Differentialgleichungen

Lineare Differentialgleichungen

■ Lineare Differentialgleichung erster Ordnung

In[1]:=
```
DSolve[y'[x] + a[x] * y[x] == b[x], y[x], x]
```

Out[1]= $\left\{\left\{y(x) \to e^{\int_1^x -a(K[1])\, dK[1]} \int_1^x b(K[2])\, e^{-\int_1^{K[2]} -a(K[1])\, dK[1]}\, dK[2] + c_1\, e^{\int_1^x -a(K[1])\, dK[1]}\right\}\right\}$

In[2]:=
```
DSolve[y'[x] + a[x] * y[x] == 0, y[x], x]
```

Out[2]= $\left\{\left\{y(x) \to c_1\, e^{\int_1^x -a(K[1])\, dK[1]}\right\}\right\}$

■ Beispiel 1.12

In[3]:=
```
DSolve[y'[x] == Sin[x] y[x], y[x], x]
```

Out[3]= $\left\{\left\{y(x) \to c_1\, e^{-\cos(x)}\right\}\right\}$

In[4]:=
```
DSolve[{y'[x] == Sin[x] y[x], y [0] == 1}, y[x], x]
```

Out[4]= $\left\{\left\{y(x) \to e^{1-\cos(x)}\right\}\right\}$

■ Variation der Konstanten

Die homogene Gleichung ist separierbar

In[5]:=
```
DE = y'[x] + a[x] * y[x] == 0
```

Out[5]= $a(x)\, y(x) + y'(x) = 0$

und hat die Lösung

In[6]:=
```
homogeneLösung = y[x] → K * Exp[∫-a[x] dx]
```

Out[6]= $y(x) \to K\, e^{-\int a(x)\, dx}$

Diese setzen wir ein und bekommen

In[7]:=
```
DE /. {homogeneLösung, D[homogeneLösung, x]}
```

Out[7]= True

Um eine Lösung der inhomogenen Differentialgleichung

In[8]:=
```
DE = y'[x] + a[x] * y[x] == b[x]
```

Out[8]= $a(x)\,y(x)+y'(x)=b(x)$

zu finden, machen wir den Ansatz (Variation der Konstanten)

In[9]:=
```
inhomogeneLösung = y[x] → K[x] * Exp[∫-a[x] ⅆx]
```

Out[9]= $y(x)\to K[x]\,e^{-\int a(x)\,dx}$

Diese setzen wir ein und bekommen

In[10]:=
```
newDE = DE /. {inhomogeneLösung, D[inhomogeneLösung, x]}
```

Out[10]= $K'(x)\,e^{-\int a(x)\,dx}=b(x)$

Diese einfache Differentialgleichung für K[x] können wir aber durch Integration lösen und wir erhalten

In[11]:=
```
spezielleLösung = y[x] → (Exp[∫-a[x] ⅆx] * ∫b[x] Exp[∫a[x] ⅆx] ⅆx)
```

Out[11]= $y(x)\to e^{-\int a(x)\,dx}\int b(x)\,e^{\int a(x)\,dx}\,dx$

Test:

In[12]:=
```
test = DE /. {spezielleLösung, D[spezielleLösung, x]}
```

Out[12]= True

DSolve kann dies auch alleine, liefert aber wieder eine kompliziert aussehende Lösung.

In[13]:=
```
DSolve[DE, y[x], x]
```

Out[13]= $\left\{\left\{y(x)\to e^{\int_1^x -a(K[1])\,dK[1]}\int_1^x b(K[2])\,e^{-\int_1^{K[2]} -a(K[1])\,dK[1]}\,dK[2]+c_1\,e^{\int_1^x -a(K[1])\,dK[1]}\right\}\right\}$

