed by any constant parameter in the exp-product.

The proof of the symmetry of $\longrightarrow_G$ is more technical. Let $L_1, L_2 \in K[\partial]$ with $L_1 \xrightarrow{r_0,r_1}_G L_2$, then we have an operator $R = r_1\partial + r_0 \in K[\partial]$ of degree one that satisfies $R(V(L_1)) = V(L_2)$. Since the dimensions of $R(V(L_1))$ and $V(L_2)$ are both two we must have $V(R) \cap V(L_1) = 0$. Otherwise R maps some solutions of L_1 to 0 and the dimension of $V(L_2)$ is at most one.

From $V(R) \cap V(L_1) = 0$ is follows that $\mathrm{GCRD}(L_1, R) = 1$. Because if there was $G = \mathrm{GCRD}(L_1, R)$ with $\deg(G) > 0$, there would exist $M_1, M_2 \in K[\partial]$ with $M_1G = L_1$ and $M_2G = R$. Hence, $Ry = L_1y = 0$ for all $y \in V(G)$. Since $\deg(G) > 0$ this would also be true for some $y \neq 0$ and then $y \in V(R) \cap V(L_1)$, which is a contradiction.

So we find $S, T \in K[\partial]$ such that $SL_1 + TR = 1$. Let $U = SL_1 + TR$. For a solution $y \in V(L_1)$ we get $y = U(y) = S(L_1(y)) + T(R(y)) = S(0) + T(R(y)) = T(R(y))$. As a result, T is the inverse operator of R. There are always infinitely many tuples S, T that solve $SL_1 + TR = 1$ and there will also be a solution with $\deg(T) < \deg(L_1) = 2$. This T is the inverse gauge transformation that makes $\longrightarrow_G$ symmetric. □

Remarks 2.6

1. The operators calculated in Theorem 2.3 are not unique.

2. The change of variables is not symmetric because that would require algebraic functions as parameter. For example, to cancel the operation $x \to x^2$, we would need $x \to \sqrt{x}$.

3. The relation $\longrightarrow$ is reflexive and transitive since it is defined as a composition of $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$.

An important question when searching for transformations between two operators L_1 and L_2 is whether we can restrict our search to a specific order of the transformations $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$.

Lemma 2.7 *Let $L_1, L_2, L_3 \in K[\partial]$ be three differential operators such that $L_1 \longrightarrow_G L_2 \longrightarrow_E L_3$. Then there exists a differential operator $M \in K[\partial]$ such that $L_1 \longrightarrow_E M \longrightarrow_G L_3$.*

Similarly, if $L_1 \longrightarrow_E L_2 \longrightarrow_G L_3$ we find M such that $L_1 \longrightarrow_G M \longrightarrow_E L_3$.

Proof. Let $r_0, r_1 \in K$ be the parameters of the gauge transformation and let $r \in K$ be the parameter of the exp-product. If $y \in V(L_1)$, then $z = \exp(\int r)\,(r_0y + r_1y')$

is a solution of L_3 which can also be rewritten:

$$z = r_0 \exp\left(\int r\right) y + r_1 \exp\left(\int r\right) y'$$

$$= (r_0 - r') \exp\left(\int r\right) y + r_1 \left(\exp\left(\int r\right) y\right)'. \tag{2.5}$$

Hence, $L_1 \xrightarrow{r}_E M \xrightarrow{s_0,s_1}_G L_3$ for some $M \in K[\partial]$ with $s_0 = r_0 - r'$ and $s_1 = r_1$.

The reverse can be derived from equation (2.5) by starting with the right-hand side. □

Definition 2.8 *The relation $\longrightarrow_{EG}$ on two differential operators $L_1, L_2 \in K[\partial]$ is defined by*

$$L_1 \longrightarrow_{EG} L_2 \Leftrightarrow \exists M \in K[\partial] : L_1 \longrightarrow_E M \longrightarrow_G L_2.$$

It follows from Lemma 2.5 and Lemma 2.7 that $\longrightarrow_{EG}$ is an equivalence relation. The problem whether two operators are connected through an exp-product and a gauge transformations is also called the *equivalence of differential operators*.

The equivalence of a two operators can equivalently be defined as follows.

Lemma 2.9 *Let $L_1, L_2 \in K[\partial]$ be given, then the following statements are equivalent*

(i) $\exists r_0, r_1 \in K : L_1 \xrightarrow{r_0,r_1}_G L_2$

(ii) $\exists G \in K[\partial], \deg(G) = 1 : L_2 G = Q L_1$ *for some operator Q.*

Furthermore, if an exp-product with parameter $r \in K$ is involved such that $L_1 \longrightarrow_G M \xrightarrow{r}_E L_2$ for some $M \in K[\partial]$ then the same statement is true for $G = \exp(\int r)\bar{G}$ with $\bar{G} \in K[\partial]$.

Proof. Let (i) be given and let $G = r_1\partial + r_0$. Then all solutions $y \in V(L_1)$ are solutions of $L_2 G$ since $L_2 G y = L_2(r_1 y' + r_0 y)$ yields zero by assumption. Thus, L_1 is a right factor of $L_2 G$, which proves (ii).

If (ii) is given and $y \in V(L_1)$, then $0 = Q L_1 y = L_2 G y$. Let $G = r_1\partial + r_0$. Then each solution y of L_1 gives a solution $Gy = r_1 y' + r_0 y \neq 0$ of L_2. Moreover, L_1 and L_2 both have degree two. Hence, $L_1 \longrightarrow_G L_2$. □

We will come back to the equivalence of operators at the end of this chapter. Now we consider the more general question whether $L_1 \longrightarrow L_2$ for two operators L_1 and L_2.

Theorem 2.10 *Let $L_1, L_2 \in K[\partial]$ such that $L_1 \longrightarrow L_2$. Then there exists an operator $M \in K[\partial]$ such that*

$$L_1 \longrightarrow_C M \longrightarrow_{EG} L_2.$$

Proof. To prove the theorem it is sufficient to show that for three operators $L_1, L_2, L_3 \in K[\partial]$ the following holds:

(i) $L_1 \longrightarrow_E L_2 \longrightarrow_C L_3 \Rightarrow \exists M \in K[\partial] : L_1 \longrightarrow_C M \longrightarrow_E L_3$, and
(ii) $L_1 \longrightarrow_G L_2 \longrightarrow_C L_3 \Rightarrow \exists M \in K[\partial] : L_1 \longrightarrow_C M \longrightarrow_G L_3$.

The rest follows from Lemma 2.7 and the definition of $\longrightarrow_{EG}$.

(i) Let r and f be the parameter of the exp-product and the change of variables respectively, and let $R = R(x) = \int r\,dx$. Then the solution space of L_3 is:

$$V(L_3) = \{\exp(R(f))g(f) \mid g(x) \in V(L_1)\}.$$

This solution space also arises from L_1 by a change of variables $x \to f$ and an exp-product with $\partial R(f)$.

(ii) Let r_0 and r_1 be the parameters of the gauge transformation and let f the parameter of the change of variables. Then the solution space of L_3 is:

$$V(L_3) = \left\{ r_0(f)g(f) + r_1(f)g'(x)\big|_{x=f} \mid g(x) \in V(L_1) \right\}.$$

Now $g'(x)\big|_{x=f} = \frac{g(f)'}{f'}$ and so the solution space

$$V(L_3) = \left\{ r_0(f)g(f) + \frac{r_1(f)}{f'}g(f)' \;\middle|\; g(x) \in V(L_1) \right\}$$

arises from L_1 by a change of variables $x \to f$ and a gauge transformation $G = \frac{r_1(f)}{f'}\partial + r_0(f)$. □

Note that the converse of (i) and (ii) is not generally true. The reason for this is the same reason why $\longrightarrow_C$ is not symmetric. We have to do the reverse operation of $r(x) \to \tilde{r}(x) = r(f(x))$, which may need algebraic functions. So we would have to allow algebraic functions as parameters in the exp-product and the gauge transformation.

The knowledge of transformations between two differential operators $L_1 \longrightarrow L_2$ reduces the problem to solve L_2 to the problem to find $V(L_1)$. If we know the transformations involved we can simply express $V(L_2)$ in terms of two independent solutions $y_1, y_2 \in V(L_1)$.

The main problem we consider in this thesis is the following: given an operator $L \in K[\partial]$ find transformations that send the modified Bessel operator L_B to L if they exist. If we found those transformations, we also found $V(L)$. Note that we also need to find the parameter ν of the Bessel functions involved.

We know from the previous theorem that if those transformations exist, there exists $M \in K[\partial]$ such that

$$L_B \longrightarrow_C M \longrightarrow_{EG} L.$$

We will address those two parts separately.

But first we will clarify the next steps using some examples.

Example 2.11

1. We consider the modified Bessel operator L_B with $\nu = 2$:

$$L = x^2\partial^2 + x\partial - \left(x^2 + 4\right).$$

Using the results of Example 1.32 we know that the generalized exponents at $x = 0$ and $x = \infty$ are

$$\begin{aligned} &\mathrm{gexp}(L,0) = \{2,-2\} \\ \text{and} \quad &\mathrm{gexp}(L,\infty) = \left\{\frac{1}{T}+\frac{1}{2}, -\frac{1}{T}+\frac{1}{2}\right\}. \end{aligned} \tag{2.6}$$

Now we apply a change of variables

$$x \to f(x) = \frac{2(x-1)(x-2)^2}{(x-3)^2}$$

to L:

```
> L:=x^2*D^2+x*D-(x^2+2^2):
> f:=2*(x-1)*(x-2)^2/(x-3)^2:
> M:=changeOfVars(L,f);
```

$$\begin{aligned} M := &(x-2)^3\left(x^2-7x+8\right)(x-3)^6(x-1)^3\partial^2 + \\ &\left(x^4-14x^3+55x^2-84x+46\right)(x-3)^5(x-1)^2(x-2)^2\partial - \\ &4\left(x^2-7x+8\right)^3\left(x^6-10x^5+42x^4-100x^3+158x^2-172x+97\right) \\ &(x-2)(x-1) \end{aligned}$$

We will now assume that M is given. We want to find the parameter of the change of variables that sends L to M, i.e. we want to find f using only the operator M.

The zeros of the leading coefficient of M are $1,2,3,\frac{7}{2}+\frac{1}{2}\sqrt{17}$ and $\frac{7}{2}-\frac{1}{2}\sqrt{17}$. The generalized exponents of the latter two points are 0 and 2:

```
> gen_exp(M,t,x=RootOf(x^2-7*x+8));
```

$$[[0,2,t=x-\mathit{RootOf}(\_Z^2-7\_Z+8)]]$$

Hence, all local solutions can be expressed as power series. These points are apparent singularities and are not considered in the following.

The other singular points are exactly the zeros and poles of f. It is not amazing that the function $I_2(f(x))$ has singularities at those points since f sends those points to 0 and ∞, which are singularities of $I_2(x)$.

Another point we have to consider is $x=\infty$. If we apply a change of variables $x \longrightarrow \frac{1}{x}$ to M, we get an operator which has a singularity at $x=0$. Hence, $x=\infty$ is also a singularity of M.

We compute the generalized exponents of M at the points $x=1$ and $x=2$ with Maple:

```
> gen_exp(M,t,x=1);
```

$$[[2,-2,t=x-1]]$$

```
> gen_exp(M,t,x=2);
```

$$[[4,-4,t=x-1]]$$

These points turn out to be regular singular since their generalized exponents are constants. Comparing the exponents with those of L at $x=0$ in equation (2.6) we see that they were multiplied by the multiplicity of the zero $x=1$ and $x=2$, respectively.

The generalized exponents at $x=3$ are the following:

```
> gen_exp(M,t,x=3);
```

$$[[-8t^{-2}-10t^{-1}+1,t=x-3],[8t^{-2}+10t^{-1}+1,t=x-3]] \tag{2.7}$$

Hence, $x=3$ is irregular singular. We compare the coefficients that appear in the generalized exponents with the partial fraction decomposition of f:

```
> convert(f,parfrac);
```

$$\frac{4}{(x-3)^2}+\frac{10}{x-3}+2+2x \tag{2.8}$$

We observe that the coefficients of $(x-3)^{-j}=t^{-j}$ in (2.8) multiplied by the corresponding exponent $-j$ appear in the first generalized exponent of (2.7). These are $4\cdot(-2)=-8$ and $10\cdot(-1)=-10$.

Similarly, we can find the coefficient of $x=\frac{1}{t_\infty}$ in (2.8) by computing the generalized exponent of M at $x=\infty$:

```
> gen_exp(M,t,x=infinity);
```

$$[[-\frac{2}{t}+\frac{1}{2},t=\frac{1}{x}],[\frac{2}{t}+\frac{1}{2},t=\frac{1}{x}]]$$

Again the coefficient of the partial fraction decomposition appears in the generalized exponent.

Concluding, we see that if $L_B\big|_{v=2} \xrightarrow{f}_C M$, the singularities and the generalized exponents of M have a lot in common with the parameter f. All this information can be used to find f. Yet, we didn't consider the constant term 2 in (2.8). We will study these facts in detail in Chapter 3.

2. Now we further apply an exp-product with parameter $r = ((x-5)(x-2))^{-1}$ to M and get:

```
> M2:=expProduct(M,((x-5)(x-2))^(-1));
```

$$\begin{aligned} M_2 = {} & (x-2)^2\left(x^2-7x+8\right)(x-1)^2(x-3)^6(x-5)^2\partial^2+ \\ & \left(x^5-21x^4+147x^3-437x^2+572x-278\right)(x-3)^5(x-5)(x-1)(x-2)\partial \\ & -4x^{14}+164x^{13}-3032x^{12}+33505x^{11}-247557x^{10}+1297816x^9- \\ & 5006810x^8+14568502x^7-32519034x^6+56185848x^5-74866424x^4+ \\ & 75029073x^3-53284469x^2+23749732x-4945502 \end{aligned}$$

So we are considering

$$L_B\big|_{v=2} \xrightarrow{f}_C M \xrightarrow{r}_E M_2$$

and want to find f by looking at M_2.

Comparing the leading coefficient of M_2 with the one we had in M we observe that there is a new singularity at $x=5$. The generalized exponents at this point are

```
> gen_exp(M2,t,x=5);
```

$$[[\frac{1}{3},\frac{4}{3},t=x-5]]$$

Hence, $x=5$ is another regular singular point and we lost the one–to–one correspondence between regular singularities and zeros of f, which we had before. The exp-product also created apparent singularities at the zeros of x^2-7x+8:

```
> gen_exp(M2,t,x=RootOf(x^2-7*x+8));
```

$$[[0,2,t=x-\mathit{RootOf}(\_Z^2-7\_Z+8)]]$$

Furthermore, the other generalized exponents were changed by the transformation, e.g. at $x=2$ we have:

```
> gen_exp(M2,t,x=2);
```

$$[[-\frac{13}{3}, \frac{11}{3}, t = x-2]]$$

Here we cannot read off the multiplicity of the factor $(x-2)$ directly.

Concluding, if an exp-product is involved, the zeros of the parameter f in the change of variables do not correspond to regular singularities and the multiplicity of the zeros can not be derived from the generalized exponents.

3. The zeros and poles of f were still among the regular and irregular singularities of M_2. But if we apply a gauge transformation to M_2 with parameters $r_0 = (x-1)^2$ and $r_1 = (x-1)^3$, the resulting operator M_3 loses also this property.

```
> M3:=gauge(M2,(x-1)^2,(x-1)^3):
```

Now we consider

$$L_B\big|_{v=2} \xrightarrow{f}_C M \xrightarrow{r}_E M_2 \xrightarrow{r_0,r_1}_G M_3.$$

The generalized exponent of M_3 at $x = 1$ is:

```
> gen_exp(M3,t,x=1);
```

$$[[0, t = x-1], [4, t = x-1]]$$

Hence, $x = 1$ becomes an apparent singularity of M_3. But we don't consider apparent singularities because there exists an operator $\tilde{M}_3$ which has the same solutions as M_3. There also might exist such an operator $\tilde{M}_3$ which has degree two. Moreover, we have seen that a change of variables and an exp-product can create apparent singularities. Therefore, apparent singularities are not taken into account. Thus, we cannot find all zeros of f by only looking at the singularities of M_3. But we will soon see that we can use a similar approach.

From the first part of the example we see that we can find the parameter f of the change of variables by looking at singularities and generalized exponents. But if exp-products and gauge transformations are involved, this is not so easy. In the next section we will develop a property which is invariant under exp-products and gauge transformations, which can then be used to find f.

2.2 The Exponent Difference

In this section we will study how exponents behave in exp-products and gauge transformations.

Lemma 2.12 *Let $L, M, \in K[\partial]$ be two differential operators such that $M \xrightarrow{r}_E L$ and let e be an exponent of M at the point p. Furthermore, let r have the series representation*

$$r = \sum_{i=m}^{\infty} r_i t_p^i, \quad m \in \mathbb{Z}.$$

Then $e + \sum_{i=m}^{-1} r_i t_p^{i+1}$ is an exponent of L at p.

Proof. Let t be the local parameter t_p. Since e is an exponent, M has a solution of the form

$$y = \exp\left(\int \frac{e}{t}\, dt\right) S,$$

for some Puiseux series $S \in k((t))[\ln(t)]$. The exp-product converts this solution into

$$z = \exp\left(\int r dt\right) \exp\left(\int \frac{e}{t}\, dt\right) S.$$

In order to determine the exponent at p we have to rewrite this expression into the form (1.26). We have to handle the positive and negative powers of t in r separately. For the power series part $\bar{r} = \sum_{i=0}^{\infty} r_i t^i$ we get

$$\exp\left(\int \bar{r}\, dt\right) = \exp\left(\sum_{i=0}^{\infty} \frac{r_i}{i+1} t^{i+1}\right).$$

With $\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ we can rewrite this as a power series in t:

$$\begin{aligned} \exp\left(\int \bar{r}\, dt\right) &= \sum_{i=0}^{\infty} \frac{1}{i!} \left(\sum_{j=0}^{\infty} \frac{r_j}{j+1} t^{j+1}\right)^i \\ &= \sum_{i=0}^{\infty} a_i t^i \quad \text{with} \quad a_i \in k, a_0 = 1. \end{aligned}$$

The negative powers of t in the series expansion of r become a part of the exponent:

$$\exp\left(\int \sum_{i=m}^{-1} r_i t^i\, dt\right) = \exp\left(\int \frac{1}{t} \sum_{i=m}^{-1} r_i t^{i+1}\, dt\right).$$

Combining the two results we get

$$z = \exp\left(\int \frac{1}{t}\left(e + \sum_{i=m}^{-1} r_i t^{i+1}\right) dt\right) \bar{S},$$

where $\bar{S} \in k((t))[\ln(t)]$ has a non-zero constant term. $\square$

Note that the result of this lemma depends only on r. Hence, each generalized exponent at a point p is shifted by the same amount.