Beispiel 1.16

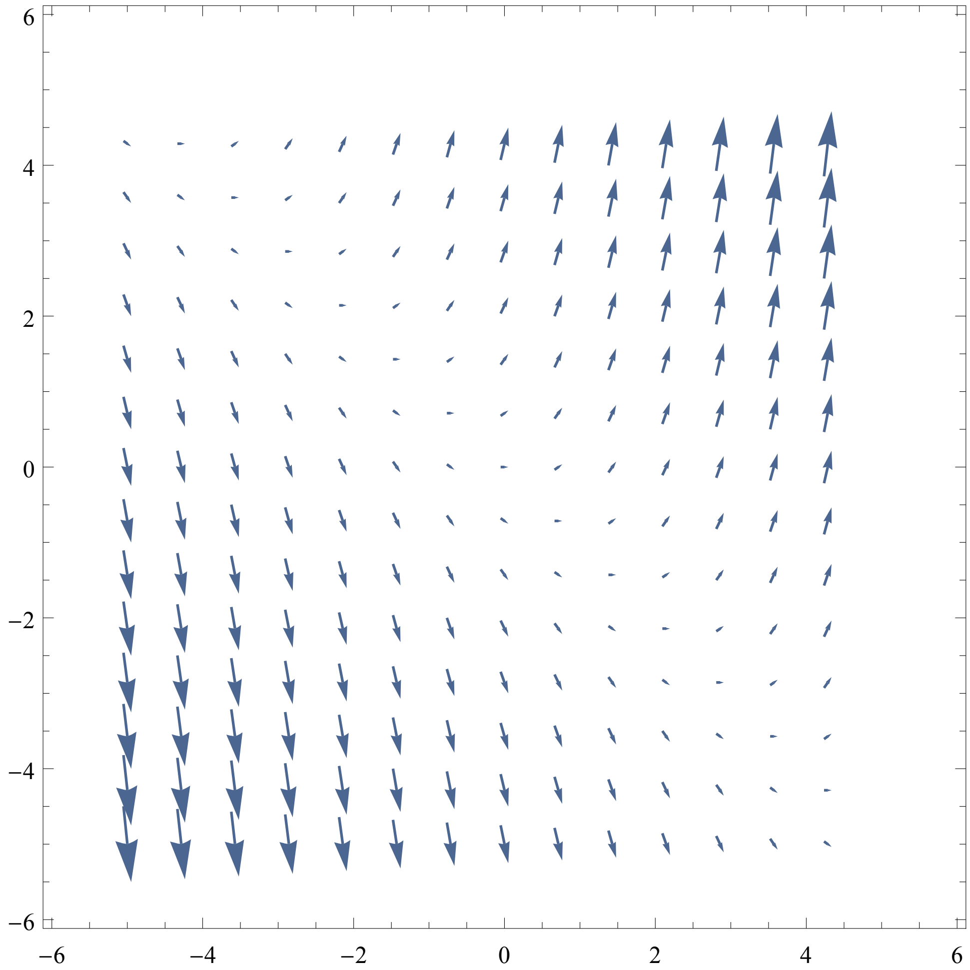
In[16]:=
```
DirectionField[DE_, y_[x_], {x_, a_, b_},
   {y_, c_, d_}, options___] := Module[{g},
   g = DE[[2]] /. y[x] → y;
   VectorPlot[{1, g}, {x, a, b}, {y, c, d}, options]
  ]
```

In[17]:=
```
DE = y'[x] == y[x] + x
```

Out[17]= $y'(x)=y(x)+x$

In[18]:=
```
plot1 = DirectionField[DE, y[x ], {x, -5, 5}, {y, -5, 5}, Frame → True]
```

Out[18]=

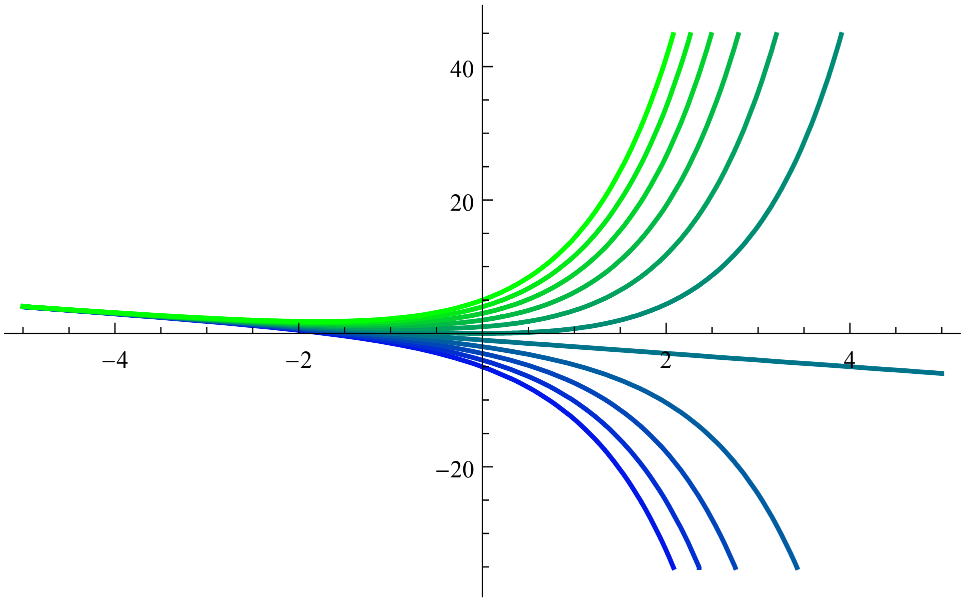


In[19]:=
```
DSolve[DE, y[x], x]
```

Out[19]= $\{\{y(x) \to c_1\, e^x - x - 1\}\}$

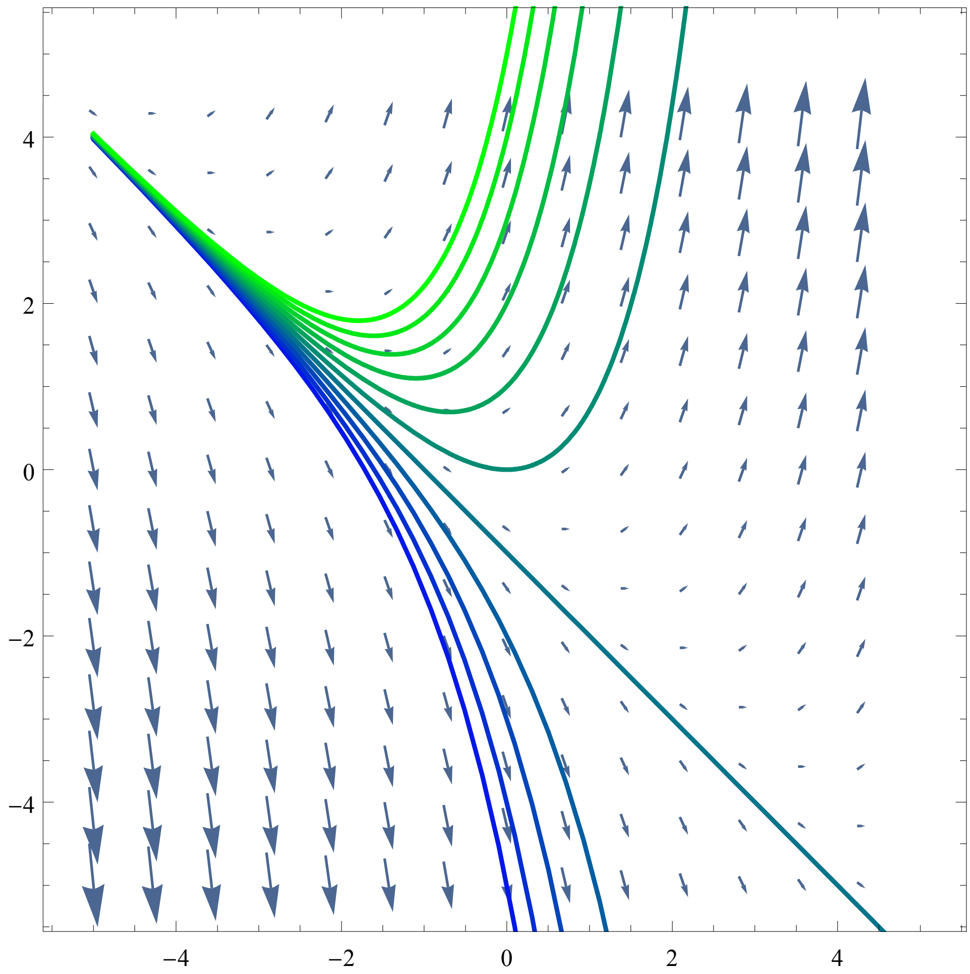
In[20]:=
```
plot2 = Plot[
  Evaluate[Table[y[x] /. DSolve[{DE, y[0] == k}, y[x], x][[1]], {k, -5, 5}]],
  {x, -5, 5},
  PlotStyle → Table[{Thickness[0.005], RGBColor[0, k/11, 1 - k/11]}, {k, 11}]]
```

Out[20]=



In[21]:= `Show[plot1, plot2, PlotRange → {-5, 5}]`

Out[21]=



nach unserer Formel:

In[22]:= `a = -1; b = x;`

Allgemeine Lösung der homogenen Differentialgleichung:

In[23]:= `hom = y1 → K * ` $e^{\int -a\,dx}$

Out[23]= $y1 \to K\,e^{x}$

Variation der Konstanten:

In[24]:= $\int b\,e^{\int a\,dx}\,dx$

Out[24]= $e^{-x}\,(-x-1)$

Spezielle Lösung der inhomogenen Differentialgleichung:

In[25]:= `var = y2 → ` $e^{\int -a\,dx} * \int b\,e^{\int a\,dx}\,dx$

Out[25]= $y2 \to -x-1$

Allgemeine Lösung der inhomogenen Differentialgleichung:

In[26]:=
```
lösung = y → y1 + y2 /. {hom, var}
```
Out[26]= $y \to K e^{x} - x - 1$

Wir lösen das Anfangswertproblem mit y(x0)=y0:

In[27]:=
```
Solve[(lösung[[2]] /. {x → x0}) == y0, K]
```
Out[27]= $\{\{K \to e^{-\text{x0}} (\text{x0} + \text{y0} + 1)\}\}$

oder mit DSolve

In[28]:=
```
lösung = DSolve[{DE, y[x0] == y0}, y[x], x]
```
Out[28]= $\{\{y(x) \to -e^{-\text{x0}} (x e^{\text{x0}} - e^{x} \text{x0} - e^{x} \text{y0} - e^{x} + e^{\text{x0}})\}\}$

In[65]:=
```
Simplify[lösung]
```

Hausaufgabe: Beispiel 1.19

In[29]:=
```
AWP = {y'[x] + Sin[x] y[x] == Sin[x], y[0] == 3}
```
Out[29]= $\{y'(x) + y(x) \sin(x) = \sin(x), y(0) = 3\}$