Definition 2.13 *Let $L \in K[\partial]$ be a differential operator, let p be any point, and let e_1 and e_2 be two generalized exponents of L at p. Then the difference $e_1 - e_2$ is called an exponent difference of L at p.*

If $\deg(L) = 2$ *there exist just two generalized exponents at each point and we define*

$$\Delta(L, p) := \pm(e_1 - e_2).$$

We define Δ modulo a factor -1 to make it well-defined because we have no ordering in the generalized exponents we compute. It follows from Lemma 2.12 that $\Delta(L_1, p) = \Delta(L_2, p)$ for $L_1 \longrightarrow_E L_2$, i.e. $\Delta(L_1, p)$ is invariant under exp-products.

Lemma 2.14 *Let $L, M \in K[\partial]$ be two differential operators such that $M \longrightarrow_G L$ and let e be a generalized exponent of M at the point p. The operator L has a generalized exponent $\bar{e}$ such that $\bar{e} = e$ mod $\mathbb{Z}$.*

Proof. Let y be a solution of M with generalized exponent e at p and let $t = t_p$. Then y has the form

$$y = \exp\left(\int \frac{e}{t} dt\right) S$$

with $S \in k((t))[\ln(t)]$.

Let $r, s \in K$ be the parameters of the gauge transformation that send M to L. Then $z = ry + sy'$ is a solution of M. Let $\bar{e}$ be the exponent of M, then our statement follows from the following facts, which are simple consequences of standard calculation rules and the series representation that we use:

1. The exponents e_r and e_s of the rational functions r and s at the point p are integers.

2. The generalized exponent of y' differs from e by an integer.

3. The generalized exponent of ry is the sum of the exponents of r and y (same holds for sy').

Then the generalized exponents of ry and sy' both differ from e by an integer, i.e.

$$ry = \exp\left(\int \frac{e + \lambda_1}{t} dt\right) S_1 \quad \text{and} \quad sy' = \exp\left(\int \frac{e + \lambda_2}{t} dt\right) S_2,$$

where $\lambda_1, \lambda_2 \in \mathbb{Z}$ and $S_1, S_2 \in k((t))[\ln(t)]$. Thus,

$$ry + sy' = \exp\left(\int \frac{e+\lambda_3}{t}\,dt\right) S_3,$$

where $\lambda_3 \in \mathbb{Z}$ depends on λ_1 and λ_2 and is such that the Puiseux series $S_3 \in k((t))[\ln(t)]$ starts with a non-zero constant term. Hence, $\bar{e}$ is a generalized exponents of L and $\bar{e} = e + \lambda_3 = e \bmod \mathbb{Z}$. □

The exponent difference Δ has the following property.

Corollary 2.15 *Two operators $L_1, L_2 \in K[\partial]$ with $L_1 \longrightarrow_{EG} L_2$ satisfy $\Delta(L_1, p) = \Delta(L_2, p) \bmod \mathbb{Z}$ for each point p, i.e. $\Delta(L_1, p) \bmod \mathbb{Z}$ is invariant under $\longrightarrow_{EG}$.*

This result will be used in the following theorem.

Theorem 2.16 *Let $L \in K[\partial]$ be a differential operator and let p be a fixed point. Then the following statements are equivalent:*

(i) There exists an operator $M \in K[\partial]$ where p is regular such that $M \longrightarrow_{EG} L$.
(ii) The solutions of L are not logarithmic and $\Delta(L, p) \in \mathbb{Z}$.

We will only prove the implication (i)⇒(ii). Above, we stated the equivalence for the sake of completeness. More details can be found in [21] and the appendix of [1].

Proof. Assume that (i) is given. Then there exist rational functions $r_0, r_1, r_2 \in K$ and $\tilde{M} \in K[\partial]$ such that

$$M \overset{r_0}{\longrightarrow}_E \tilde{M} \overset{r_1,r_2}{\longrightarrow}_G L.$$

Furthermore, let p be a regular point of M. The generalized exponents at p are 0 and 1. Hence, $\Delta(M, p) \in \mathbb{Z}$ and from the previous corollary it follows that $\Delta(L, p) \in \mathbb{Z}$.

Let

$$y_1 = \sum_{i=0}^{\infty} a_i t_p^i, a_0 \neq 0 \quad \text{and} \quad y_2 = \sum_{i=1}^{\infty} b_i t_p^i, b_1 \neq 0$$

be linear independent local solutions of M at the point p, i.e.

$$V(M) = \{c_1 y_1 + c_2 y_2 \mid c_1, c_2 \in K\}.$$

After the exp-product the solution space is

$$V(\tilde{M}) = \left\{ \exp\left(\int r_0\right)(c_1 y_1 + c_2 y_2) \,\middle|\, c_1, c_2 \in K \right\}$$

and the gauge transformation changes this into

$$V(L) = \left\{ r_1 z + r_2 z' \mid z \in V(\tilde{M}) \right\}.$$

Since we have no logarithms in $V(M)$ we will also have none in $V(\tilde{M})$ or $V(L)$. Therefore, the solutions of L are not logarithmic and we have proven (ii). □

For us the important direction is $(i) \Rightarrow (ii)$. Consider the situation $M \longrightarrow_{EG} L$ with fixed M. We are interested in the singularities of M. The theorem tells us that the points p where (ii) does not hold are singularities of M.

Note that we do not know all singularities of M. Assume that (ii) holds at a point p. From the theorem we know the existence of some $\tilde{M}$ such that $\tilde{M} \longrightarrow_{EG} L$ and $\tilde{M}$ is regular at p. However, we can not make a choice for M in our situation. Hence, we can not say whether p is regular point of M or not.

Definition 2.17 *The singular points of an operator L where (ii) of the last theorem holds are called exp-apparent singularities.*

Considering $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$ this means that all singularities of L which are not exp-apparent are singularities of M which can be used to compute f. Once we found f, we can compute M. The remaining problem is the equivalence between M and L.

2.3 Equivalence of Differential Operators

The problem whether two operators are equivalent can be solved for example using the algorithm by Barkatou and Pflügel described in [3] which is implemented in the `ISOLDE` package for Maple.

In this section we want to introduce a simpler algorithm through an example, which is called the *cyclic vector method.*

But before be can apply this method we need to reduce our problem to a system of linear differential equations.

Theorem 2.18 *The question whether two operators $L_1, L_2 \in K[\partial]$ are equivalent can be reduced to a system of linear differential equations with hyperexponential solutions.*

Proof. From Lemma 2.9 we know that the operators satisfy $L_1 \longrightarrow_G L_2$ if and only if there exists an operator $G \in K[\partial]$ of order one such that L_1 is a right factor of $L_2 G$.

If $L_1 \longrightarrow_{EG} L_2$, then the same is true for an operator $G = \exp(\int r)\bar{G}$ with $r \in K$ and $\bar{G} \in K[\partial]$ of order one.

We start with an operator $G = r_1\partial + r_0$, where $r_0 = \exp(\int r)s_0$ and $r_1 = \exp(\int r)s_1$. So in both cases we search for hyperexponential functions r_0 and r_1. Then the rest R after a right division of L_2G by L_1 must be zero. The operator R is a operator of degree one and the coefficients are a K-linear combination of $r_0, r_0', r_0'', r_1, r_1'$ and r_1''. Equating these coefficients with zero yields a system of two differential equations of order two with two variables.

Furthermore, replacing r_0' and r_1' by two new variables $\bar{r}_0$ and $\bar{r}_1$ transforms this system into a system of differential equations of order one. We add two equations $r_0' - \bar{r}_0 = 0$ and $r_1' - \bar{r}_1 = 0$ and finally get a system of four order one equations in four variables. In matrix representation such a system is written as $Y' - AY = 0$, where A is a 4×4 matrix and Y is the vector including the undetermined functions $r_0, r_1, \bar{r}_0$ and $\bar{r}_1$. □

These *hyperexponential solutions* can be found with the *cyclic vector method*.

Definition 2.19 *Let V be a n-dimensional vector space and $\partial : V \to V$ an endomorphism. A vector $v \in V$ is called a cyclic vector of V if*

$$\{z, \partial z, \partial^2 z, \ldots, \partial^{n-1} z\}$$

is a basis of V.

Definition 2.20 *Let*

$$L = a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0\partial^0$$

be a differential operator. We define the adjoint operator

$$L^* := (-1)^n \left((-\partial)^0 a_0 + \ldots + (-\partial)^{n-1} a_{n-1} + (-\partial)^n a_n\right).$$

The adjoint operator satisfies $L^{**} = L$ and $(L_1L_2)^* = L_2^*L_1^*$.

The cyclic vector method works as follows.

Theorem 2.21 *Let A be the 4×4 matrix of the equation $Y' - AY = 0$ and let $\partial y = y' - Ay$. Then we can compute a hyperexponential solution by the following steps:*

1. *Pick a random element $v \in K^4$.*
2. *Check whether v is cyclic, otherwise repeat step one and two.*
3. *Compute $L = a_0 + a_1\partial + a_2\partial^2 + a_3\partial^3 + \partial^4$ such that $Lv = 0$.*
4. *Compute a hyperexponential solution s of L^* (e.g. use* `expsols` *in Maple).*
5. *Compute R such that $L = \left(\partial + \frac{s'}{s}\right)\left(\frac{1}{s}R\right)$.*

6. Let $R = y_0 + y_1\partial + y_2\partial^2 + y_3\partial^3$ and $y = y_0v + y_1\partial v + y_2\partial^2 v + y_3\partial^3 v$.

Then y is a hyperexponential solution of $Y' - AY = 0$.

Proof. Let v be a cyclic vector, then $v, \partial v, \partial^2 v$ and $\partial^3 v$ are a basis of K^4. Adding the vector $\partial^4 v$ this set is linear dependent and we can find a linear combination $a_0v + a_1\partial v + a_2\partial^2 v + a_3\partial^3 v + a_4\partial^4 v$ with coefficients in K which is zero. We can also find a combination with $a_4 = 1$ and the corresponding coefficients define the operator $L = a_0 + a_1\partial + a_2\partial^2 + a_3\partial^3 + \partial^4$.

A solution of the matrix equation is a vector $y = \exp(\int r)\bar{y}$ with $\bar{y} \in K^4, r \in K$ for which $\partial y = 0$. The vector $\bar{y}$ can be represented with the basis of the cyclic vector v such that

$$\begin{aligned} \bar{y} &= b_0v + b_1\partial v + b_2\partial^2 v + b_3\partial^3 v \\ \text{and} \quad y &= c_0v + c_1\partial v + c_2\partial^2 v + c_3\partial^3 v \end{aligned} \tag{2.9}$$

with $b_i \in K$ and $c_i = \exp(\int r)b_i$. The coefficient c_3 cannot be zero. If it is zero, the equation $\partial y = 0$ is a differential equation of order < 4 for v, which is zero. Since we can divide by the exponential part $\exp(\int r)$ this also yields a differential equation of order less than four with coefficients in K. But then v would not be cyclic. Hence, $c_3 \neq 0$ and we factor out c_3 in (2.9). We can also rewrite ∂y:

$$0 = \partial y = c_3(d_0v + d_1\partial v + d_2\partial^2 v + d_3\partial^3 v + \partial^4 v) = c_3\tilde{L}v$$

where $\tilde{L} = d_0 + d_1\partial + d_2\partial^2 + d_3\partial^3 + \partial^4$ and $d_i \in K$. The equations $Lv = 0$ and $\tilde{L}v = 0$ both are a linear combination of $v, \partial v, \ldots, \partial^4 v$ that is zero. The leading coefficients are both one and therefore $L = \tilde{L}$ and $\partial y = c_3Ly$. For the corresponding operator we obtain

$$\partial(c_0 + c_1\partial + c_2\partial^2 + c_3\partial^3) = c_3L.$$

The left-hand side of this equation can be rewritten as

$$\partial c_3\left(\frac{c_0}{c_3} + \frac{c_1}{c_3}\partial + \frac{c_2}{c_3}\partial^2 + \partial^3\right) = c_3\left(\partial + \frac{c_3'}{c_3}\right)\left(\frac{c_0}{c_3} + \frac{c_1}{c_3}\partial + \frac{c_2}{c_3}\partial^2 + \partial^3\right).$$

Hence, $\partial + \frac{c_3'}{c_3}$ is a left factor of L and $\left(\partial + \frac{c_3'}{c_3}\right)^* = \partial - \frac{c_3'}{c_3}$ is a right factor of L^*. This factor has a hyperexponential solution c_3 which is also a solution of L^*.

Thus, the solution s that is computed in step four is in fact $s = c_3$. The fifth steps yields $R = c_0 + c_1\partial + c_2\partial^2 + c_3\partial^3$ and from (2.9) we know that the vector y in step six must be a solution of the matrix equation. $\square$

Example 2.22
Let $L_1 = \partial^2 + x$ and $L_2 = x^2\partial^2 - 2x\partial + 2 + x^3$. We want to compute the gauge transformation that sends L_1 to L_2.

```
> L1:=D^2+x:
> L2:=x^2*D^2-2*x*D+2+x^3:
```

At first, we transform this problem into a system of differential equations. Let $G = r_1\partial + r_0$, we compute the rest R of the right division of L_2G by L_1 and introduce new variables $r_2 = r_0'$ and $r_3 = r_1'$.

```
> R:=numer(op(2,rightdivision(mult(L2,G),L1))):
> R:=subs({diff(r0(x),x)=r2(x),diff(r1(x),x)=r3(x)
  },R);
```

$$\left(2\,r1\,(x) - 2\,r0\,(x)\,x + \left(\frac{d}{dx}r3\,(x)\right)x^2 + 2\,r2\,(x)\,x^2 - 2\,r3\,(x)\,x\right)\partial + \left(\frac{d}{dx}r2\,(x)\right)x^2 + r1\,(x)\,x^2 + 2\,r0\,(x) - 2\,r2\,(x)\,x - 2\,r3\,(x)\,x^3$$

Hence, we have differential equations

$$\begin{aligned} 0 &= x^2r_3' + 2r_1 - 2xr_0 + 2x^2r_2 - 2xr_3, \\ 0 &= x^2r_2' + x^2r_1 + 2r_0 - 2xr_2 - 2x^3r_3, \\ 0 &= r_0' - r_2, \\ \text{and} \quad 0 &= r_1' - r_3, \end{aligned}$$

which can be written as $Y' - AY = 0$ with

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2x^{-1} & -2x^{-2} & 2x^{-1} & -2 \\ -2x^{-2} & -1 & 2x & 2x^{-1} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

Next, we will use the cyclic vector method as described in Theorem 2.21. Let $v_0 = [1,0,0,0]$ and $v_i = v' - Av$ for $1 \le i \le 3$. We compute L:

```
> s:=solve({add(a[i]*V[i],i=0..4),a[4]=1},{a[0]
  ,a[1],a[2],a[3],a[4]}):
> L:=eval(add(a[i]*D^i,i=0..4),s);
```

$$10x^{-1} + 6\,\frac{\left(-1+3x^3\right)\partial}{x^3} + 2\,\frac{\left(3+2x^3\right)\partial^2}{x^2} + 7\,\frac{\partial^3}{x} + \partial^4$$

The vector v must be cyclic since the general solution s gave no further choice for the coefficients.

Now we compute one exponential solutions of the adjoint operator L^*.

```
> La:=adjoint(L);
```

$$\partial^4 - 7\,\frac{\partial^3}{x} + \frac{\left(27+4x^3\right)\partial^2}{x^2} - 10\,\frac{\left(6+x^3\right)\partial}{x^3} + 10\,\frac{6+x^3}{x^4}$$

```
> c[3]:=op(1,expsols(diffop2de(La,y(x)),y(x)));
```

$$c[3] := x$$

Finally, we compute the operator R

```
> R:=collect(c[3]*op(1,leftdivision(L,
  adjoint(D-normal(diff(c[3],x)/c[3])))),D,normal);
```

$$R := x\partial^3 + 6\,\partial^2 + 2\,\frac{\left(3+2x^3\right)\partial}{x} + 10x$$

and the solution y:

```
> y:=normal(expand(add(coeff(R,D,i)*v[i],i=0..3)));
```

$$y := [6x, 0, 6, 0]$$

We are just interested in the first two components of y, $r_0 = 6x$ and $r_1 = 0$. Hence, $G = 0 \cdot \partial + 6x$. Since we can change G by a constant factor we get

$$L_1 \xrightarrow{0,x}_G L_2.$$

We can verify this by:

```
> gauge(L1,x,0);
```

$$x^2\partial^2 - 2x\partial + 2 + x^3$$

This result is equal to L_2, so if $y \in V(L_1)$, then $xy \in V(L_2)$.

There are faster methods to compute hyperexponential solutions than the cyclic vector method. But this gets more important if the systems are bigger. A problem is that the direct algorithm by Barkatou and Pflügel just works with $\mathbb{Q}$ as the field of constants. Therefore, our `equiv` implementation uses the cyclic vector method with some more modifications that we will not consider here.

Example 2.23

We can use `equiv` to compute the reverse transformations from Example 2.4:

```
> LB:=x^2*D^2+x*D-(x^2+v^2):
> L:=gauge(subs(v=0,LB),0,1):
> equiv(L,subs(v=0,LB));
```

$$-\frac{1}{x}, x\partial + 1$$

The output r, G represents an operator of the form $\exp(\int r)G$. Hence, the reverse transformation to ∂ is $\exp(-\int \frac{1}{x}\,dx)(x\partial + 1) = 1\partial + 1/x$. Note that this result depends on L_B, which can be seen from the formulas derived in the proof of Theorem 2.3.

3

Solving in Terms of Bessel Functions

Let L_{in} be a differential operator. In this chapter we will solve the question whether there exist transformations such that $L_B \longrightarrow L_{in}$. Using the results of the last chapter we just have to consider

$$L_B \xrightarrow{f}_C M \longrightarrow_{EG} L_{in}. \tag{3.1}$$

The operator L_{in} is the only input to the algorithm. We define k to be the field such that L_{in} is defined over $K = k(x)$, i.e. $L_{in} \in K[\partial]$.

In the next section we will take a closer look at the part $L_B \longrightarrow_C M$. Once we found the Bessel parameter ν and the parameter f we can obtain M from L_B. For fixed $M \in K[\partial]$ we can already solve the question of equivalence between M and L_{in}. We can then finally solve (3.1).

3.1 Change of Variables

Let $M \in K[\partial]$ be given. We want to know whether there exists $f = f(x) \in K$ and $\nu \in \mathbb{C}$ such that

$$L_B \xrightarrow{f}_C M \tag{3.2}$$

holds.