In[30]:=
```
DSolve[AWP, y[x], x]
```
Out[30]= $\left\{\left\{y(x) \to \frac{2 e^{\cos(x)} + e}{e}\right\}\right\}$

schrittweise Lösung: homogene Lösung

In[31]:=
```
sol = DSolve[y'[x] + Sin[x] y[x] == 0, y[x], x]
```
Out[31]= $\{\{y(x) \to c_1 e^{\cos(x)}\}\}$

In[32]:=
```
y1 = y[x] /. sol[[1]]
```
Out[32]= $c_1 e^{\cos(x)}$

Variation der Konstanten

In[33]:=
```
K = ∫Sin[x] Exp[-Cos[x]] dx
```
Out[33]= $e^{-\cos(x)}$

In[34]:=
```
y2 = K Exp[Cos[x]]
```
Out[34]= 1

In[35]:=
```
y = y1 + y2
```
Out[35]= $c_1 e^{\cos(x)} + 1$

Anfangswert

In[36]:=
```
sol = Solve[(y /. x → 0) == 3, C[1]]
```
Out[36]= $\left\{\left\{c_1 \to \frac{2}{e}\right\}\right\}$

In[37]:= `y /. sol[[1]]`

Out[37]= $2\, e^{\cos(x)-1}+1$

Hausaufgabe: Beispiel 1.23

In[39]:= `Clear[y]`

In[40]:= `gleichung =` $\frac{1}{2}\int \frac{2\,y-1}{y^2-y-2}\,dy == \int x\,dx$

Out[40]= $\frac{1}{2}\log\left(-y^2+y+2\right)=\frac{x^2}{2}$

In[41]:= `Solve[` $-y^2+y+2==0$, `y]`

Out[41]= $\{\{y\to -1\},\{y\to 2\}\}$

In[42]:= `Solve[gleichung, y]`

Out[42]= $\left\{\left\{y\to \text{ConditionalExpression}\left[\frac{1}{2}\left(1-\sqrt{9-4\,e^{x^2}}\right), -\pi<\text{Im}\left(x^2\right)\le\pi\right]\right\},\right.$
$\left.\left\{y\to \text{ConditionalExpression}\left[\frac{1}{2}\left(\sqrt{9-4\,e^{x^2}}+1\right), -\pi<\text{Im}\left(x^2\right)\le\pi\right]\right\}\right\}$

In[43]:= `DE = y'[x] ==` $x\,\frac{2\,y[x]^2-2\,y[x]-4}{2\,y[x]-1}$

Out[43]= $y'(x)=\frac{x\left(2\,y(x)^2-2\,y(x)-4\right)}{2\,y(x)-1}$

In[44]:= `DSolve[DE, y[x], x]`

Out[44]= $\left\{\left\{y(x)\to\frac{1}{2}\left(1-\sqrt{9-4\,e^{c_1+x^2}}\right)\right\},\left\{y(x)\to\frac{1}{2}\left(\sqrt{9-4\,e^{c_1+x^2}}+1\right)\right\}\right\}$

In[45]:= `DSolve[{y'[x] ==` $x\,\frac{2\,y[x]^2-2\,y[x]-4}{2\,y[x]-1}$`, y[x0] == y0}, y[x], x]`

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

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Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. >>

Out[45]= $\left\{\left\{y(x)\to\frac{1}{2}\left(1-\sqrt{9-\left(-4\,\text{y0}^2+4\,\text{y0}+8\right)e^{x^2-\text{x0}^2}}\right)\right\},\left\{y(x)\to\frac{1}{2}\left(\sqrt{9-\left(-4\,\text{y0}^2+4\,\text{y0}+8\right)e^{x^2-\text{x0}^2}}+1\right)\right\}\right\}$

In[46]:=
```
DSolve[{y'[x] == x (2 y[x]^2 - 2 y[x] - 4)/(2 y[x] - 1), y[0] == 3}, y[x], x]
```

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

DSolve::bvnul : For some branches of the general solution, the given boundary conditions lead to an empty solution. ≫

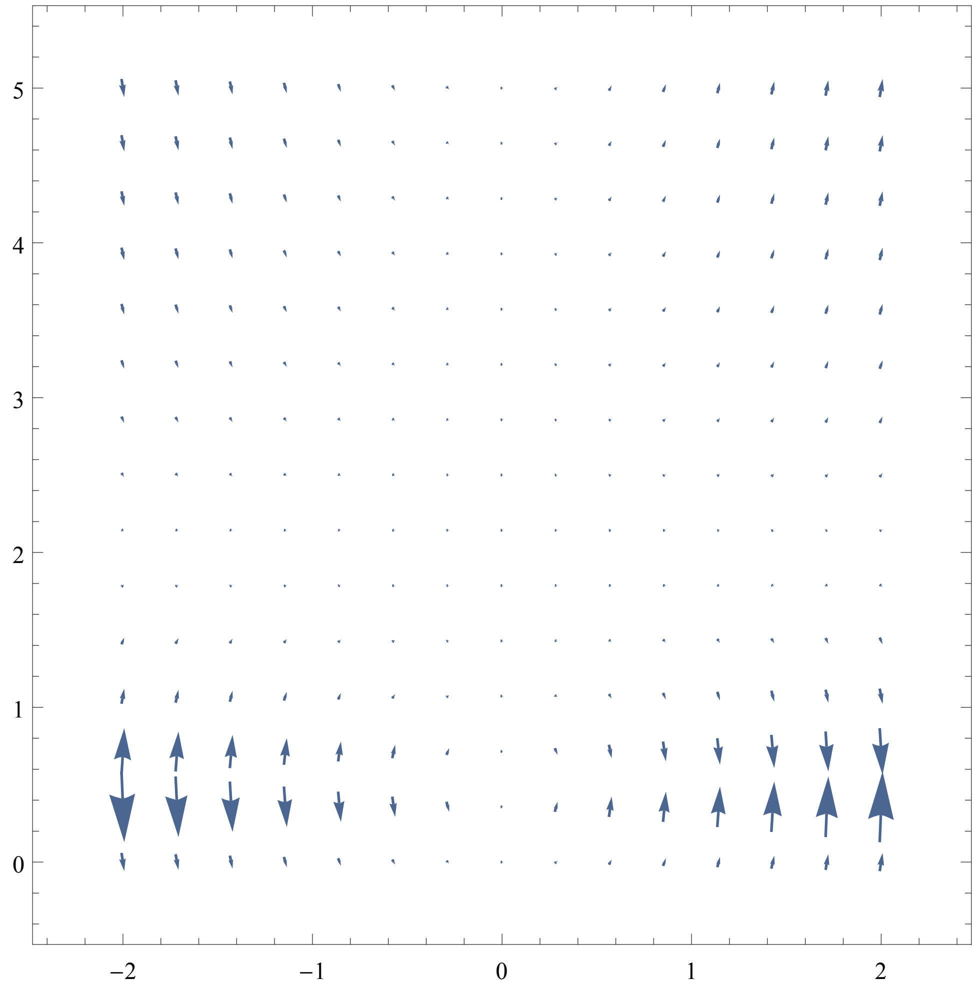
Solve::ifun : Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

General::stop : Further output of Solve::ifun will be suppressed during this calculation. ≫

Out[46]= $\left\{\left\{y(x) \to \frac{1}{2}\left(\sqrt{16\, e^{x^2} + 9} + 1\right)\right\}\right\}$

In[47]:=
```
plot1 = DirectionField[DE, y[x], {x, -2, 2}, {y, 0, 5}, Frame → True]
```