Let us assume that $B_\nu(x)$ is a solution of L_B. Then $B_\nu(f(x))$ is a solution of M. Since $B_\nu(x)$ has singularities at 0 and ∞ it is obvious that the singularities of $B_\nu(f(x))$ are at those points p were $f(p) = 0$ or $f(p) = \infty$, i.e. at the zeros and poles of $f(x)$.

We will now analyze $\Delta(M, p)$ because we can then apply results not only to (3.2) but also to (3.1).

Theorem 3.1 *Let L_B be a Bessel operator and let $M \in K[\partial]$ be such that $L_B \xrightarrow{f}_C M, f \in K$.*

(a) *If p is a zero of f with multiplicity $m \in \mathbb{N}$, then p is a regular singularity of M and $\Delta(M,p) = 2m\nu$.*

(b) *If p is a pole of f with multiplicity $m \in \mathbb{N}$ such that*

$$f = \sum_{i=-m}^{\infty} f_i t_p^i, \tag{3.3}$$

then p is an irregular singularity of M and

$$\Delta(M,p) = 2 \sum_{i=-m}^{-1} i f_i t_p^i. \tag{3.4}$$

Proof. Let t be the local parameter t_p.

(a) Let p be a zero of f with multiplicity $m > 0$, then f has the representation $f = t^m \sum_{i=0}^{\infty} f_i t^i$ with $f_i \in k$ and $f_0 \neq 0$. Furthermore, let $y \in V(L_B)$ be a local solution at $x = 0$ of the form

$$y = x^\nu \sum_{i=0}^{\infty} a_i x^i, \quad a_i \in k, a_0 \neq 0.$$

If we now replace x by f, we get

$$z = f^\nu \sum_{i=0}^{\infty} a_i f^i \tag{3.5}$$

which is a local solution of M at p. Hence, we can rewrite z as a series in t, i.e.

$$z = t^e \sum_{i=0}^{\infty} b_i t^i, \quad b_i \in k, b_0 \neq 0 \tag{3.6}$$

and e is the exponent of z. Now $f^i = t^{mi} \bar{f}$ where the constant coefficient of $\bar{f} \in k[[t]]$ is non-zero and comparing the representations (3.5) and (3.6) of z yields $e = m\nu$.

Similarly, for the second independent local solution of L_B at $x = 0$, which has exponent $-\nu$, we obtain the generalized exponent $e = -m\nu$. Hence, the singularity p is regular and $\Delta(M,p) = 2m\nu$.

If $\nu \in \mathbb{Z}$ the second independent solution contains a logarithm $\ln(x)$. However, we can still do the same computations. The solution z would then involve $\ln(t)$ and the result for the exponent is still true.

(b) A similar approach works in second case. Let p be a pole of f with multiplicity $m \in \mathbb{N}$. Then representation (3.3) can also be written as $f = t^{-m} \sum_{i=0}^{\infty} f_{i-m} t^i$ with $f_i \in k, f_{-m} \neq 0$.

We start with a local solution y of L_B at $x = \infty$ corresponding to the exponent $\frac{1}{t_\infty} + \frac{1}{2}$. There exists a series $S \in k[[t_\infty]]$ such that

$$y = \exp\left(\int \frac{1}{t_\infty^2} + \frac{1}{2t_\infty} dt_\infty\right) S = \exp\left(-\frac{1}{t_\infty}\right) t_\infty^{1/2} S \tag{3.7}$$

is a solution of L_B. In order to get a solution z of M we have to replace x by f, i.e. $t_\infty = \frac{1}{x}$ by $\frac{1}{f}$. Hence, we do the following substitutions:

$$\begin{aligned} t_\infty &\longrightarrow \frac{1}{f} = t^m \sum_{i=0}^{\infty} \tilde{f}_i t^i, \quad \tilde{f}_i \in k, \tilde{f}_0 \neq 0 \\ \frac{1}{t_\infty} &\longrightarrow f, \\ \text{and} \quad t_\infty^{1/2} &\longrightarrow \frac{1}{f^{1/2}} = t^{m/2} \sum_{i=0}^{\infty} \bar{f}_i t^i, \quad \bar{f}_i \in k, \bar{f}_0 \neq 0. \end{aligned} \tag{3.8}$$

We apply these substitutions to (3.7) and get a local solution z of M at $x = p$:

$$z = \exp(-f)\, t^{m/2} \tilde{S}, \quad \tilde{S} \in k[[t]],$$

where $\tilde{S}$ combines all the new series that we obtain from (3.8). As we did in the proof of Lemma 2.12 we can rewrite $\exp(\sum_{i=0}^{\infty} f_i t^i)$ as power series in t. The negative powers of t remain in the exponential part, which then is

$$\begin{aligned} &\exp\left(-\sum_{i=-m}^{-1} f_i t^i\right) t^{m/2} = \exp\left(-\sum_{i=-m}^{-1} f_i t^i + \frac{m}{2}\ln(t)\right) \\ =&\exp\left(\int \left(\sum_{i=-m}^{-1} -i f_i t^{i-1} + \frac{m}{2t}\right) dt\right) = \exp\left(\int \frac{1}{t}\left(\sum_{i=-m}^{-1} -i f_i t^i + \frac{m}{2}\right) dt\right). \end{aligned}$$

Thus, z has the generalized exponent $\sum_{i=-m}^{-1} -i f_i t^i + \frac{m}{2}$.

If we start with the second independent solution with generalized exponent $-\frac{1}{t_\infty} + \frac{1}{2}$ we similarly get $\sum_{i=-m}^{-1} i f_i t^i + \frac{m}{2}$. Hence, p is an irregular singularity of M and $\Delta(L,p) = 2\sum_{i=-m}^{-1} i f_i t^i$. □

Now we apply this result to (3.2) where the rational function f is unknown. The theorem tells us that we find the poles and zeros of f by looking at the singularities of M. For each pole p we can find the negative coefficients in the series representation (3.3) of f. And the exponent difference of the other singularities will give us the multiplicities of the zeros of f. All this information together will almost completely give f.

Example 3.2
As in Example 2.11 we start with the modified Bessel operator L_B with $\nu = 2$. We do the same change of variables with

$$f = \frac{2(x-1)(x-2)^2}{(x-3)^2} = \frac{2x^3 - 10x^2 + 16x - 8}{x^2 - 6x + 9}$$

and get the operator

```
> LB:=x^2*Dx^2+x*Dx-(x^2+2^2):
> f:=2*(x-1)*(x-2)^2/(x-3)^2:
> L:=changeOfVars(LB,f);
```

$$L := (x-2)^3 \left(x^2 - 7x + 8\right) (x-3)^6 (x-1)^3 \partial^2 + \\ \left(x^4 - 14x^3 + 55x^2 - 84x + 46\right) (x-3)^5 (x-1)^2 (x-2)^2 \partial - \\ 4 \left(x^2 - 7x + 8\right)^3 \left(x^6 - 10x^5 + 42x^4 - 100x^3 + 158x^2 - 172x + 97\right) \\ (x-2)(x-1)$$

The singularities of this operator are $1, 2, 3, \infty, \frac{7}{2} + \frac{1}{2}\sqrt{17}$ and $\frac{7}{2} - \frac{1}{2}\sqrt{17}$. We already discovered in Example 2.11 that the latter two singularities are apparent, 1 and 2 are the regular singularities, and 3 and ∞ are irregular singularities.

We will now use part (b) of the theorem to find f. In order to do this we compute the generalized exponent at $x = 3$ using Maple:

```
> gen_exp(L,t,x=3);
```

$$[[-\frac{8}{t^2} - \frac{10}{t} + 1, t = x - 3], [\frac{8}{t^2} + \frac{10}{t} + 1, t = x - 3]] \tag{3.9}$$

The exponent difference is

$$\Delta(L, 3) = -\frac{16}{t_3^2} - \frac{20}{t_3}.$$

If we divide $\Delta(L, 3)$ by two and each coefficient by its corresponding degree we get

$$f_3 = -\frac{4}{t_3^2} - \frac{10}{t_3}.$$

This will be the polar part corresponding to t_3 in the partial fraction decomposition of f.

We do the same computations at the point $x = \infty$:

```
> gen_exp(L,t,x=infinity);
```

$$[[-\frac{2}{t} + \frac{1}{2}, t = \frac{1}{x}], [\frac{2}{t} + \frac{1}{2}, t = \frac{1}{x}]] \tag{3.10}$$

$$\Rightarrow \quad \Delta(L,\infty) = -4t_\infty^{-1}.$$

Hence, $f_\infty = -2t_\infty^{-1} = -2x$ is the polar part corresponding to t_∞.

Maple's output in (3.9) and (3.10) is not ordered. Therefore, we defined Δ modulo a factor -1. So, we don't know whether the coefficients of f_3 and f_∞ are 1 or -1 and we also don't know the constant part of f. But we know that the regular singularities 1 and 2 must be zeros of f. So we compute:

$$\begin{aligned} f_3 + f_\infty|_{x=1} &= 6, \quad & f_3 + f_\infty|_{x=2} &= 10, \\ f_3 - f_\infty|_{x=1} &= 2, \quad & f_3 - f_\infty|_{x=2} &= 2. \end{aligned}$$

We do not need the possibilities $-f_3 + f_\infty$ and $-f_3 - f_\infty$ since $L_B \xrightarrow{-x}_C L_B$. In other words, if we find a solution were $\tilde{f}$ is the parameter of the change of variables, we will also find a solution if we take the parameter $-\tilde{f}$ in the change of variables.

In the first possibility $f_3 + f_\infty$ we would need two different constants in order to make both $x = 1$ and $x = 2$ a zero of f. So the only possibility that remains is

$$\tilde{f} = f_3 - f_\infty - 2 = -\frac{4}{(x-3)^2} - \frac{10}{x-3} - 2x - 2 = -2\,\frac{x^3 - 5x^2 + 8x - 4}{(x-3)^2}$$

which, in fact, is equal to $-f$.

As a result we can represent the solutions of L with the modified Bessel functions $I_2(\tilde{f})$ and $K_2(\tilde{f})$.

At the end of the last chapter we defined exp-apparent points. These are singularities p of an operator L where $\Delta(L,p) \in \mathbb{Z}$ and the solutions at p are not logarithmic. We will now distinguish between two more cases which will correspond to zeros and poles of f.

Definition 3.3 *Let p be a singularity of the operator $L \in K[\partial]$ which is not exp-apparent. Then p is called*

(i) exp-regular $\Leftrightarrow \Delta(L,p) \in \mathbb{C}$,
(ii) exp-irregular $\Leftrightarrow \Delta(L,p) \in \mathbb{C}[1/t_p]\backslash\mathbb{C}$.

We denote the set of singularities that are exp-regular by S_{reg} and those that are exp-irregular by S_{irr}.

In situation (3.2) this would mean that all poles of f become exp-irregular points of M. Since we will look at the exponent differences modulo $\mathbb{Z}$, we might lose some information about the zeros of f. Depending on ν and the multiplicity of the zero, their exponent difference can be an integer. Hence, the zeros of f can become either exp-regular or exp-apparent points of M. However, we know that each exp-regular point of M corresponds to a zero of f.

We now consider the situation (3.1) again, where $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L_{in}$. Since $\Delta(L,p)$ modulo $\mathbb{Z}$ is invariant under exp-products and gauge transformations we get $\Delta(M,p) = \Delta(L_{in},p) \bmod \mathbb{Z}$. As a result, what was said about zeros and poles of f concerning M now also holds for L_{in}.

Corollary 3.4 *In situation (3.1) the following holds:*

(i) $p \in S_{irr} \Leftrightarrow \Delta(L_{in},p) \in \mathbb{C}[1/t_p]\backslash\mathbb{C} \Leftrightarrow$ *p is a pole of f, and*

(ii) $p \in S_{reg} \Leftrightarrow \Delta(L_{in},p) \in \mathbb{C}\backslash\mathbb{Z}$ *or L_{in} is logarithmic at $p \Rightarrow p$ is a zero of f.*

Remarks 3.5

1. The important fact that gives this result is that for the irregular singularity ∞ of the Bessel operator, we have not only $e_1, e_2 \in \mathbb{C}[t_p^{-1}]\backslash\mathbb{C}$ but also $e_1 - e_2 \in \mathbb{C}[t_p^{-1}]\backslash\mathbb{C}$. Thus, the exponent difference depends on the local parameter and this fact is finally used to separate S_{reg} and S_{irr}.

2. Since we have equivalence in (i) we can always find possibilities for the parameter f using S_{irr} up to a constant.

3. In some cases, when all exponent differences are integers, we have no exp-regular points. These cases will be very hard because we then have no information about the zeros of f.

We have already seen in the last example that we can find the polar part of f and will now summarize this in the following algorithm.

Algorithm 1: BesselSubst

Input: A differential operator $L_{in} \in K[\partial]$

Output: A list $\mathfrak{F}$ for which the following holds: If $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L_{in}$ for some $\nu \in \mathbb{C}, f \in K$ and $M \in K[\partial]$, there exists a constant $c \in \mathbb{C}$ such that $f - c \in \mathfrak{F}$.

compute singularities S of L and extract S_{irr}
for each $s \in S_{irr}$
    $d_s := \Delta(L_{in}, s)$
    let $d_s = \sum_{i=-m}^{-1} a_i t_s^i$
    $p_s := \frac{1}{2}\sum_{i=-m}^{-1} \frac{a_i}{i} t_s^i$
$\mathfrak{F} = \{\sum_{s \in S_{irr}} \pm p_s\}$
return $\mathfrak{F}$

Example 3.6
Actually, in the implementation we do not return a list of possibilities since this list can easily become huge. Instead a list of the polar parts is returned from which we can create the possibilities.

We take the operator L from the last example:

```
> LB:=x^2*Dx^2+x*Dx-(x^2+2^2):
> f:=2*(x-1)*(x-2)^2/(x-3)^2:
> L:=changeOfVars(LB,f):
```

First we compute the irregular singularities and their exponent differences with the function `irreguarSing`. It takes the operator L, a variables t and a list of roots that generate the field of constants. In our example this list is empty:

```
> Sirr:=irregularSing(L,t,{});
```

$$Sirr := [[\infty, x^{-1}, -4t^{-1}], [3, x-3, -16t^{-2} - 20t^{-1}]]$$

The output is a list of elements of the form $(p, t_p, d_p))$, where each element contains a irregular singularity p, the local parameter t_p, and the exponent difference $d_p = \Delta(L, p)$.

This result can now be passed to the `besselsubst` algorithm:

```
> besselsubst(Sirr,t,{});
```

$$[-\frac{4}{(x-3)^2} - \frac{10}{x-3}, -2x]$$

The list contains f_3 and f_∞ and now any combination $\pm f_3 \pm f_\infty$ is a possibility in $\mathcal{F}$. Nevertheless, such a list of polar parts will be considered as a list of possibilities.

3.2 Finding the Parameter ν

Lemma 3.7 *Let $L \in K[\partial]$ be a differential operator. Assume that there are transformations such that $L_B \longrightarrow L$. Then the following statements are equivalent:*

(a) The Bessel parameter is an integer, i.e. $\nu \in \mathbb{Z}$.
(b) There is an exp-regular singularity p of L such that L is logarithmic at p.

Proof. (a) $\Rightarrow$ (b) It follows from Corollary 1.23 that the solutions of L_B with $\nu \in \mathbb{Z}$ are gauge transformations of the solutions of L_B with $\nu = 0$. So, we can assume that $\nu = 0$ and compute the local solutions of L_B at $x = 0$. We have already seen in Example 1.32 that L_B has a local solution y at $x = 0$ which is logarithmic. Since $L_B \longrightarrow L$ there are three transformations connection L_B and L, i.e. $L_B \longrightarrow_C M_1 \longrightarrow_E M_2 \longrightarrow_G L$. These transformations will send y to a solution of L.

Initially, we apply a change of variables to y as we did in the proof of Theorem 3.1. For a fixed zero p of the parameter f we get a local solution y_1 of M_1 at $x = p$ which is logarithmic. This solution is obtained by taking y, substituting $x \to f$, and writing everything locally at $x = p$. This has already been done in the proof of Theorem 3.1. The logarithm changes as follows:

$$\begin{aligned}\ln(x) \to \ln(f) &= \ln\left(t_p^m c \sum_{i=0}^{\infty} f_i t_p^i\right), \quad \text{where } c \in k \text{ is such that} f_0 = 1 \\ &= \ln(c) + m\ln(t_p) + \ln\left(1 + \sum_{i=1}^{\infty} f_i t_p^i\right).\end{aligned}$$

The last logarithm can then be rewritten as a power series in t_p and the only logarithm that remains is $\ln(t_p)$. Hence, the local solution y_1 of M_1 at $x = p$ is logarithmic.

The exp-product and the gauge transformation transform the solution y_1 into a solution y_2 of M_2 and a solution y_3 of L. These two transformations will not change the logarithm which appeared in y_1, and y_3 is still logarithmic. Moreover, y_3 will be a local solution at $x = p$. So we have found a logarithmic solution of L at $x = p$. Hence, p is not a regular point of L and p cannot be exp-apparent since L is logarithmic at p. Furthermore, we know that p was a zero of f, which makes p an exp-regular point of L.

(b) $\Rightarrow$ (a) Assume that $\nu \notin \mathbb{Z}$. Then L_B has the exponents ν and $-\nu$ at the point $x = 0$. Since these two exponents are different modulo $\mathbb{Z}$ the solution space breaks down into two spaces. From Remark 1.26 we know that we can not have logarithms in the solutions. So there are two independent local solutions y_1, y_2 of L_B at the point $x = 0$ without logarithms.

Let p be an exp-regular point of L. Then p is also a zero of f. As before, we can now apply the three transformations to y_1 and y_2. This gives two independent solutions z_1 and z_2 of L which are local solutions at p. These solutions are not logarithmic, since they are obtained from y_1 and y_2. The solutions space of L at p will thus not contain any logarithmic solution. Hence, L is not logarithmic at p.

This is true for any exp-regular point p of L, which proves the statement. □

Remark 3.8

The proof showed that in situation (3.1) the local solutions of the exp-regular points arise from the local solution of L_B at the point $x = 0$. As a result, if there is one exp-regular point p at which L_{in} is logarithmic, then L_{in} is logarithmic at every exp-regular point.

The case where there are logarithmic solutions in L_{in} has been solved with Lemma 3.7 and Corollary 1.23. We will always find a solution if we take $\nu = 0$.

If there are no logarithmic solutions, we can still distinguish between different cases for ν. They are summarized in the following lemma.