Out[47]=



In[48]:=
```
plot2 = Plot[
  Evaluate[Table[y[x] /. DSolve[{DE, y[0] == k/5}, y[x], x][[1]], {k, 20}]],
  {x, -2, 2},
  PlotStyle → Table[{Thickness[0.005], RGBColor[k/20, 0, 1 - k/20]}, {k, 20}]]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

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General::stop : Further output of Solve::ifun will be suppressed during this calculation. ≫

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General::stop :
Further output of DSolve::bvnul will be suppressed during this calculation. ≫

Part::partw : Part 1 of {} does not exist. ≫

ReplaceAll::reps :
{{}⟦1⟧} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. ≫

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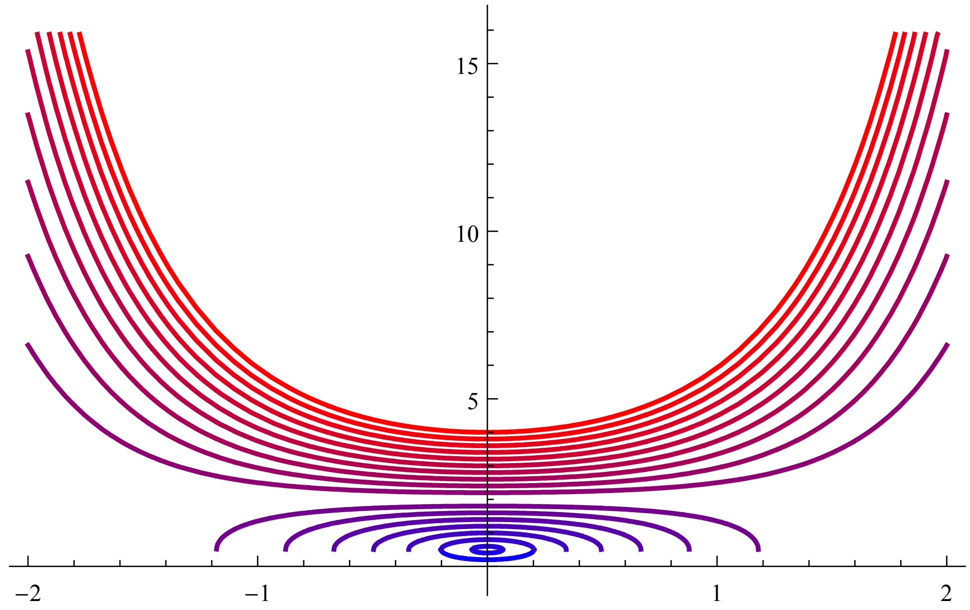
{{}⟦1⟧} is neither a list of replacement rules nor a valid dispatch table, and so cannot be used for replacing. ≫

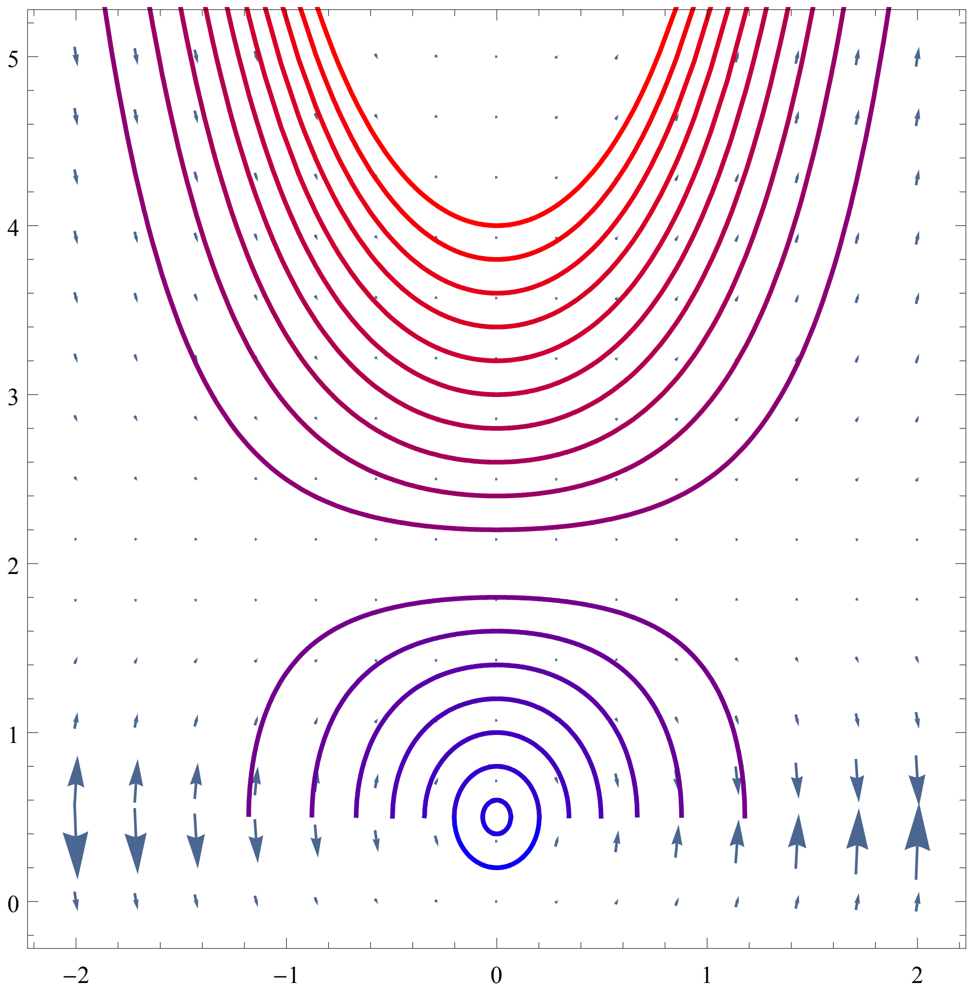
General::stop :
Further output of ReplaceAll::reps will be suppressed during this calculation. ≫

Out[48]=



In[49]:= **Show[plot1, plot2, PlotRange → {0, 5}]**

Out[49]=



Ein Beispiel, bei welchem *Mathematica* die Lösung (nach Auflösen nach y[x]) durch spezielle Funktionen ausdrückt.

In[50]:=
```
DE1 = y'[x] == (y[x] + 1)/(y[x] - 1)
```

Out[50]= $y'(x) = \frac{y(x)+1}{y(x)-1}$

In[51]:=
```
DSolve[DE1, y[x], x]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[51]= $\left\{\left\{y(x) \to -2\, W\left(-\frac{1}{2}\sqrt{e^{-c_1-x-1}}\right) - 1\right\}, \left\{y(x) \to -2\, W\left(\frac{1}{2}\sqrt{e^{-c_1-x-1}}\right) - 1\right\}\right\}$

In[52]:=
```
FullForm[%]
```

Out[52]//FullForm=

List[List[Rule[y[x], Plus[−1, Times[−2, ProductLog[Times[Rational[−1, 2],
Power[Power[E, Plus[−1, Times[−1, x], Times[−1, C[1]]]], Rational[1, 2]]]]]]]],
List[Rule[y[x], Plus[−1, Times[−2, ProductLog[Times[Rational[1, 2],
Power[Power[E, Plus[−1, Times[−1, x], Times[−1, C[1]]]], Rational[1, 2]]]]]]]]]

In[53]:=
```
? ProductLog
```

ProductLog[z] gives the principal solution for w in $z = we^w$.
ProductLog[k, z] gives the k^{th} solution. ≫

Zum Schluss noch ein Beispiel, bei welchem *Mathematica* fälschlicherweise keine korrekte Fallunterscheidung vornimmt.