Lemma 3.9 *The following statements are true for all $s \in S_{reg}$:*

(a)	*L_{in} logarithmic at s*	$\Leftrightarrow$	$\nu \in \mathbb{Z}$
(b)	$S_{reg} = \emptyset$	$\Rightarrow$	$\nu \in \mathbb{Q} \backslash \mathbb{Z}$
(c)	$\Delta(L_{in}, s) \in \mathbb{Q}$	$\Rightarrow$	$\nu \in \mathbb{Q} \backslash \mathbb{Z}$
(d)	$\Delta(L_{in}, s) \in k \backslash \mathbb{Q}$	$\Leftrightarrow$	$\nu \in k \backslash \mathbb{Q}$
(e)	$\Delta(L_{in}, s) \notin k$	$\Leftrightarrow$	$\nu \notin k$

Here we exclude logarithmic solutions from cases (b) to (e) so that we are always in exactly one of the cases. And k is the field such that $L_{in} \in k(x)[\partial]$ as we introduced in the beginning of this chapter.

Proof. Case (a) has already been proven in Lemma 3.7

Let s be a zero of the parameter f of the change of variables, then

$$\Delta(L_{in}, s) = 2m_s\nu + z \tag{3.11}$$

where m_s is the multiplicity of the zero s and $z \in \mathbb{Z}$ is some arbitrary integer. If $S_{reg} = \emptyset$, then $\Delta(L_{in}, s) \in \mathbb{Z}$ for all zeros s. Therefore $\nu \in \mathbb{Q}$ in case (b).

In the other cases equation (3.11) still holds. Since m_s and z are rational numbers, it follows that ν is always in the same field that $\Delta(L_{in}, s)$ is in and vice versa.

From the logarithmic case (a) it follows that $\nu \notin \mathbb{Z}$ in cases (b) and (c).

Note that in cases (d) and (e) it is enough to check the exponent difference at one point $s_0 \in S_{reg}$, to know whether $\nu \in k$ or not. □

In the following we will separate the five cases from the previous lemma:

(a) logarithmic case,
(b) integer case,
(c) rational case,
(d) base field case, and
(e) irrational case.

The *base field* refers to the field of constants k which is defined by the coefficients of L_{in}. Note that the integer case does not imply $\nu \in \mathbb{Z}$. It only implies $\nu \in \mathbb{Q}$ but unlike the rational case we have $S_{reg} = \emptyset$, i.e. $\Delta(L_{in}, s) \in \mathbb{Z}$ for all $s \notin S_{irr}$. This means that there can not be any exp-regular point.

The integer case is the only case where we have no information about the zeros of f. In all other cases there exists at least one $s_0 \in S_{reg}$ which must be a zero of f. We can use it to compute the constant part c of the partial fraction decomposition of f. With the other points $s \in S_{reg}$ we can verify the constant or exclude some possibilities for f.

In the logarithmic case nothing else has to be done since we already know $\nu = 0$. In cases (c), (d) and (e) we will use the exponent difference of each $s \in S_{reg}$ to compute possibilities for ν. In the integer case we need a different approach since $S_{reg} = \emptyset$.

Definition 3.10 *Consider (3.1) and let $s \in S_{reg}$. Then s is a zero of the parameter $f \in K$. Let m_s be the multiplicity of s. We define*

$$\mathcal{N}_s := \left\{ \frac{\Delta(L_{in}, s) + i}{2m_s} \,\middle|\, 0 \le i \le 2m_s - 1 \right\} \tag{3.12}$$

$$\textit{and} \quad \mathcal{N} := \left\{ \nu \in \mathbb{C}/\mathbb{Z} \,\middle|\, \forall s \in S_{reg} \exists z_s \in \mathbb{Z} : \nu + z_s \in \mathcal{N}_s \right\}.$$

We will now prove the following statements. For every singularity s the Bessel parameter ν appears in $\mathcal{N}_s$ modulo some integer. But it is enough if ν is correct modulo some integer. So every set $\mathcal{N}_s$ is a set of possibilities for ν. The solution, of course, must appear modulo an integer in every set $\mathcal{N}_s$. These possibilities are combined in $\mathcal{N}$. Therefore, the set $\mathcal{N}$ can be regarded as the intersection of all $\mathcal{N}_s$ modulo $\mathbb{Z}$.

Lemma 3.11 *Consider (3.1) and assume $S_{reg} \neq \emptyset$. Then there exists some integer $z \in \mathbb{Z}$ such that $\nu + z \in \mathcal{N}$.*

Proof. Let $s \in S_{reg}$ be a zero of f. Then there is some integer $\ell \in \mathbb{N}$ such that $\Delta(L_{in}, s) = 2m_s\nu + \ell$, i.e. $\nu = \frac{\Delta(L_{in},s)-\ell}{2m_s}$. We can always write $\frac{-\ell}{2m_s} = z_s + \frac{i}{2m_s}$ with $i, z_s \in \mathbb{Z}$ and $0 \le i \le 2m_s - 1$. Thus, $\nu = \frac{\Delta(L_{in},s)+i}{2m_s} + z_s$ and $\nu - z_s \in \mathcal{N}_s$.

That way, we find such an integer z_s for every singularity $s \in S_{reg}$. From the definition of $\mathcal{N}$ it follows that $\nu + z \in \mathcal{N}$ for some $z \in \mathbb{Z}$. □

Since we only need to find ν modulo an integer we can regard $\mathcal{N}$ as a set of possibilities for ν. However, this does not work in the integer case because then the condition in the definition of $\mathcal{N}$ is empty, which gives an infinitely large set $\mathcal{N} = \mathbb{C}/\mathbb{Z}$.

If we are not in the integer case, we already know the constant part of every possibility $f \in \mathcal{F}$. We can compute all multiplicities, all the sets $\mathcal{N}_s$, and finally also $\mathcal{N}$. We then have to try all possibilities of $\nu \in \mathcal{N}$. If there is a solution to (3.1), it must be among them.

3.3 The Algorithm

The input of our algorithm is a differential operator L_{in} and we want to know whether the solutions can be expressed in terms of Bessel functions. We assume that we are in situation (3.1). If we find a solution to that problem, then we also find the solution space of L_{in}. If we do not succeed, we know that the solutions of L_{in} can not be expressed with Bessel functions.

We will first assume $k = \mathbb{C}$ and will deal with more general fields k in the next section. Let L_{in} be a differential operator of degree two with coefficients in $K = \mathbb{C}(x)$.

Let's summarize the steps of the algorithm that we know from previous results:

A. We can compute the singularities S of L_{in} by factoring the leading coefficient of L_{in} and the denominators of the other coefficients into linear factors.

B. For each $s \in S$ we compute $d_s = \Delta(L_{in}, s)$, isolate exp-apparent points with $d_s \in \mathbb{Z}$, and differ between exp-regular singularities S_{reg} with $d_s \in \mathbb{C}$ and exp-irregular singularities S_{irr} with $d_s \in \mathbb{C}[t_s^{-1}] \backslash \mathbb{C}$.

C. We can use the exponent differences d_s for $s \in S_{irr}$ to compute possibilities $\mathcal{F}$ for the parameter f up to a constant $c \in k$.

D. In all cases but the integer case we know at least one zero of f by picking some $s_0 \in S_{reg}$. So we can also compute the missing constant c for each $\tilde{f} \in \mathcal{F}$.

E. The set $\mathcal{N}$ is a set of possibilities for ν. When not being in the integer case, this set is finite. But the set might depend on the possibility $f \in \mathcal{F}$.

F. For each pair $(\nu, f) \in \mathcal{N} \times \mathcal{F}$ we can compute an operator $M = M_{(\nu,f)}$ such that $L_B \xrightarrow{f}_C M$.

G. For each M we can decide whether $M \longrightarrow_{EG} L_{in}$ and compute the transformations.

Steps D and E have to be done by case differentiation. The basic procedure will look like this.

Algorithm 2: dsolveBessel

Input: An operator $L_{in} \in K[\partial]$.
Output: $V(L_{in})$ when it can be represented in terms of Bessel functions and *FAIL* otherwise.

compute singularities S of L_{in} and S_{reg}, S_{irr}
$\mathcal{F} = \textbf{besselSubst}(S_{irr})$
$P = \{\}$

4 **for each** $f \in \mathcal{F}$
5     $P = P \cup$ **findBesselνf**(f, S_{reg})
6 **for each** $(\nu, f) \in P$
7     $M =$ **changeOfVar**(L_B, f)
8   **if** $\exists r_0, r_1, r_2 \in K : M \xrightarrow{r_0}_E \tilde{M} \xrightarrow{r_1, r_2}_G L_{in}$ for some $\tilde{M} \in K[\partial]$ **then**
9   **return** $V(L_{in})$
10 **return** *FAIL*

The steps A and B are done in line 1, step C is done in line 2, steps D and E are separated in the function findBesselνf, and step F is done in line 7. The first solution to step G in line 8 yields a representation of the solutions of L_{in}. If at the end no pair $(\nu, f) \in P$ gave an operator M that was equivalent to L_{in}, then *FAIL* is returned.

We will now focus on how we compute the possibilities (ν, f) in the procedure findBesselνf using the case differentiation of Lemma 3.9. Particularly the integer case is still a problem. In the other cases we are going to try to reduce the number of possibilities in P since step G is the bottleneck of the algorithm.

Note that there will be no irrational case since $k = \mathbb{C}$ is the *base field*.

For each case we describe the algorithm findBesselνf which will take a possibility $f \in \mathcal{F}$ and the exp-regular points S_{reg}, and will return a set of pairs (ν, f).

3.3.1 Logarithmic Case

The logarithmic case is the easiest one because without loss of generality we can assume $\nu = 0$. We only use S_{reg} to reduce the possibilities $\mathcal{F}$.

Take a fixed possibility $f \in \mathcal{F}$. We can compute the constant by taking one $s_0 \in S_{reg}$. But the other points in S_{reg} must also be zeros and if they are not we can exclude f. This condition can also be used in the other cases where $S_{reg} \neq \emptyset$. In the logarithmic case this condition is very strong because we know that the zeros of f are *exactly* the singularities S_{reg}.

This case can be summarized as follows:

Algorithm 3: findBesselνf logarithmic case

pick one $s_0 \in S_{reg}$
$c :=$**solve**$(f\,|_{x=s_0} + c = 0, c)$
if $f\,|_{x=s} + c = 0$ for all $s \in S_{reg}$ **then**
    return $\{(0, f + c)\}$
else
    return $\{\}$

Example 3.12
We start with the modified Bessel operator L_B with $\nu = 0$ and apply a change of variables

$$x \to f = \frac{(x+2)^2(x-2)^2}{(x-1)(x-3)(x-4)}.$$

to get the operator L:

```
> LB:=x^2*D^2+x*D-(x^2+0^2):
> f:=(x+2)^2*(x-2)^2/((x-1)*(x-3)*(x-4)):
> L:=changeOfVars(LB,f):
```

Now we compute the exp-irregular singularities and their corresponding parts:

```
> Sirr:=irregularSing(L,t,{}):
> besselsubst(Sirr,t,{});
```

$$[\frac{3}{2(x-1)}, \frac{25}{2(x-3)}, \frac{48}{x-4}, x]$$

This yields the polar parts f_1, f_3, f_4 and $f\infty$ corresponding to the exp-irregular singularities $1,3,4$ and ∞. Thus, we have a set of 16 possibilities for f. Since we just need one possibility of a pair $f, -f$ there remain 8 possibilities.

We check whether the formal solutions at $x = 2$ and $x = -2$ are logarithmic:

```
> formal_sol(L,`has logarithm?`,x=2);
```

true

```
> formal_sol(L,`has logarithm?`,x=-2);
```

true

Hence, $x = 2$ and $x = -2$ are exp-regular and must be zeros of f. We evaluate the possibilities at these points to compute the constant part:

possibility $\tilde{f}$	$\tilde{f}\vert_{x=2}$	$\tilde{f}\vert_{x=-2}$
$f_1 + f_3 + f_4 + f_\infty$	-13	-33
$f_1 + f_3 + f_4 - f_\infty$	-9	-37
$f_1 + f_3 - f_4 + f_\infty$	3	15
$f_1 + f_3 - f_4 - f_\infty$	7	11
$f_1 - f_3 + f_4 + f_\infty$	-12	-36
$f_1 - f_3 + f_4 - f_\infty$	-8	-40
$f_1 - f_3 - f_4 + f_\infty$	4	12
$f_1 - f_3 - f_4 - f_\infty$	8	8

The only possibility which has both zeros is

$$\tilde{f} = f_1 - f_3 - f_4 - f_\infty - 8 = -f.$$

Hence, the algorithm findBesselνf returns only one pair $(0,-f)$ from which we can compute the solution space $V(L)$.

If $\nu \neq 0$, then guessing $\nu = 0$ can generate gauge transformations which may not be needed. In order to simplify the output of the algorithm we can take the average of exponent differences each divided by the corresponding multiplicity. If there is no gauge transformation needed to transform L_B into L_{in} this average will give the correct ν so that no gauge transformation will appear in the output.

Example 3.13
We consider the operator L that is obtained from L_B with $\nu = 2$, a change of variables $x \to f = (x+1)^2(x-5)^3$, and a gauge transformation $G = \partial + 1$:

```
> f:=(x+1)^2*(x-5)^3:
> M:=changeOfVars(x^2*D^2+x*D-(x^2+2^2),f):
> L:=gauge(M,1,1):
```

Since the only pole of f is at ∞ there is only one exp-irregular singularity of L at $x = \infty$. This gives us just one possibile $f \in \mathcal{F}$:

```
> Sirr:=irregularSing(L,t,{}):
> f:=besselsubst(Sirr,t,{})[1];
```

$$f := x^5 - 13x^4 + 46x^3 + 10x^2 - 175x$$

Since we are in the logarithmic case the zeros -1 and 5 will be exp-regular points of L. Evaluating the possibility $f \in \mathcal{F}$ at either of this points yields 125. So the parameter of the change of variables is $f = x^5 - 13x^4 + 46x^3 + 10x^2 - 175x - 125$.

We can assume $\nu = 0$ and compute M such that $L_B\big|_{\nu=0} \xrightarrow{f}_C M$. The equivalence between M and L then yields:

```
> M:=changeOfVars(x^2*D^2+x*D-(x^2+0^2),f):
> r,G:=equiv(M,L);
```

$$\begin{aligned} r,G := &-\frac{55x^2 - 160x + 73}{5x^3 - 27x^2 + 3x + 35}, \big(-109393 - 228097x + 21873x^2 + 245625x^3 \\ &+ 14250x^4 - 100290x^5 + 15558x^6 + 15762x^7 - 7707x^8 + 1487x^9 - 137x^{10} \\ &+ 5x^{11} \big)\partial + (5x-7)\left(x^2 - 14x + 9\right)(x+1)^3(x-5)^5 \end{aligned}$$

This represents the exp-product and gauge transformation $\exp(\int r)G$.

If we look at the exponent differences of the exp-regular points me might be able to specify ν more exactly:

```
> gen_exp(L,t,x=-1);
```

$$[[-5, 3, t = x + 1]]$$

```
> gen_exp(L,t,x=5);
```

$$[[-7, 5, t = x - 5]]$$

Hence, $\Delta(L, -1) = 8$ and $\Delta(L, 5) = 12$. Using the multiplicity of the corresponding zero we get $8 = 2 \cdot 2 \cdot \nu$ and $12 = 2 \cdot 3 \cdot \nu$ and therefore we take $\nu = 2$. So let M be such that $L_B\big|_{\nu=2} \xrightarrow{f}_C M$ and compute the equivalence:

```
> M:=changeOfVars(x^2*D^2+x*D-(x^2+2^2),f):
```

```
> r,G:=equiv(M,L);
```

$$r, G := 0, \partial + 1$$

This result is not only much simpler than the result we obtained before but it is also computed much faster.

3.3.2 Integer Case

This is the only case where $S_{reg} = \emptyset$. Here we have absolutely no information about the zeros of f. We know, however, that in general the multiplicity of a zero of f divides the degree of the numerator of f. For $m \in \mathbb{N}$ we can define

$$\mathcal{N}(m) := \left\{ \frac{i}{2m}, i = 1, \ldots, 2m - 1 \right\}, \tag{3.13}$$

which is similar to $\mathcal{N}_s$ in Definition 3.10. If s is a zero of f with multiplicity m and $\Delta(L_{in}, s) \in \mathbb{Z}$, then $\mathcal{N}(m) = \mathcal{N}_s$ modulo $\mathbb{Z}$.

The following lemma will help us to find ν.

Lemma 3.14 *Consider the situation (3.1). Let $\nu \in \mathbb{Q}$ and $\Delta(L_{in}, s) \in \mathbb{Z}$ for all $s \notin S_{irr}$, i.e. $S_{reg} = \emptyset$. Let n be the degree of the numerator of f. Then there exists $p, \ell \in \mathbb{Z}$ such that $p \mid n$ and $\nu + \ell \in \mathcal{N}(p)$.*

Proof. Let $\nu \in \mathbb{Q}$, then we can find $z \in \mathbb{Z}$ and $\nu_1, p \in \mathbb{N}$ such that

$$\nu = z + \frac{\nu_1}{2p}, \quad 0 < \nu_1 < 2p, \gcd(\nu_1, p) = 1. \tag{3.14}$$

So $\nu - z \in \mathcal{N}(p)$.

Let s be a zero of f with multiplicity m, then the exponent difference $\Delta(L_{in}, s) = 2m\nu$ is an integer by condition. Using the representation (3.14) of ν we get

$$2m\nu = 2mz + \frac{m\nu_1}{p} \in \mathbb{Z}.$$

Since $z, m \in \mathbb{Z}$ and $p \nmid \nu_1$ this is equivalent to $p \mid m$. So p divides all multiplicities of the zeros of f. The degree of the numerator of f is equal to the sum of these multiplicities and we get $p \mid \deg(\text{numer}(f))$, where $\text{numer}(f)$ denotes the numerator of f.

Hence, p and $\ell = -z$ satisfy the statement. $\square$

The purpose of the lemma is the following. Assume that we know the degree of the numerator $n = \deg(\text{numer}(f))$. We can then take a divisor p of n and check whether for certain constants c the monic part of the numerator of f becomes a p-th power. This can simply be done with linear algebra[1] and leads to a nonlinear system of equations for the constant c. Solving these equations gives us a set $\mathcal{C}$ of possible values for c. At the end for each $p \mid n$ we get a finite set $\mathcal{N}_p$ and a finite set $\mathcal{C}_p$. We define the sets $\mathcal{N}$ and $\mathcal{C}$ to be the union of those sets, respectively. If there is a solution of (3.1), there must also be a solution corresponding to a pair $(\nu, c) \in \mathcal{N} \times \mathcal{C}$.

If $p = 1$, we cannot do this because every polynomial is a first power of itself. Hence, we do not get any equation for c. In this case we have $\nu \in \mathcal{N}(1) = \{\frac{1}{2}\}$. For $\nu = \frac{1}{2}$ the solutions of L_B are hyperexponential solutions and $V(L_{in})$ can be found by using `DFactor` or `dsolve` but not with our algorithm.

Example 3.15
Let L be the modified Bessel operator with parameter $\nu = \frac{1}{2}$:

```
> L:=x^2*D^2+x*D+(x^2-(1/2)^2);
```

$$L := x^2\partial^2 + x\partial + x^2 - \frac{1}{4}$$

We can factor this operator with Maple

```
> L2:=DFactor(L);
```

$$L_2 := [\partial + \frac{RootOf\left(\_Z^2+1\right)\left(2x - RootOf\left(\_Z^2+1\right)\right)}{2x}, \\ \partial - \frac{RootOf\left(\_Z^2+1\right)\left(2x + RootOf\left(\_Z^2+1\right)\right)}{2x}]$$

and compute the solutions

```
> dsolve(diffop2de(op(1,Lfactorized),y(x)),y(x));
```

$$y(x) = \frac{\_C1 \exp(-RootOf\left(\_Z^2+1\right)x)}{\sqrt{x}}$$

```
> dsolve(diffop2de(op(2,Lfactorized),y(x)),y(x));
```

$$y(x) = \frac{\_C1 \exp(RootOf\left(\_Z^2+1\right)x)}{\sqrt{x}}$$

[1] see appendix for details on the algorithm `ispower`

We can also find the solutions of L directly

```
> dsolve(diffop2de(L,y(x)),y(x));
```

$$y(x) = \frac{\_C1 \sin(x)}{\sqrt{x}} + \frac{\_C2 \cos(x)}{\sqrt{x}}$$

For all $\nu = n + \frac{1}{2}$ with $n \in \mathbb{Z}$ we can compute such elementary solutions, see [24, 7.3] for more details.

The problem remains how to find the degree n of the numerator of f without knowing the constant part c.

Lemma 3.16 *Consider the integer case of (3.1), where $\Delta(L_{in}, s) \in \mathbb{Z}$ for all singularities $s \notin S_{irr}$ and $2\nu = \frac{\nu_1}{p}$ for some $\nu_1 \in \mathbb{Z}, p \in \mathbb{N}$ and $\gcd(\nu_1, p) = 1$.*

(a) If $\infty \in S_{irr}$, then $\deg(\mathrm{numer}(f)) = \deg(\mathrm{numer}(f+c))$ for all $c \in \mathbb{C}$.

(b) If $\infty \notin S_{irr}$, then $p \mid \deg(\mathrm{numer}(f)) \Leftrightarrow p \mid \deg(\mathrm{denom}(f))$.

Here, $\mathrm{denom}(f)$ denotes the denominator of f.

Proof. After using the exp-irregular points S_{irr} to find the polar parts f has the form

$$f = \frac{f_1}{f_2} + c + f_3, \tag{3.15}$$

where $f_1, f_2, f_3 \in k[x]$ and $\deg(f_1) < \deg(f_2)$ or $f_1 = 0$. The polar parts for $s \in S_{irr} \backslash \{\infty\}$ are combined in $\frac{f_1}{f_2}$. The polynomial f_3 is the polar part of $\infty \in S_{irr}$.

(a) In this case $\infty \in S_{irr}$ and hence $f_3 \neq 0$. So c does not effect the degree of the numerator of f.

(b) Since $\infty \notin S_{irr}$, $f_3 = 0$ in equation (3.15) and $f = \frac{f_1}{f_2} + c$ with $\deg(f_1) < \deg(f_2)$.

Case 1: If $c \neq 0$, then $\deg(\mathrm{numer}(f)) = \deg(\mathrm{denom}(f))$ and nothing remains to be proven.

Case 2: If $c = 0$, then ∞ is a zero of f. The multiplicity m must be a multiple of p. Otherwise $\Delta(L_{in}, \infty) \notin \mathbb{Z}$ since $\Delta(L_{in}, \infty) = 2m\nu + z$ for some $z \in \mathbb{Z}$ and $2\nu = \frac{\nu_1}{p}$. Hence, $m = kp$ for some $k \in \mathbb{N}$.

This multiplicity of the point ∞ is $m = \deg(\mathrm{denom}(f)) - \deg(\mathrm{numer}(f))$. This can be seen if $f(\frac{1}{x})$ is written as power series at the point 0. In total we get $\deg(\mathrm{numer}(f)) = \deg(\mathrm{denom}(f)) - kp$ for some $k \in \mathbb{N}$ and this proves (b). $\square$

Algorithm 4: findBesselνf integer case

$P := \{\}$
if $\infty \in S_{irr}$ **then**
  $n := \deg(\text{numer}(f))$
else
  $n := \deg(\text{denom}(f))$
for each $p \mid n$
  $g :=$ **ispower**$(\text{numer}(f+c), x, p)$ *(here c is a variable)*
  $\mathcal{C} :=$ **solve**$(\text{numer}(f+c) - g^p = 0, c)$ *(find solutions $c \in k$)*
  for each $c \in \mathcal{C}$
    $P := P \cup \{(\nu, f+c) \mid \nu \in \mathcal{N}(p)\}$
return P

Example 3.17
1. Let L be such that $L_B\big|_{\nu=\frac{1}{4}} \xrightarrow{f}_C L$ with $f = 3(x-2)^2$.

```
> LB:=x^2*D^2+x*D-(x^2+nu^2);
> f:=3*(x-2)^2:
> L:=changeOfVars(subs(nu=1/4,LB),f:
```

The only exp-irregular singularity is $\infty \in S_{irr}$ so we get just one possible $f \in \mathcal{F}$:

```
> Sirr:=irregularSing(L,t,{}):
> F:=besselsubst(Sirr,t,{});
```

$$F := [3x^2 - 12x]$$

Hence, $f = 3x^2 - 12x + c$ for some constant c. Since the degree n of f is 2 the only divisor we have to check is $p = 2$. The monic part of f is $x^2 - 4x + \frac{c}{3}$ and it must be a power of $x-2$:

```
> g:=ispower(x^2-4x+c/3,x,2);
```

$$g := x - 2$$

Hence, we get $f + c - 3g^n = c - 12$ and therefore $c = 12$. The possible pairs that the algorithm returns are

$$\left\{ \left(\frac{1}{4}, 3x^2 - 12x + 12\right), \left(\frac{1}{2}, 3x^2 - 12x + 12\right), \left(\frac{3}{4}, 3x^2 - 12x + 12\right) \right\}.$$

The second pair can be excluded since our algorithm doesn't work for $\nu = \frac{1}{2}$. In the remaining two cases we can find a solution. For $\nu = \frac{1}{4}$ no gauge transformation or exp-product is needed:

```
> M:=changeOfVars(subs(nu=1/4,LB),f):
```

```
> equiv(L,M);
```

$$0,1$$

Therefore,

$$V(L)=\left\{C_1\,I_{1/4}(3x^2-12x+12)+C_2\,K_{1/4}(3x^2-12x+12)\;\middle|\;C_1,C_2\in\mathbb{C}\right\}.$$

For $\nu=\frac{3}{4}$ the solution is:

```
> M:=changeOfVars(subs(nu=3/4,LB),f):
> equiv(L,M);
```

$$-2\,(x-2)^{-1},(2x-4)\,\partial+1$$

Although the second solution is more complex, we get solutions in terms of Bessel functions in both cases.

2. We consider a second operator L:

```
> f:=(x-2)^4/(x-1):
> L:=changeOfVars(subs(nu=1/4,LB),f):
```

Since 1 is a pole of f and the degree of the numerator if greater than the degree of the denominator we have irregular singularities at 1 and ∞ with the following polar parts:

```
> Sirr:=irregularSing(L,t,{}):
> besselsubst(Sirr,t,{});
```

$$[x^3-7x^2+17x,-\frac{1}{x-1}]$$

Hence, we have two possibilities

$$f_1=x^3-7x^2+17x-\frac{1}{x-1}+c=\frac{x^4-8x^3+24x^2-17x-1+cx-c}{x-1}$$

$$\text{and}\quad f_2=x^3-7x^2+17x+\frac{1}{x-1}+c=\frac{x^4-8x^3+24x^2-17x+1+cx-c}{x-1}.$$

The degree of the numerator of both possibilities is $n=4$. For the possibility f_1 there will be no constants $c\in k$ satisfying the conditions. So we only do the computations for f_2. We first check $p=2$:

```
> f2:=x^3-7*x^2+17*x+1/(x-1)+c:
> g:=ispower(numer(f2),x,2);
```

$$g:=x^2-4x+4$$

```
> factor(expand(numer(f2)-g^2));
```

$$(15+c)(x-1)$$

So $\mathcal{C}_2 = \{-15\}$. This will also work for $p = 4$:

```
> g:=ispower(numer(f2),x,4);
```

$$g := x-2$$

```
> factor(expand(numer(f2)-g^4));
```

$$(15+c)(x-1)$$

Therefore, we also have $\mathcal{C}_4 = \{-15\}$ and every $\nu \in \left\{\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}\right\}$ gives a possible pair $(\nu, f_2 - 15)$. From these six possibilities only $\nu = \frac{1}{4}$ and $\nu = \frac{3}{4}$ will lead to a result. As in the previous example we will need a gauge transformation and an exp-product to represent the solutions of L with $\nu = \frac{3}{4}$:

```
> M:=changeOfVars(subs(nu=3/4,LB),subs(c=-15,f2)):
> equiv(L,M);
```

$$-\frac{12x^2-21x+10}{(x-2)(x-1)(3x-2)}, 4\,(x-1)(x-2)\,\partial + 3x - 2$$

For $\nu = \frac{1}{4}$ no gauge transformation needed and we get the solution space:

$$V(L) := \left\{ C_1 I_{1/4}\left(\frac{(x-2)^4}{(x-1)}\right) + C_2 K_{1/4}\left(\frac{(x-2)^4}{(x-1)}\right) \;\middle|\; C_1, C_2 \in \mathbb{C} \right\}.$$

3.3.3 Rational Case

In the rational case we can use some of the results of the integer case to exclude some possibilities (ν, f). From Lemma 3.9 we know that $\nu \in \mathbb{Q}$.

Let a possible $f \in \mathcal{F}$ be fixed. The constant part of f can be computed using one exp-regular point and, as in the logarithmic case, we can check if all exp-regular points become zeros of f. We can then also determine the multiplicity m_s of each exp-regular point $s \in S_{reg}$.

Since ν is rational the exponent difference $\Delta(L_{in}, s) = 2m_s\nu$ can be an integer for some zero s if the corresponding multiplicity m_s is a multiple of the denominator of 2ν.

Let $h = \mathrm{numer}(f)/\prod_{s\in S_{reg}}(x-s)^{m_s}$. The exp-apparent zeros of f are zeros of this polynomial. Now similar arguments as in the integer case are used.

Consider the case $\deg(h) > 0$ and let z be a zero of h which is also a zero of f. Since $z \notin S_{reg}$ we know for the exponent difference $\Delta(L_{in}, z) = 2m_z\nu \in \mathbb{Z}$. So m_z

must be a multiple of $p = \text{denom}(2\nu)$. Since this is true for all zeros the monic part of the polynomial h must be a p-th power.

This information is used in the following algorithm.

Algorithm 5: findBesselνf rational case

$P := \{\}$
$c :=$ **solve**$(f\,|_{x=s_0} + c = 0, c)$
if $f\,|_{x=s} + c = 0$ for all $s \in S_{reg}$ **then**
    $h := \text{numer}(f + c)$
    for each $s \in S_{reg}$
        $h := h/(x-s)^{m_s}$
    $\mathcal{N} := \cap_{s \in S_{reg}} \mathcal{N}_s \mod \mathbb{Z}$
    for each $\nu \in \mathcal{N}$
        $p := \text{denom}(2\nu)$
        if $h = g^p$ for some $g \in k[x]$ **then**
            $P := P \cup \{(\nu, f + c)\}$
return P

Example 3.18
Let L be such that $L_B\big|_{\nu=2/3} \xrightarrow{f}_C L$ with $f = (x-2)^2(x-3)^3(x-5)$.

```
> LB:=x^2*D^2+x*D-(x^2+nu^2):
> f:=(x-2)^2*(x-3)^3*(x-5):
> L:=changeOfVars(subs(nu=2/3,LB),f):
```

Then L has one exp-irregular singularity from which we get one possible $f \in \mathcal{F}$:

```
> Sirr:=irregularSing(L,t,{}):
> besselsubst(Sirr,t,{}):
```

$$[x^6 - 18x^5 + 132x^4 - 506x^3 + 1071x^2 - 1188x]$$

The generalized exponents at the points 2, 3 and 5 are:

```
> gen_exp(L,t,x=2);
```

$$[[-\frac{4}{3}, t = x-2], [\frac{4}{3}, t = x-2]]$$

```
> gen_exp(L,t,x=3);
```

$$[[-2, 2, t = x-3]]$$

```
> gen_exp(L,t,x=5);
```

$$[[-\frac{2}{3}, t = x-5], [\frac{2}{3}, t = x-5]]$$

So the point 3 is exp-apparent since $\Delta(L,3) \in \mathbb{Z}$. The other two points are exp-regular and the differences are $\Delta(L,2) = \frac{8}{3}$ and $\Delta(L,5) = \frac{4}{3}$. One of these two points can be used to find the constant part of the possibility which becomes

$$f_1 = x^6 - 18x^5 + 132x^4 - 506x^3 + 1071x^2 - 1188x + 540.$$

To compute the multiplicities of the zeros 2 and 5 we divide f_1 by $x-2$ and $x-5$ successively until the remainder is non-zero. That way we find $m_2 = 2, m_5 = 1$ and the remainder

```
> h:=normal(f1/(x-2)^2/(x-5));
```

$$h := x^3 - 9x^2 + 27x - 27$$

With $\Delta(L,2) = \frac{8}{3}$ and $\Delta(L,5) = \frac{4}{3}$ we can determine the sets $\mathcal{N}_2$ and $\mathcal{N}_5$ with equation (3.12):

$$\mathcal{N}_2 = \left\{\frac{2}{3}, \frac{11}{12}, \frac{7}{6}, \frac{17}{12}\right\}$$
$$\text{and} \quad \mathcal{N}_5 = \left\{\frac{2}{3}, \frac{7}{6}\right\}.$$

The intersection is $\mathcal{N} = \{\frac{2}{3}, \frac{7}{6}\}$ and we get two possibilities $\nu \in \mathcal{N}$. In both cases the denominator of 2ν is 3. Since $h = (x-3)^3$ the alogrithm returns two pairs: $(\frac{2}{3}, f)$ and $(\frac{7}{6}, f)$.

For $\nu = \frac{7}{6}$ the operator M with $L_B\big|_{\nu=\frac{7}{6}} \xrightarrow{f}_C M$ is not equivalent to L:

```
> M:=changeOfVars(subs(nu=7/6,LB),f):
> equiv(L,M);
```

$$0$$

The first pair with $\nu = \frac{2}{3}$ will certainly be a solution since it matches the values we started with.

3.3.4 Base Field Case

Let $\nu \in k, \nu \notin \mathbb{Q}$, then for all zeros z of f we have $\Delta(L_{in}, z) \in k \backslash \mathbb{Q}$, i.e. $z \in S_{reg}$. In this case we know all the zeros of f and we have the following statement for their multiplicities.

Lemma 3.19 *Consider (3.1). Let $\nu \in k \backslash \mathbb{Q}$, $S_{reg} = \{s_1, \ldots, s_n\}$ and $d_i = \Delta(L_{in}, s_i)$. Then we can do the following steps:*

1. *Compute $r_i, t_i \in \mathbb{Q}$ such that $d_i = r_i d_1 + t_i$.*

2. *Let* $a_i, b_i \in \mathbb{Z}$ *be such that* $r_i = \frac{a_i}{b_i}$ *and* $\gcd(a_i, b_i) = 1$.
3. *Let* $\ell = \operatorname{lcm}(b_i, 1 \le i \le n)$.

Then the monic part of the numerator of f *is a power of* $h \in k[x]$ *where*

$$h = \prod_{i=1}^{n} (x - s_i)^{\ell r_i}. \tag{3.16}$$

Proof. Let m_i be the multiplicity of s_i as a zero of f. Since a gauge transformation can change the exponent difference by an integer we know

$$d_i = 2m_i\nu + z_i \text{ for some } z_i \in \mathbb{Z}. \tag{3.17}$$

This equation yields for $i = 1$ the equation

$$\nu = \frac{d_1 - z_1}{2m_1}.$$

Plugging this into (3.17) we get

$$d_i = \frac{m_i}{m_1} d_1 + z_i - \frac{m_i z_1}{m_1}.$$

So the numbers

$$r_i = \frac{m_i}{m_1} \quad \text{and} \quad t_i = z_i - \frac{m_i z_1}{m_1} \tag{3.18}$$

satisfy the equation in step 1. Since $d_i \notin \mathbb{Q}$ the rational factor r_i is unique.

Now let $a_i, b_i \in \mathbb{Z}$ be such that $r_i = \frac{a_i}{b_i}$ and $\gcd(a_i, b_i) = 1$. Then $m_i = \frac{a_i}{b_i} m_1$. Since $m_i \in \mathbb{Z}$ and $b_i \nmid a_i$ we obtain $b_i \mid m_1$. Then also $\ell := \operatorname{lcm}(b_i, 1 \le i \le n) \mid m_1$. We use $m_i = r_i m_1$ and finally get $\ell r_i \mid m_i$. So the exponents in (3.16) each divide the multiplicity in the numerator of f.

Let $p_i \in \mathbb{N}$ be such that $\ell r_i p_i = m_i$. To prove (3.16) we have to see that all p_i are equal. Using the equation for r_i in (3.18) yields $\ell p_i = m_1$. So $p = p_i = \frac{m_1}{\ell}$ is independent of i and the numerator of f must be a scalar multiple of a p-th power of h. $\square$

Algorithm 6: findBesselνf base field case

$P := \{\}$
let $S_{reg} = \{s_1, \ldots, s_n\}$
for $i = 1, \ldots, n$
    $d_i := \Delta(L_{in}, s_i)$
    compute r_i, t_i, such that $d_i = r_i d_1 + t_i$
$l := \mathrm{lcm}(\mathrm{denom}(r_i), i = 1, \ldots, n)$
$h := \prod_{i=1}^{n} (x - s_i)^{l r_i}$
$c :=$**solve**$(f\,|_{x=s_0} + c = 0, c)$
if $f\,|_{x=s} + c = 0$ for all $s \in S_{reg}$
  and $\mathrm{numer}(f + c) = h^p$ for some $p \in \mathbb{N}$ **then**
    $\mathcal{N} := \cap_{s \in S_{reg}} \mathcal{N}_s \mod \mathbb{Z}$
    for each $\nu \in \mathcal{N}$
        $P := P \cup \{(\nu, f)\}$
return P

Note that h only has to be computed once, and is independent of the possibility f that we are dealing with.