In[54]:=
```
DE = y'[x] == x Sqrt[1 + y[x]]
```

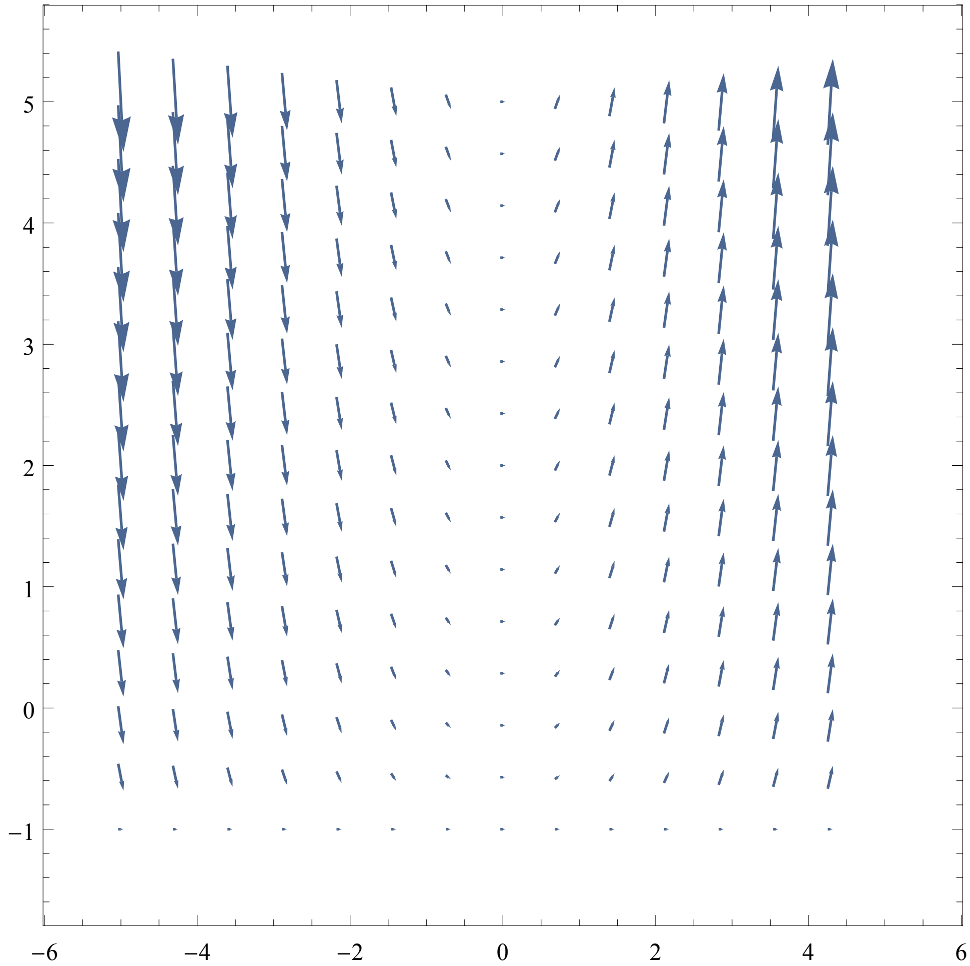
Out[54]= $y'(x) = x\sqrt{y(x)+1}$

In[55]:=
```
DSolve[DE, y[x], x]
```

Out[55]= $\left\{\left\{y(x) \to \frac{1}{16}\left(4\, c_1\, x^2 + 4\, c_1^2 + x^4 - 16\right)\right\}\right\}$