Example 3.20
Let L be the operator that we obtain from L_B with the change of variables

$$x \to f = \frac{(x-1)^2 (x-2)^4}{(x+3)(x+4)}$$

and the undetermined constant Bessel parameter ν.

```
> LB:=x^2*D^2+x*D-(x^2+nu^2):
> f:=(x-1)^2*(x-2)^4/((x+3)*(x+4)):
> L:=changeOfVars(LB,f):
```

Since ν will occur in L the field of constants we work with ist $k = \mathbb{Q}(\nu)$. From the exp-irregular singularities we get the polar parts f_∞, f_4 and f_3:

```
> Sirr:=irregularSing(L,t,{}):
> besselsubst(Sirr,t,{});
```

$$[x^4 - 17x^3 + 148x^2 - 920x, -\frac{32400}{x+4}, -\frac{10000}{x+3}]$$

Thus, we get the four possibilities

$$f_1 = f_\infty + f_4 + f_3, \qquad f_2 = f_\infty + f_4 - f_3,$$
$$f_3 = f_\infty - f_4 + f_3, \quad \text{and } f_4 = f_\infty - f_4 - f_3.$$

The exp-regular singularities 1 and 2 have the following generalized exponents:

```
> g:=gen_exp(L,t,x=1);
```

$$g := [[2\nu, t = x-1], [-2\nu, t = x-1]]$$

```
> g:=gen_exp(L,t,x=2);
```

$$g := [[4\nu, t = x-2], [-4\nu, t = x-2]]$$

Hence, $\Delta(L,1) = 4\nu$ and $\Delta(L,2) = 8\nu$. From $\Delta(L,2) = 2\Delta(L,1) + 0$ we get

$$h = (x-1)(x-2)^2 = x^3 - 5x^2 + 8x - 4.$$

The only possibility where both exp-regular points become a zero and the numerator is a power of h is $f_2 + 4768$. From the sets

$$\mathcal{N}_1 = \left\{\nu, \nu+\frac{1}{4}, \nu+\frac{1}{2}, \nu+\frac{3}{4}\right\}$$
$$\text{and} \quad \mathcal{N}_2 = \left\{\nu, \nu+\frac{1}{8}, \nu+\frac{1}{4}, \nu+\frac{3}{8}, \nu+\frac{1}{2}, \nu+\frac{5}{8}, \nu+\frac{3}{4}, \nu+\frac{7}{8}\right\}$$

we finally get the possibilities for ν:

$$\mathcal{N} = \left\{\nu, \nu+\frac{1}{4}, \nu+\frac{1}{2}, \nu+\frac{3}{4}\right\}.$$

Hence, we have four pairs we have to try using `equiv`. The only possibility that yields a solution is $(\nu, f_2 + 4768)$.

The condition that the numerator of f is a power of h didn't really help in the last example since the other possibilities could already be excluded due to the other condition. In the following example with just one exp-regular point this is not the case.

Example 3.21
Let L be the operator that we obtain from L_B with the change of variables

$$x \to f = \frac{(x-2)^2}{x-1}$$

and $\nu = \sqrt{2} + \frac{1}{2}$. We obtain:

```
> LB:=x^2*D^2+x*D-(x^2+nu^2):
```

```
> f:=(x-2)^2/(x-1):
```

```
> L:=changeOfVars(subs(nu=sqrt(2)+1/2,LB),f);
```

$$L := 4x(x-2)^2(x-1)^4\partial^2 + 4\left(-2+x^2\right)(x-2)(x-1)^3\partial \\ -x^3\left(4x^4-32x^3+105x^2-146x+73+4\sqrt{2}x^2-8\sqrt{2}x+4\sqrt{2}\right)$$

Therefore the field of constants is $k = \mathbb{Q}(\sqrt{2})$. The exp-regular singularity at $x = 2$ has the generalized exponents:

```
> g:=gen_exp(L,t,x=2);
```

$$[[1+2\sqrt{2}, t = x-2], [-1-2\sqrt{2}, t = x-2]]$$

Since this is the only exp-regular singularity there is just one factor in (3.16) and we get $h = x-2$.

From the exp-irregular points we get $\mathcal{F}$:

```
> Sirr:=irregularSing(L,t,{}):
> besselsubst(Sirr,t,{});
```

$$[-x, -\frac{1}{x-1}]$$

Since $x = 2$ must be a zero we get the two possibilities

$$f_1 = -\frac{x^2-4x+4}{x-1} \quad \text{and} \quad f_2 = -\frac{x(x-2)}{x-1}.$$

Now f_2 can clearly be excluded because the numerator is not a power of $h = x-2$. In f_1 the multiplicity of the zero at 2 is $m_2 = 2$.

Finally, we divide $\Delta(L,2) = 2+4\sqrt{2}$ by $2m_2$ to get

$$\nu \in \mathcal{N} = \mathcal{N}_2 = \left\{\sqrt{2}+\frac{1}{2}, \sqrt{2}+\frac{3}{4}, \sqrt{2}+1, \sqrt{2}+\frac{5}{4}\right\}.$$

So there are four pairs (ν, f) from which just $(\sqrt{2}+\frac{1}{2}, f_1)$ will yield a solution with `equiv`.

3.4 Solving Over a General Field k

For now, we were just working over the constant field $\mathbb{C}$ and we haven't thought of the speed of the algorithm yet. We started by computing all the singularities of L and did some computations with them. So what we actually did is factor the leading coefficient $l(x)$ of L into linear factors. This can be very expensive and will lead to a huge (but finite) extension of $\mathbb{Q}$, in which all the other computations take place.

But this is not necessary. In this section we will discuss how we can work over a finite extension of $\mathbb{Q}$, which is much smaller than the earlier one.

We will use the following setting. Let k be a finite extension of $\mathbb{Q}$ such that the input operator L is defined over $K = k(x)$. So k is the field of constants and L has coefficients in K. Let $\prod_{i=1}^{n} l_i(x)$ be a factorization of $l(x)$ over K. For each $l_i(x)$ we pick one zero p_i. Furthermore, let $\sigma \in \mathrm{Hom}_k(k(s), \bar{k})$ be an embedding of $k(p)$ in $\bar{k}$ that keeps k fixed and we define the *trace* of a element $a \in k(s)$:

$$\mathrm{Tr}(a) := \sum_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} \sigma(a).$$

We will now focus on each step of the algorithm and the changes that have to be made.

A. **Singularities**
When we factor the coefficients of L in $k[x]$ we get irreducible factors whose degree can be greater than one. For each irreducible factor, we fix one zero. The singularities then are

$$S = \{\sigma(s) \mid s \text{ zero of irred. factor}, \sigma \in \mathrm{Hom}_k(k(s), \bar{k})\}.$$

Now fix a factor $q(x) = \sum_{i=0}^{n} q_i x^i$, and let s be a zero of $q(x)$ and $\sigma \in \mathrm{Hom}_k(k(s), \bar{k})$.

B. **Generalized exponents**
In the computation of the generalized exponent at the point $x = s$ the field $k(s)$ is taken as the field of constants. An important fact that we will use is

$$\mathrm{gexp}(L, \sigma(s)) = \sigma(\mathrm{gexp}(L, s)). \tag{3.19}$$

Similarly, if y is a local solution at the point $x = s$, then $\sigma(y)$ is a local solution at $x = \sigma(s)$ because the operator cannot distinguish between the points s and $\sigma(s)$. Hence, $\Delta(L, \sigma(s)) = \sigma(\Delta(L, s))$.

Since all our results were based on generalized exponents and exponent differences we can use (3.19) to transfer results for s to $\sigma(s)$. The sets S_{reg} and S_{irr} always just contain one zero for each irreducible factor $q(x)$.

C. **besselSubst**
Let $s \in S_{irr}$. We can compute the polar part f_s corresponding to s. The polar part corresponding to $\sigma(s)$ is $f_{\sigma(s)} = \sigma(f_s)$. The reason is obvious: if $f \in k(x)$ and $s \notin k$, then the series expansion of f at s and at $\sigma(s)$ are equal modulo σ. This also follows from (3.19) since the polar parts depend on the generalized exponents.

There was a factor -1 allowed for every polar part. We had to do this, because we have no ordering in the output of `gen_exp`. But now we use σ and the factor is the same for f_s and $f_{\sigma(s)}$. So we can simply apply trace to f_s:

$$\mathrm{Tr}(f_s) = \sum_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} \sigma(f_s) = \sum_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} f_{\sigma(s)}.$$

The result is the polar part of f corresponding to the irreducible polynomial $q(x)$.

D. **Compute constant of f**
Let $f = \tilde{f} + c$ for some $\tilde{f} = \tilde{f}(x) \in \mathcal{F}$ be a possibility for the parameter in the change of variables. If $S_{reg} \neq \emptyset$, then we know at least one zero of f. Assume $s \in S_{reg}$, then we compute c such that $f(s) = 0$. If $s \notin k$, we would get $c \notin k$ in general. But, we also know that $f(\sigma(s)) = \sigma(f(s))$, so all $\sigma(s)$ must be zeros of f. Thus, $\mathrm{Tr}(f(s))$ must be zero. Since $\mathrm{Tr}(f(s)) \in k(c)$, we can compute a constant $c \in k$ that satisfies the conditions.

When we have computed the constant, we always check whether another point $s \in S_{reg}$ is also zero. All $\sigma(s)$ have to be zeros of f, so the minimal polynomial $\mathrm{minpol}(s)$ of s has to be a factor of the numerator of f.

E. **The set $\mathcal{N}$**
For these computations we only used exp-regular points $s \in S_{reg}$ with exponent difference $\Delta(L_{in}, s) = 2m_s\nu$. Since $m_{\sigma(s)} = m_s$, we have not only $\Delta(L_{in}, \sigma(s)) = \sigma(\Delta(L_{in}, s))$ but also $\Delta(L_{in}, \sigma(s)) = \Delta(L_{in}, s)$ for all $s \in S_{reg}$. So $\mathcal{N}_s = \mathcal{N}_{\sigma(s)}$ and we do not need the other zeros to compute $\mathcal{N}$.

F, G. **Compute M, exp-product and gauge transformation**
When we have computed the parameter ν and parameter f of the change of variables, we can determine M and decide whether there exists an exp-product and a gauge transformation of the desired kind. So, from here, everything works as before.

3.4.1 Changes in the Case Separation

We will now focus on the changes that have to be done during the case separation.

1. **Logarithmic case**
It has already been said, that we compute the constant $c \in k$ by taking the trace $\mathrm{Tr}(f_s)$ for some $s \in S_{reg}$ and that we can determine whether $s \in S_{reg}$ is a zero, by checking whether $\mathrm{minpol}(s)$ is a factor of f.

2. **Integer case**
   This case does not use S_{reg}, so we do not have to make a change.

3. **Rational case**
   After computing the constant c, we used the multiplicities m_s. Let h be the numerator of f. The multiplicity m_s can be defined as the biggest $m_s \in \mathbb{N}$ such that $\mathrm{minpol}(s)^{m_s} \mid h$ and we can compute $h/\mathrm{minpol}(s)^{m_s}$, which is a polynomial[2].

   Since $\mathcal{N}_s = \mathcal{N}_{\sigma(s)}$, the rest of the algorithm does not depend on s.

4. **Base field case**
   The steps that were explained in Lemma 3.19 were based on the relation between two singularities. So the results will still be true if just a subset of the singularities is used and we can take one singularity for each irreducible factor. Since s and $\sigma(s)$ have the same multiplicity and $\mathrm{numer}(f) \in k[x]$, we get
   $$h \mid \mathrm{numer}(f) \quad \Leftrightarrow \quad \mathrm{N}(h) \mid \mathrm{numer}(f)$$
   for any $h \in \mathbb{C}[x]$. Here, $\mathrm{N}(a)$ is the norm of an element a such that $\mathrm{N}(a) = \prod_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} \sigma(a)$ for any $a \in k(s)$. Both divisions should be considered in $\mathbb{C}[x]$. But since $\mathrm{N}(h)$ and $\mathrm{numer}(f)$ both are polynomials in $k[x]$ the right-hand division can be done in $k[x]$.

   So, we only have to take the norm in line 7 of Algorithm 6 and the rest will work as before.

3.4.2 Irrational Case

As in Lemma 3.19 of the base field case we can find rational factors connecting the exponent differences.

Lemma 3.22 *Let $\nu \in \bar{k}$ and $\nu^2 \in k$, $S_{reg} = \{s_1, \ldots, s_n\}$ and $d_i = \Delta(L_{in}, s_i)$. Then we can find $r_i, t_i \in \mathbb{Q}$ such that $d_i = r_1 d_1 + t_i$ and the numerator of f has the form*

$$h = \prod_{i=1}^{n} (x - s_i)^{\ell r_i},$$

where $\ell = \mathrm{lcm}(b_i, 1 \le i \le n)$ for $r_i = \frac{a_i}{b_i}$.

---
[2] compare with line 6 in Algorithm 5

Proof. The proof is an analogue to the proof of Lemma 3.19. We only compute the rational number r_i differently:

We know that the exponent difference $d_i = \Delta(L_{in}, s_i)$ satisfies $d_i = 2m_i\nu + z_i$ for some $z_i \in \mathbb{Z}$. The integer part z_i can easily be removed from d_i since it is the only constant c such that $d_i - c$ is a zero of a polynomial of the from $x^2 - p_i \in k[x]$. We compute those $p_i = 4m_i^2\nu^2$.

Finally $r_i = \sqrt{\frac{p_i}{p_1}} = \frac{m_i}{m_1}$ and the rest works as in the base field case. $\square$

The algorithm we obtain is very similar to the algorithm in the base field case. The two differences are the computation of the rational factor r_1 and that we can directly specify ν.

Algorithm 7: findBesselνf irrational case

$P := \{\}$
let $S_{reg} = \{s_1, \dots, s_n\}$
$d_i := \Delta(L_{in}, s_i)$
for $i = 1, \dots, n$
  compute polynomial $x^2 - p_i$
  $r_i := \sqrt{\frac{p_i}{p_1}}$
$l := \mathrm{lcm}(\mathrm{denom}(r_i), i = 1, \dots, n)$
$h := \prod_{i=1}^{n}(x - s_i)^{lr_i}$
$c :=$**solve**$(f\,|_{x=s_0} + c = 0, c)$
if $f\,|_{x=s} + c = 0$ for all $s \in S_{reg}$
  and $\mathrm{numer}(f) = h^p$ for some $p \in \mathbb{N}$ **then**
  $\nu := \frac{\sqrt{p_1}}{2m_1}$
  $P := P \cup \{(\nu, f)\}$
return P

3.4.3 Constant Factor of f

Remember that in general we allow any $f \in \mathbb{C}(x)$. In the steps we listed above we will only find this parameter if $f \in k(x)$. Yet, we didn't prove that this is enough. We do not know whether there exist an operator $L_{in} \in k(x)[\partial]$ which can be derived from L_B with a parameter $f \notin k(x)$ in the change of variables.

Example 3.23
We take the Bessel operator L_B. We apply a change of variables with $f = cx$, where c is some constant:

```
> LB:=x^2*D^2+x*D-(x^2+nu^2):
```

```
> L:=changeOfVars(LB,c*x);
```

$$L := x^2\partial^2 + x\partial - c^2x^2 - \nu^2$$

We see that the constant c appears squared in the operator L. So if k is the field over which L is defined, then $c^2 \in k$.

This example shows that we have to consider constant factors of f which are not in k. We will restrict the parameter of the change of variables to cf with $c^2 \in k$ and $f \in k(x)$. However, it is still to prove that this is sufficient.

We have seen in Algorithm 1 that we find the parameter f up to some signs and a constant. So one would assume that this still works with a constant factor.

The constant c appears in the generalized exponent of L at $x = \infty$:

```
> gen_exp(L,t,x=infinity);
```

$$[[-\frac{c}{t} + \frac{1}{2}, t = \frac{1}{x}], [\frac{c}{t} + \frac{1}{2}, t = \frac{1}{x}]]$$

Excluding the constant term $\frac{1}{2}$, c is a factor of each monomial in the generalized exponents. So c will also be a factor of $\Delta(L,\infty)$ and also a factor of every possibility $f \in \mathcal{F}$. But unfortunately this is not always the case.

Next we consider the operator L which is obtained from L_B with $\nu = 2$ and the change of variables

$$x \rightarrow f = \frac{\sqrt{2}}{(x^2-2)(x-1)}.$$

```
> L:=changeOfVars(subs(nu=2,LB),f):
```

$$\begin{aligned} L := &\left(3x^2-2x-2\right)\left(x^2-2\right)^4(x-1)^4\partial^2 + \\ &\left(3x^4-4x^3+2x^2-8x+8\right)\left(x^2-2\right)^3(x-1)^3\partial - \\ &2\left(9+2x^6-4x^5-6x^4+16x^3-16x\right)\left(3x^2-2x-2\right)^3 \end{aligned}$$

Note that the field of constants, which is always defined by the constants in L, is $k = \mathbb{Q}$, but $f \notin \mathbb{Q}(x)$. The generalized exponent at the root $\sqrt{2}$ of x^2-2 is:

```
> g:=gen_exp(L,t,x=sqrt(2)):
```

$$[[-\frac{1}{2t} - \frac{\sqrt{2}}{2t} + \frac{1}{2}, t = x - \sqrt{2}], [\frac{1}{2t} + \frac{\sqrt{2}}{2t} + \frac{1}{2}, t = x - \sqrt{2}]]$$

Although we would expect a constant factor, the first term of each generalized exponents does not have a factor $\sqrt{2}$.

More importantly, $f_{\sigma(s)} = \sigma(f_s)$ does not hold for the exp-irregular singularity $s = \sqrt{2}$ and $\sigma \in \mathrm{Hom}_k(k(s),k)$ since $f \notin k(x)$. This can be seen from the series expansion of f at those points:

```
> series(f,x=sqrt(2),0);
```

$$\frac{1}{2}\left(\sqrt{2}-1\right)^{-1}\left(x-\sqrt{2}\right)^{-1}+O(1)$$

```
> series(f,x=-sqrt(2),0);
```

$$-\frac{1}{2}\left(-\sqrt{2}-1\right)^{-1}\left(x+\sqrt{2}\right)^{-1}+O(1)$$

Here $f_{\sigma(\sqrt{2})} = -\sigma\left(f_{\sqrt{2}}\right)$ for $\sigma : \sqrt{2} \to -\sqrt{2}$. Hence, one of the polar parts has the coefficient 1 while the other coefficient is -1. So in part C of the algorithm we cannot work with trace in this case.

If the other computations should still work correctly we have to find the constant factor before we start the algorithm and divide all generalized exponents by this factor.

There might be points were the constant factor is a factor of the exponent difference. But in the example above we have seen that this factor cannot always be determined easily.

Let c be the constant factor we search for and let p be a singularity. For each point we have the constant fields $k \subset k(c) \subset k(c,p) = k_p$. The generalized exponent at the point p will be represented in k_p. So we have to find a algebraic extension $\tilde{k}$ of k of degree two such that $\tilde{k} \subset k_p$ for all p. All the constants $c \in \tilde{k}$ for which $c^2 \in k$ are possible values for the constant factor.

In most cases we can read off the constant factor and in other cases there is just one possibility for $\tilde{k}$ and we just need one of a pair $c, -c$. But very rarely there might be more than one possibility for $\tilde{k}$ and in that case we have to start the algorithm with a modified list of generalized exponents for each possibility. At the end the result is just a longer list of possibilities for f.

Example 3.24

Let L be the operator obtained from L_B with undetermined parameter ν and a change of variables

$$x \to f = \frac{\sqrt{5}}{x^2+3x-2}.$$

```
> LB:=x^2*D^2+x*D-(x^2+nu^2):
> f:=sqrt(5)/((x^2+3*x-2)):
> L:=changeOfVars(LB,f);
```

$$L := (2x+3)\left(x^2+3x-2\right)^4\partial^2+\left(2x^2+6x+13\right)\left(x^2+3x-2\right)^3\partial -$$
$$\left(5+\nu^2x^4+6\nu^2x^3+5\nu^2x^2-12\nu^2x+4\nu^2\right)(2x+3)^3$$

The field of constants defined by L is $k = \mathbb{Q}(\nu)$ and $c = \sqrt{5} \notin k$ is the constant we need to find from the singularities of L.

The zeros of $q_1 = x^2 + 3x - 2$ are exp-irregular singularities of L. Their generalized exponent is

```
> q1:=x^2+3*x-2:
```

```
> gen_exp(L,t,x=RootOf(q1));
```

$$[[\frac{RootOf\left(-5+17\_Z^2\right)}{t}+\frac{1}{2}, t = x - RootOf\left(\_Z^2+3\_Z-2\right)]]$$

Let p be a zero of q_1. Then

$$d_p := \Delta(L,p) = \frac{2\sqrt{85}}{17t} = \frac{1}{t}\sqrt{\frac{20}{17}}.$$

So the exponent difference d_p is defined over the field of constants generated by $q_2 = 17x^2 - 20$. To get a description of the field $k_p = k(c,p)$ we also need the point p. This is done with the command `Primfield` in Maple:

```
> q2:=17*x^2-20:
```

```
> r:=evala(Primfield({RootOf(q1),RootOf(q2)})):
```

```
> r:=sub(_Z=x,op(1,lhs(r[1,1])));
```

$$r := 289x^4 + 1734x^3 + 765x^2 - 5508x - 2864$$

Now k_p is generated over $k = \mathbb{Q}(\nu)$ by one of the zeros of r. We finally compute the subfields of k_p which have degree 2 over k:

```
> evala(Subfields({r},2,{},x));
```

$$\left\{RootOf\left(\_Z^2-85\right), RootOf\left(-5+\_Z^2\right), RootOf\left(\_Z^2-17\right)\right\}$$

So the constant factor c might be either $\sqrt{85}, \sqrt{5}$ or $\sqrt{17}$. For each of these possibilities we divide the exponent differences at the exp-irregular points by this constant c and compute possibilities for ν and f. Afterwards each $f \in \mathfrak{F}$ is multiplied by c again. Luckily, there will be no pairs with $c = \sqrt{17}$ and $c = \sqrt{85}$ that satisfy all the conditions. So there are not too many possibilities we have to check with `equiv`. At the end we find the solutions

$$V(L) = \left\{ C_1 I_\nu\left(\frac{\sqrt{5}}{x^2+3x-2}\right) + C_2 K_\nu\left(\frac{\sqrt{5}}{x^2+3x-2}\right) \,\middle|\, C_1, C_2 \in \mathbb{C} \right\}.$$

Remark 3.25
The change of variables $x \to \sqrt{x}$ applied to the Bessel operator still creates an operator $L \in K[\partial]$:

```
> LB:=x^2*D^2+x*D-x^2-nu^2:
> L:=changeOfVars(LB,sqrt(x));
```

$$L := 4x^2\partial^2 + 4x\partial - x - \nu^2$$

This operator L will still have solutions which can be expressed with Bessel functions. But we don't consider algebraic functions as parameters.

3.5 Whittaker Functions

The algorithm can easily be adapted to Whittaker functions. The Whittaker function are defined by the differential operator

$$L_W := D^2 - \frac{1}{4} + \frac{\mu}{x} + \frac{\frac{1}{4} - \nu^2}{x^2}$$

which has the two independent solutions

$$M_{\mu,\nu}(x) = \exp\left(-\frac{1}{2}x\right) x^{\frac{1}{2}+\nu} M\left(\frac{1}{2} + \nu - \mu, 1 + 2\nu, x\right)$$
$$W_{\mu,\nu} = \exp\left(-\frac{1}{2}x\right) x^{\frac{1}{2}+\nu} U\left(\frac{1}{2} + \nu - \mu, 1 + 2\nu, x\right),$$

where $M(\mu,\nu,x)$ and $U(\mu,\nu,x)$ are the Kummer functions. The following example will show how closely related Whittaker and Bessel functions are.

Example 3.26
Consider the Whittaker operator with parameter $\mu = 0$:

```
> L:=D^2-1/4+0/x+(1/4-nu^2)/x^2;
```

$$L := \partial^2 - \frac{1}{4} + \frac{\frac{1}{4} - \nu^2}{x^2}$$

The solutions of L can be expressed by Bessel functions:

```
> dsolve(diffop2de(L,y(x)),y(x))
```

$$y(x) = \_C1\,\sqrt{x}\,I_\nu\left(\frac{x}{2}\right) + \_C2\,\sqrt{x}\,K_\nu\left(\frac{x}{2}\right)$$

This is also true for any $\mu \in \mathbb{Z}$.

The generalized exponents also remind of Bessel functions. The Whittaker operator has two singularities, $x = 0$ and $x = \infty$.

At $x = 0$ the generalized exponents are

```
> LW:=D^2-1/4+mu/x+(1/4-nu^2)/x^2;
> gen_exp(LW,t,x=0);
```

$$[[\frac{1}{2} - \nu, t = x], [\frac{1}{2} + \nu, t = x]]$$

and the exponent difference is

$$\Delta(L_W, 0) = 2\nu.$$

At $x = \infty$ we have the generalized exponents

```
> gen_exp(LW,t,x=infinity);
```

$$[[\frac{1}{2t} - \mu, t = \frac{1}{x}], [-\frac{1}{2t} + \mu, t = \frac{1}{x}]]$$

and the exponent difference is

$$\Delta(L_W, \infty) = \frac{1}{t} - 2\mu.$$

If $\nu = \frac{1}{2}$, then L_W has a logarithmic solution at $x = 0$:

```
> formal_sol(subs(nu=1/2,LW),`has logarithm?`,x=0);
```

true

If $\nu \notin \frac{1}{2}\mathbb{Z}$, then the generalized exponents at $x = 0$ already tell us that we cannot have logarithmic solutions because the exponents are different modulo $\mathbb{Z}$.

We now want to apply the same or a similar algorithm that we developed for Bessel functions to Whittaker functions.

Remember that we just had to find ν modulo $\mathbb{Z}$ for Bessel functions since a shift $\nu \to \nu + 1$ just changed the gauge transformation involved. Similar statements also hold for Whittaker functions.[3] If either $\mu \to \mu + 1$ or $\nu \to \nu + 1$ the solution space changes by a gauge transformation. The same is true for the simultaneous shifts $\mu \to \mu + \frac{1}{2}$ and $\nu \to \nu + \frac{1}{2}$.

Hence, it is sufficient to compute one of the parameters modulo $\frac{1}{2}\mathbb{Z}$ and the other modulo $\mathbb{Z}$. And as another consequence L_W has logarithmic solutions at $x = 0$ for all $\nu \in \frac{1}{2}\mathbb{Z}$.

Considering the change of variables

$$L_W \xrightarrow{f}_C M, \quad f \in K,$$

we can make a similar statement as in Theorem 3.1 for the Whittaker operator. In fact, case (a) of this theorem still holds.

[3] Formulas are given in Exercises 6.3 to 6.7 in [24].

Theorem 3.27 *Let $M \in K[\partial]$ be such that $L_W \xrightarrow{f}_C M, f \in K$.*

(a) If p is a zero of f with multiplicity m, then p is a regular singularity of M and $\Delta(M,p) = 2m\nu$.

(b) If p is a pole of f with multiplicity m such that

$$f = \sum_{i=-m}^{\infty} f_i t_p^i, \tag{3.20}$$

then p is an irregular singularity of M and

$$\Delta(M,p) = 2m\mu + \sum_{i=-m}^{-1} i f_i t_p^i. \tag{3.21}$$

Proof. The proof is analogous to the proof of Theorem 3.1.

The constant $\frac{1}{2}$ in the generalized exponent of L_W at $x = 0$ disappears when we take the exponent difference and thus we have the same result as in the Bessel case.

At the point $x = \infty$ the results of the Bessel case can be devolved to the Whittaker case with some minor changes in the formula. □

If we remove the constant term $2m\mu$ from the exponent differences at the irregular singular points, then we have the same conditions as in the Bessel case. Especially Corollary 3.4 holds and since there is no difference in the exponent differences at the exp-regular points we can apply all the cases we developed in Section 3.2.

The only problem that remains is to compute the parameter μ. But this is easier than in the Bessel case. We know all the irregular singularities and from each $s \in S_{irr}$ we can determine the constant term c_s of $\Delta(L_{in}, s) \in \mathbb{C}[t^{-1}]$. Since gauge transformations can change this constant by an integer we know $c_s = 2m_s\mu$ mod $\mathbb{Z}$. We now define sets

$$\mathcal{M}_s := \left\{ \frac{c_s + i}{2m_s} \,\middle|\, 0 \le i \le 2m_s - 1 \right\}$$

which satisfy the corresponding statement for $\mathcal{M}_s$ to Lemma 3.11, i.e. for each $s \in S_{reg}$ there is an integer $z_s \in \mathbb{Z}$ such that $\mu + z_s \in \mathcal{M}_s$. The intersection modulo $\mathbb{Z}$ yields a set $\mathcal{M}$ of possible values for μ.

Example 3.28
Let L be the operator obtained from L_W with $\mu = \frac{5}{8}$ and the change of variables $x \to f = x^2 + 5x + 3$:

```
> LW:=D^2-1/4+mu/x+(1/4-nu^2)/x^2;
> f:=x^2+5*x+3:
> L:=changeOfVars(subs(mu=5/8,LW),f):
```

The generalized exponent at the exp-irregular singularity $x = \infty$ is:

```
> gen_exp(L,t,x=infinity);
```

$$[[t^{-2}+5/2t^{-1}-5/4, t = x^{-1}], [-t^{-2}-5/2t^{-1}+5/4, t = x^{-1}]]$$

and $\Delta(L,\infty) = \frac{2}{t^2} + \frac{5}{t} - \frac{5}{2}$. The constant term in $\Delta(L,\infty)$ is $c_\infty = -\frac{5}{2}$ and the multiplicity is $m_\infty = 2$. Therefore, the set of possibilities for μ is

$$\mathcal{M} = \left\{ -\frac{5}{8}, -\frac{3}{8}, -\frac{1}{8}, \frac{1}{8} \right\}.$$

If we then start the Bessel algorithm with modified exponent difference at $x = \infty$, we also get possibilities for f and ν. If we try all combinations with $\mu \in \mathcal{M}$, we will finally find the solutions of L:

```
> dsolveBessel(L);
```

$$\_C1\, M_{5/8,\nu}\left(x^2+5x+3\right) + \_C2\, W_{5/8,\nu}\left(x^2+5x+3\right)$$

Hence, the solutions of L can be expressed with Whittaker functions.

3.6 Two Final Examples

The following example occurred in research of W. N. Everitt [10] and was completely solved after a contribution[4] of M. van Hoeij. The complete result can be found in the follow-up [11].

Example 3.29
We consider the differential equation from [11]:

$$\left(xy''(x)\right)'' - \left(\left(\frac{9}{x} + \frac{8}{M}x\right) y'(x)\right)' = \lambda^2 \left(\lambda^2 + \frac{8}{M}\right) xy(x)$$

for all $x \in (0,\infty)$, whereas M and λ are constant parameters. The corresponding differential operator is

```
> L:=x*D^4+2*D^3-(9*M+8*x^2)/(x*M)*D^2-)-(-9*M+
  8*x^2)/(x^2*M)*D-(lambda^2*x*(lambda^2*M+8))/M;
```

$$L := x\partial^4 + 2\partial^3 - \frac{\left(9M+8x^2\right)}{xM}\partial^2 - \frac{\left(-9M+8x^2\right)}{x^2M}\partial - \frac{\lambda^2 x\left(\lambda^2 M+8\right)}{M}$$

We can factor[5] the operator L:

[4] Personal contribution at the International Conference on Difference Equations, Special Functions and Applications, Technical University Munich, Germany: July 2005

[5] you should use Maple 10 or higher

```
> LL:=DFactorLCLM(L);
```

$$[\partial^2+\frac{\left(\lambda^4M^2x^2+8\,\lambda^2Mx^2+16x^2-48M\right)\partial}{x\left(\lambda^4M^2x^2+8\,\lambda^2Mx^2+16x^2-16M\right)}-\frac{4\,\lambda^4M^3+32\,\lambda^2M^2+16\,\lambda^4M^2x^2+128x^2+80\,\lambda^2Mx^2+x^2\lambda^6M^3}{M\left(\lambda^4M^2x^2+8\,\lambda^2Mx^2+16x^2-16M\right)},$$
$$\partial^2+\frac{\left(\lambda^4M^2x^2+8\,\lambda^2Mx^2+16x^2-48M\right)\partial}{x\left(\lambda^4M^2x^2+8\,\lambda^2Mx^2+16x^2-16M\right)}+\frac{\lambda^2\left(-4\,\lambda^2M^2-32M+16x^2+8\,\lambda^2Mx^2+\lambda^4M^2x^2\right)}{\lambda^4M^2x^2+8\,\lambda^2Mx^2+16x^2-16M}]$$

Hence, L is least common left multiple of two operators of degree two. This means that the solutions of L are generated by the solutions of these factors. The two operators can now be solved in terms of Bessel functions:

```
> L2:=LL[2]:
> dsolveBessel(L2);
```

$$\frac{\_C1}{x}\left(-2\lambda M\,J_1\left(x\lambda\right)+\left(\lambda^2M+4\right)xJ_0\left(x\lambda\right)\right)+\frac{\_C2}{x}\left(-2\lambda M\,Y_1\left(x\lambda\right)+\left(\lambda^2M+4\right)xY_0\left(x\lambda\right)\right)$$

The solutions of the operator L_2 match the solutions (2.2) and (2.4) in [11]. And the solutions of the operator L_1 match those of (2.7) and (2.8):

```
> L1:=LL[1];
```

```
> dsolveBessel(L2);
```

$$\frac{\_C1}{x}\left(-2\left(\lambda^2M+8\right)M\,I_1\left(\frac{x\sqrt{\lambda^2M^2+8M}}{M}\right)+x\sqrt{\left(\lambda^2M+8\right)M}\left(\lambda^2M+4\right)I_0\left(\frac{x\sqrt{\lambda^2M^2+8M}}{M}\right)\right)+\frac{\_C2}{x}\left(-2\left(\lambda^2M+8\right)M\,K_1\left(\frac{x\sqrt{\lambda^2M^2+8M}}{M}\right)-x\sqrt{\left(\lambda^2M+8\right)M}\left(\lambda^2M+4\right)K_0\left(\frac{x\sqrt{\lambda^2M^2+8M}}{M}\right)\right)$$

Together $V(L_1)$ and $V(L_2)$ generate the solution space $V(L)$.

Since Whittaker can be written as Kummer functions and vice versa we can also solve operators in terms of Kummer functions. A little heuristic switches to the Kummer representation if the output is probably shorter.

Example 3.30
We consider the following operator which occurred in W. Koepf's and M. Foupouagnigni's research about orthogonal polynomials:

```
> L:=(4*x^4-12*x^2+3)*D^2-2*x*(4*x^4+4*x^2-21)*D+
  (64*x^4-96*x^2+8*n*x^4-24*n*x^2+6*n);
```

$$L := \left(4x^4 - 12x^2 + 3\right)\partial^2 + \left(-8x^5 - 8x^3 + 42x\right)\partial + 64x^4 - 24nx^2 + 6n - 96x^2 + 8nx^4$$

The solutions can be expressed with Kummer functions:

```
> dsolveBessel(L);
```

$$\_C1\left(\left(-9n - 33 + 4x^4 + 4nx^4 - 4x^2 - 4nx^2\right)M\left(-2 - \frac{n}{2}, \frac{1}{2}, x^2\right) - 4\left(-3 + 2x^2\right)(4+n)M\left(-1 - \frac{n}{2}, \frac{1}{2}, x^2\right)\right) + \_C2\left(\left(-9n - 33 + 4x^4 + 4nx^4 - 4x^2 - 4nx^2\right)U\left(-2 - \frac{n}{2}, \frac{1}{2}, x^2\right) + 2(4+n)(n+3)\left(-3 + 2x^2\right)U\left(-1 - \frac{n}{2}, \frac{1}{2}, x^2\right)\right)$$

For $n \in 2\mathbb{N}$ the first parameter of these hypergeometric functions will be a negative integer and the series will break down into polynomials.

4

Conclusion

We developed an algorithm to solve differential equations $Ly = 0$ for an operator L of degree two in terms of Bessel functions. If $L \in k(x)[\partial]$ and the field of constants k is defined by the coefficients in L then we will find solutions of the form

$$\exp\left(\int r\right)\left(r_0 B_\nu(f) + r_1 B'_\nu(f)\right),$$

where $B_\nu(x)$ is a Bessel function, $r, r_0, r_1 \in k(x)$ and $f = c \cdot \bar{f}$ for some $\bar{f} \in k(x)$ and $c^2 \in k$. The parameter ν can either be a constant $\nu \in \mathbb{C}$ or a transcendental symbol. For $\nu \in \frac{1}{2}\mathbb{Z}$ our algorithm could not find a solutions. But in that case it turned out that L is reducible and has hyperexponential solutions.