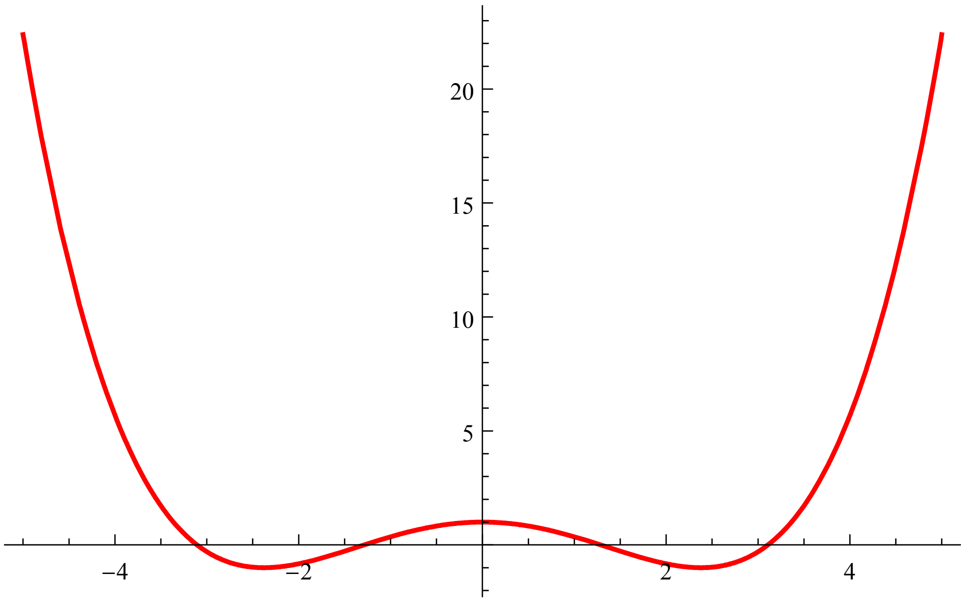
In[56]:= `plot1 = DirectionField[DE, y[x], {x, -5, 5}, {y, -1, 5}, Frame → True]`

Out[56]=



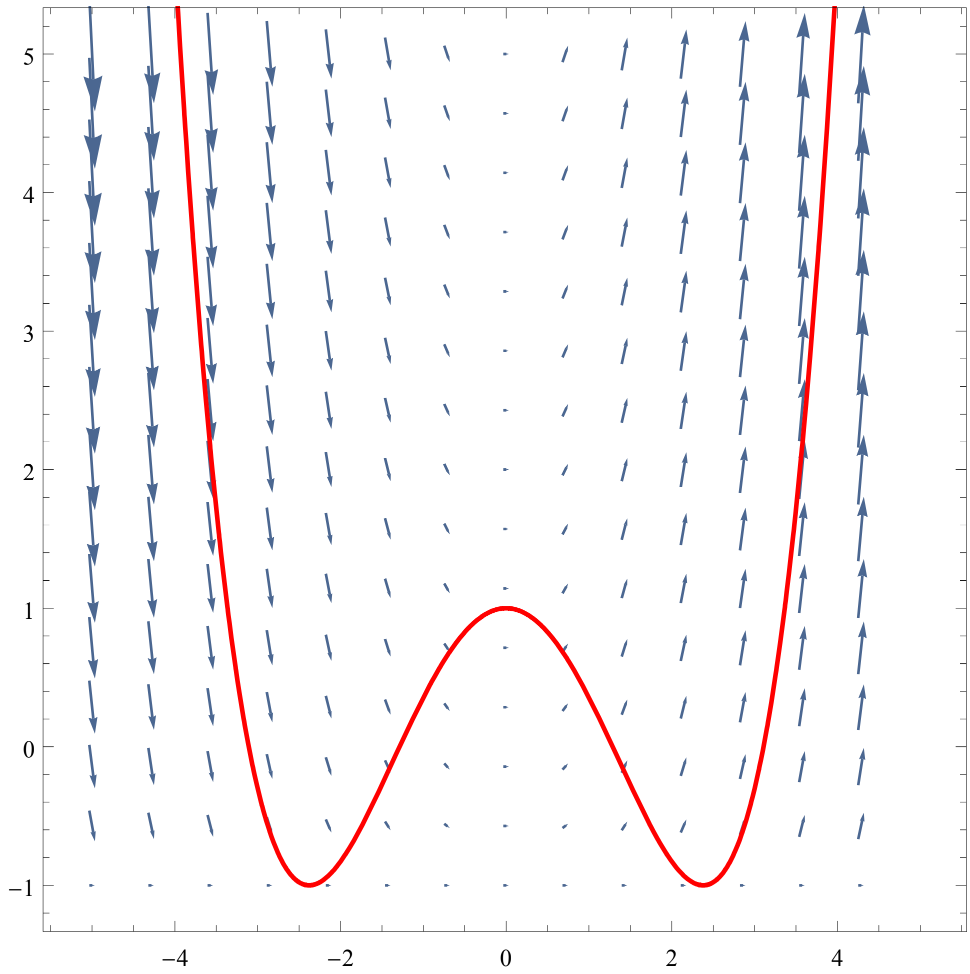
In[57]:= `plot2 = Plot[Evaluate[y[x] /. DSolve[{DE, y[0] == 1}, y[x], x][[1]]], {x, -5, 5}, PlotStyle → {Thickness[0.005], RGBColor[1, 0, 0]}]`

Out[57]=



In[58]:= `Show[plot1, plot2, PlotRange → {-1, 5}]`

Out[58]=



Hausaufgabe: Bernoulli-Differentialgleichung

In[59]:= `DE = y'[x] == (y[x]^2 + x^2 y[x]) / x^3`

Out[59]= $y'(x) = \frac{x^2\, y(x) + y(x)^2}{x^3}$

In[60]:= `DSolve[DE, y[x], x]`

Out[60]= $\left\{\left\{y(x) \to \frac{x^2}{c_1\, x + 1}\right\}\right\}$