After studying transformation of differential operators we restricted the problem to

$$L_B \xrightarrow{f}_C M \longrightarrow_{EG} L.$$

We used generalized exponents of L and their corresponding exponent differences to make statements about zeros and poles of f. As a result we had a set $\mathcal{F}$ of possibilities for f and a set $\mathcal{N}$ of possibilities for ν. For each pair we could then compute the operator M and solve the equivalence between M and L.

We finally discussed how the algorithm can be extended to Whittaker functions which also includes solutions in terms of Kummer functions.

The next step would be to extend the algorithm to all ${}_2F_1$-functions. One problem we have to handle is that the ramification index might be two, e.g. in the generalized exponent of the irregular singularity in the ${}_0F_1$-case (see Example 1.29). Hence, we also have to deal with fractional exponent in the Puiseux series.

Moreover, we have to work with different kinds of singularities. The general ${}_2F_1$-function has three regular singularities and no irregular singularities. So the idea of computing polar parts of the parameter f in the change of variables will not work. On the other hand, ${}_0F_1$-functions have just one irregular singularity and no regular singularities. This will cause difficulties in finding zeros of f. Therefore, it seems that we will have to distinguish between several cases and it is unlikely that we find an approach that works in every one of them.

Another interesting challenge is to prove the completeness of the algorithm. If an operator $L \in k(x)[\partial]$ is given, we just search for solutions with $r, r_0, r_1 \in k(x)$ and $f = c\bar{f}$ with $\bar{f} \in k(x)$ and $c^2 \in k$. However, we do not know if this operator can be obtained from the Bessel operator if we allow other parameters. In other words, can we apply transformations with parameters with other parameters and still get an operator L which is defined over $k(x)$. If this is not the case we know that our algorithm is complete and that we will always find a Bessel solution if such a solution exists.

A

Appendix

A.1 Transformations

Form Theorem 2.3 we can derive algorithms that apply a change of variables, an exp-product or a gauge transformation to a differential operator.

Algorithm 8: changeOfVars

Input: operator $L \in k(x)[\partial]$ of degree two and rational function f
Output: operator $\tilde{L} \in k(x)[\partial]$ of degree two such that $y(f) \in V(\tilde{L})$ for every $y(x) \in V(L)$

$l :=$**lcoeff**(L, ∂)
$a_0, a_1 :=$**coeffs**$(L, \partial)/l$
$b_0 := \left.\frac{a_0}{a_2}\right|_{x=f} (f')^2$
$b_1 := \frac{1}{f'}\left(\left.\frac{a_1}{a_2}\right|_{x=f} (f')^2 + f''\right)$
return $\left(\textbf{collect}\left(\textbf{numer}\left(\partial^2 + b_1\partial + b_0\right), \partial\right)\right)$

Algorithm 9: expProduct

Input: operator $L \in k(x)[\partial]$ of degree two and rational function r
Output: operator $\tilde{L} \in k(x)[\partial]$ of degree two such that $exp(\int r)y \in V(\tilde{L})$ for every $y \in V(L)$

$l :=$**lcoeff**(L, ∂)
$a_0, a_1 :=$**coeffs**$(L, \partial)/l$
$b_1 := -2r + a_1$
$b_0 := -r' - r^2 + a_0 + b_1 r$
return $\left(\textbf{collect}\left(\textbf{numer}\left(\partial^2 + b_1\partial + b_0\right), \partial\right)\right)$

Algorithm 10: gauge

Input: operator $L \in k(x)[\partial]$ of degree two and two rational functions r_0, r_1
Output: operator $\tilde{L} \in k(x)[\partial]$ of degree two such that $r_0 y + r_1 y' \in V(\tilde{L})$ for every $y \in V(L)$

$l :=$**lcoeff**(L, ∂)

$a_0, a_1 :=$**coeffs**$(L, \partial)/l$

$b_0 := -\big(-r_1 a_0 r_1'' - 3 r_1 a_0 r_0' + r_1^2 a_0 a_1' - r_1 a_0 a_1 r_0 + r_0' r_1 a_1^2 - 2 r_0' r_1' a_1 - r_0' r_1 a_1' + r_0' r_1'' - r_0' a_1 r_0 + 2 r_0'^2 + a_0 r_0^2 - r_0'' r_0 - r_1 a_0 r_1' a_1 + r_1 a_0' r_0 + 3 a_0 r_0 r_1' + a_0^2 r_1^2 - r_0'' r_1' + 2 r_1'^2 a_0 + r_1 a_0' r_1' + r_0'' r_1 a_1 - r_1^2 a_0' a_1\big)/\big(-r_0^2 - r_0 r_1' + r_0 r_1 a_1 + r_1 r_0' - r_1^2 a_0\big)$

$b_1 := (r_0 r_1'' + 2 r_0 r_0' + r_0 r_1 a_1^2 - 2 r_0 r_1' a_1 - r_0 r_1 a_1' - a_1 r_0^2 - a_0 r_1^2 a_1 + r_1^2 a_0' - r_1 r_0'' + 2 r_1 r_1' a_0)/(-r_0^2 - r_0 r_1' + r_0 r_1 a_1 + r_1 r_0' - r_1^2 a_0)$

return $\big($**collect** $\big($**numer** $\big(\partial^2 + b_1 \partial + b_0\big), \partial\big)\big)$

A.2 IsPower

In the integer case of the algorithm, which was discussed in section 3.3.2, we had to determine whether a monic polynomial is a p-th power of another polynomial.

Algorithm 11: ispower

Input: a monic polynomial $f \in K[x]$ and $p \in \mathbb{N}$
Output: $g \in K[x]$ with the following property: if a solution for $y^p = f$ exists, then g is a solution.

1 **if** $p = 1$ **then return** f
2 $d :=$**degree**(f, x)
3 $n := d/p$
4 **if** $n \notin \mathbb{Z}$ **then return** *FAIL*
5 $A := x^n + \sum_{i=0}^{n} a_i x^i$
6 **for** $i = 1 \ldots n$
7 $\quad a_{n-i} :=$**solve**(**coeff**$(A^p, x, d-i)-$**coeff**$(f, x, d-i), a_{n-i})$
(solves a linear equation in one unknown)

Return: A

The only thing we have to prove is that the equation in line 7 introduces one new variable each time.

Let $a = \sum_{i=0}^{n} a_i x^i$. Then the p-th power is:

$$a^p = a_n^p x^{np} + a_n^{p-1} a_{n-1} x^{np-1} + (a_n^{p-1} a_{n-2} + a_n^{p-2} a_{n-1}^2) x^{np-2} + \cdots.$$

Let us take a closer look at the coefficients of $a^p = \sum_{i=0}^{np} b_i x^i$. If we choose p integers $m_i \in \{0,\dots,n\}$ and define $c_i := a_{m_i}$, then $c_1 \cdots c_p$ is part of the coefficient of x^k with $k = m_1 + \cdots + m_p$. The coefficients b_i are then sums of such products.

We can express this more exactly. Let $\mathcal{P}$ be the power set of $\{0,\dots,n\}$ and define $\mathcal{P}_k := \{P \in \mathcal{P} \mid \sum_{j\in P} j = k \text{ and } |P| = p\}$. So $\mathcal{P}_k$ contains set with p elements whose sum is k. In the notation above we had $\{m_1,\dots,m_p\} \in \mathcal{P}_k$. Then

$$b_k = \sum_{P\in\mathcal{P}_k} \prod_{m\in P} a_m.$$

Now fix a coefficient a_k. Then there exists $P \in \mathcal{P}_{np-j}$ such that $k \in P$, only if $k \geq n-j$. If $k < n-j$, then the highest coefficient, in which a_k is involved is $b_{(p-1)n+k}$. Since $(p-1)n+k = np-n+k < np-j$ we get $k \notin \mathcal{P}_{np-j}$ and a_k is not involved in b_{np-j}.

Let's look at the coefficients $b_{np}, b_{np-1}, b_{np-2} \dots b_{np-n}$ successively. In b_{np} just a_n will appear, in b_{np-1} we will have a_n and a_{n-1} and so on. So each equation in line 7 introduces one *new* variable step by step. So we can solve them one by one to find $a_n, a_{n-1}, \dots, a_0$.

The algorithm does not check whether the solution is correct, because in the integer case there was still a unknown constant c in the input f. The output g gives us an equation $g^p - f$ which should be zero for some value of c.

A.3 Package Description

In this chapter we will give an overview over the functions implemented in the package.

If the base field k is needed, it is passed through a set of *RootOf*-structures which is read from the input using the `indets` command.

besselequiv
Input: An operator $L \in K[\partial]$, a rational function $f \in K$, and a constant $\nu \in \mathbb{C}$.
Output: A sequence $M \in K[\partial], [y_1, y_2]$ such that y_1 and y_2 are the (modified) Bessel function of the first and second kind and $M(y)$ is a solution of L. If such a solution does not exist 0 is returned.

besselsubst
(implementation of Algorithm 1 on page 54)
Input: S_{irr} and their exponent differences, local parameter t, the field k
Output: A list $[f_1,\dots,f_n]$ that corresponds to possibilities $\sum_{i=1}^n \pm f_i$.

changeconstant
Input: A rational function $f \in K = k(x)$, a point p.
Output: A rational function $g = g(x) \in K$ such that $g = f + c$ for some $c \in k$ and $g(p) = 0$. If $p = \infty$, $g = f$ is returned.

compare
Input: Two constants $a, b \in k$.
Output: Two rational numbers $r, s \in \mathbb{Q}$ such that $a = rb + s$.

dsolve_bessel
Input: (i) A differential operator $L \in K[\partial]$ and optionally the domain
(ii) A differential equation and the dependent variable
Output: The solution space if it can be expressed by Bessel or Whittaker functions.

equiv
Input: Two operators $L_1, L_2 \in K[\partial]$ of degree two.
Output: An operator M such that $My \in V(L_2)$ for every $y \in V(L_1)$. If a solution $M \neq 0$ was found, a sequence $r \in K, G \in K[\partial]$ which satisfies $M = \exp(\int r)G$ would be returned.

findBesselvf
Input: An integer that indicates the case we are in, S_{irr}, S_{reg}, the field k, and the variable t for the local parameter.
Output: A list of pairs (ν, f).

findBesselvfint
(implementation of Algorithm 4 on page 66)
Input: $\mathcal{F}$, boolean b_∞ that indicates whether $\infty \in S_{irr}$, the field k, and S_{reg}.
Output: A list of pairs (ν, f).

findBesselvfirrat
(implementation of Algorithm 7 on page 78)
Input: $S_{reg}, \mathcal{F}$, b_∞, and the field k.
Output: A list of pairs (ν, f).

findBesselvfK
(implementation of Algorithm 6 on page 72)
Input: $S_{reg}, \mathcal{F}$, b_∞, and the field k.
Output: A list of pairs (ν, f).

findBesselvfln
(implementation of Algorithm 3 on page 60)
Input: $S_{reg}, \mathcal{F}$, and the field k.
Output: A list of pairs (ν, f).

findBesselvfrat
(implementation of Algorithm 5 on page 69)
Input: $S_{reg}, \mathcal{F}$, and the field k.
Output: A list of pairs (ν, f).

findWhittaker
Input: $L, S_{irr}, S_{reg}, k, t$ and x
Output: The solution space of L if it can be expressed by Whittaker functions.

ispower
(implementation of Algorithm 11 on page 92)
Input: A monic polynomial $f \in k[x]$ and $p \in \mathbb{N}$
Output: $g \in k[x]$ with the following property: if a solution for $y^p = f$ exists, then g is a solution.

kummerequiv
Input: An operator $L \in K[\partial]$, two constants $\mu, \nu \in \mathbb{C}$, a rational function $f \in K = k(x)$, and the variable x.
Output: A sequence $M \in K[\partial], [y_1, y_2]$ such that y_1 and y_2 are the Kummer function of the first and second kind and $M(y)$ is a solution of L. If such a solution does not exist 0 is returned.

poly_d
Input: A constant $c \in \bar{k}$, a variable x, and the field k.
Output: A polynomial $p = x^2 - d \in k[x]$ if the minimal polynomial of c is a shift of p.

possibility
Input: A list $[f_1, \ldots, f_n]$, and an integer $m \in \mathbb{N}$, $1 \leq m \leq 2^n$.
Output: The m-th possibility for $\sum_{i=1}^{n} \pm f_i$. More precisely, it returns $\sum_{i=1}^{n} a_i f_i$ where $a_i = (-1)^{b_i}$ and b_i is the i-th digit in the binary representation of m.

Same_5_curvature
Input: Two rational functions $a, b \in K$, and a variable x.
Output: A boolean which indicates whether $\partial + a = \partial + b \bmod 5$.

Same_p_curvature
Input: Two rational functions $a, b \in K$, and a variable x.
Output: A boolean which indicates whether $\partial + a = \partial + b \bmod 3$. If the comparison modulo 3 fails the comparison modulo 5 is used.

SimplifyAnswer
Input: $d \in k(x), L$ and a list of functions F
Output: A list of function obtained by applying the operator $\exp(\int d)L$ to the functions in F

singgenexp
Input: $L \in k(x)[\partial], k$, a variable t, and an optional parameter to pass some informations about singularities
Output: A list of elements of the form $[p, t, D, p, n]$ such that: p is a singularity of L, p is a polynomial over k with zero p, $n = \deg(p)$, and D is the exponent

difference $d = \Delta(L, p)$. If $d \notin k(p,t)$ then D is either a list $[c, d']$, if $d = c\sqrt{(}d')$, or the minimal polynomial of d over k.

SqrtConst

Input: S_{irr}, k, t, x

Output: A set of pairs S'_{irr}, c where $\sqrt{c}$ is a possible constant factor of the parameter f in the change of variables and S'_{irr} is the set of updated exponent differences at the exp-irregular points.

testzeros

Input: $f \in k(x)$ and a set of points

Output: *True* if all points are zeros of f and *false* otherwise.

whittakerequiv

Input: An operator $L \in K[\partial]$, two constants $\mu, \nu \in \mathbb{C}$, a rational function $f \in K = k(x)$, and a constant c.

Output: A sequence $M \in K[\partial], [y_1, y_2]$ such that y_1 and y_2 are the Whittaker or the Kummer function of the first and second kind and $M(y)$ is a solution of L. If such a solution does not exist 0 is returned.

Bibliography

[1] ABRAMOV, S. A., BARKATOU, M. A., AND VAN HOEIJ, M. Apparent singularities of linear differential equations with polynomial coefficients. *AAECC 17* (2006), 117–133.

[2] ABRAMOWITZ, M., AND STEGUN, I. A. *Handbook of Mathematical Functions*, vol. 55 of *Applied Mathamatics Series*. Tenth Printing, 1972.

[3] BARKATOU, M. A., AND PFLÜGEL, E. On the equivalence problem of linear differential systems and its application for factoring completely reducible systems. In *Proceedings of ISSAC 1998* (New York, USA, 1998), ACM Press, pp. 268–275.

[4] BRONSTEIN, M. *Symbolic Integration I: Transcendental Functions*, 2nd ed. Springer, Berlin, 1997.

[5] BRONSTEIN, M., AND LAFAILLE, S. Solutions of linear ordinary differential equations in terms of special functions. In *Proceedings of ISSAC'2002* (2002), ACM Press, pp. 23–28.

[6] CARLSON, B. C. *Special Function of Applied Mathematics*. Academic Press, London, 1977.

[7] CHAN, L., AND CHEB-TERRAB, E. S. Non-liouvillian solutions for second order linear odes. In *Proceedings of ISSAC 2004* (New York, NY, USA, 2004), ACM Press, pp. 80–86.

[8] CLUZEAU, T., AND VAN HOEIJ, M. A modular algorithm to compute the exponential solutions of a linear differential a modular algorithm to compute the exponentail solution of a linear differential operator. *J. Symb. Comp. 38* (2004), 1043–1076.

[9] CURTISS, J. H. *Introduction to Function of a Complex Variable*. Marcel Dekker, 1978.

[10] EVERITT, W. N., AND MARKETT, C. On a generalization of bessel functions satisfying higher-order differential equations. *J. Comput. Appl. Math. 54*, 3 (1994), 325–349.

[11] EVERITT, W. N., SMITH, D. J., AND VAN HOEIJ, M. The fourth-order type linear ordinary differential equations. arXiv:math/0603516, March 2006.

[12] FISCHER, W., AND LIEB, I. *Funktionentheorie*, 8th ed. Vieweg, 2003.

[13] HENDRIKS, P. A., AND VAN DER PUT, M. Galois actions on solutions of a differential equation. *J. Symb. Comp. 19*, 6 (June 1995), 559–576.

[14] HUNT, R. W. *Differential Equations*. Wadsworth Publishing, Belmont, California, 1973.

[15] INCE, E. L. *Ordinary Differential Equations*. Dover Publications Inc., 1956.

[16] KAMKE, E. *Differentialgleichungen, Lösungemethoden und Lösungen, Gewöhnliche Differentialgleichungen*. Becker Erler Kom.-Ges., Leipzig, 1943.

[17] KOEPF, W. *Hypergeometric Summation*. Vieweg, Braunschweig / Wiesbaden, 1998.

[18] PFLÜGEL, E. An algorithm for computing exponential solutions of first order linear differential systems. In *Proceedings of ISSAC 1997* (New York, USA, 1997), ACM Press, pp. 164–171.

[19] PRUDNIKOV, A. P., BRYCHKOV, Y. A., AND MARICHEV, O. I. *Integrals and Series*, vol. 3: More Special Functions. Gordon and Breach Science, 1990.

[20] SLAVYANOV, S. Y., AND LAY, W. *Special Functions*. Oxford University Press, 2000.

[21] TSAI, H. Weyl closure of linear differential operator. *J. Symb. Comp. 29* (2000), 747–775.

[22] VAN DER PUT, M., AND SINGER, M. F. *Galois Theory of Linear Differential Equations*, vol. 328 of *A Series of Comprehensive Studies in Mathematics*. Springer, Berlin, 2003.

[23] VAN HOEIJ, M. *Factorization of Linear Differential Operators*. PhD thesis, Universitijt Nijmegen, 1996.

[24] WANG, Z. X., AND GUO, D. R. *Special Functions*. World Scientific Publishing, Singapore, 1989.

[25] WILLIS, B. L. An extensible differential equation solver. *SIGSAM Bulletin 35*, 1 (March 2001), 3–7.

Index