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Fourth-order difference equation for the associated classical discrete orthogonal polynomials

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Abstract

We derive the fourth-order difference equation satisfied by the associated order r of classical orthogonal polynomials of a discrete variable.

The coefficients of this equation are given in terms of the polynomials σ and τ which appear in the discrete Pearson equation $\Delta(\sigma\rho)=\tau\rho$ defining the weight $\rho(x)$ of the classical discrete orthogonal polynomials.

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1. Introduction

The fourth-order difference equation for the associated Meixner and Charlier polynomials were given for all r (order of association) in [6], using an explicit solution of the recurrence relation built from the symmetry properties of this recurrence. On the other hand, the equation for the first associated ($r=1$) of all classical discrete polynomials was obtained in [10] using a useful relation proved in [1].

In this work, we give a single fourth-order difference equation which is valid for all integers r and for all classical discrete orthogonal polynomials. This equation is important in birth and death processes [6] and also for some connection coefficient problems [7].

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Let $(P_n)_n$ be a sequence of monic orthogonal polynomials of degree n with respect to the regular linear functional $\mathscr{L}$. $(P_n)_n$ satisfies the following second-order recurrence relation:

$$P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad n \geqslant 1, \ \gamma_n \neq 0,$$
$$P_0(x) = 1, \qquad P_1(x) = x - \beta_0.$$

The associated orthogonal polynomials of order $r, P_n^{(r)}$, are defined by the shifted recurrence relation ($n \to n+r$ in β_n and γ_n)

$$P_{n+1}^{(r)}(x) = (x-\beta_{n+r})P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x), \quad n \geqslant 1,$$
$$P_0^{(r)}(x) = 1, \qquad P_1^{(r)}(x) = x - \beta_r, \quad r \geqslant 0.$$

When the family $(P_n)_n$ is classical (continuous), the polynomials P_n are solutions of the hypergeometric equation

$$L_{2,0}[y] \equiv \sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where σ is a polynomial of degree at most two, τ is a polynomial of degree one, and λ_n is a constant [8].

From the following coupled second-order relations [2, 9]:

$$L_{2,r}[P_n^{(r)}] = K_r[P_{n-1}^{(r+1)}]', \qquad L_{2,r}^*[P_{n-1}^{(r+1)}] = K_r^*[P_n^{(r)}]' \tag{1}$$

with

$$L_{2,r} = \sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (x-\beta_r)\ (r\sigma'' + \tau')\frac{\mathrm{d}}{\mathrm{d}x} + (\lambda_n - nr\sigma''),$$
$$L_{2,r}^* = \sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2} - [(x-\beta_r)\ (r\sigma''+\tau') - 2\sigma']\frac{\mathrm{d}}{\mathrm{d}x} + (\lambda_n^* - (n+1)r\sigma''), \tag{2}$$
$$K_r^* = -[(2r-1)\sigma'' + 2\tau'], \qquad K_r = \begin{cases} \gamma_r[(2r-3)\sigma'' + 2\tau'] & \text{if } r \geqslant 1, \\ 0 & \text{if } r = 0 \end{cases}$$

in [11], using the representation of γ_r and β_r in terms of σ and τ the generic fourth-order differential equation $M_4^{(r)}(P_n^{(r)}(x)) = 0$ satisfied by the associated of any integer order r of the classical class was derived, where

$$\begin{aligned} M_4^{(r)} = {} & \sigma^2\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 5\sigma\sigma'\frac{\mathrm{d}^3}{\mathrm{d}x^3} + [-\tau^2 + 2\tau\sigma' + 3\sigma'^2 - (2n+4r)\sigma\tau' \\ & + (4 + n - n^2 + 4r - 2nr - 2r^2)\sigma\sigma'']\frac{\mathrm{d}^2}{\mathrm{d}x^2} \\ & - \frac{3}{2}[2\tau\tau' + (2n-2+4r)\sigma'\tau' - 2\tau\sigma'' + (n^2 - n - 4r + 2nr + 2r^2)\sigma'\sigma'']\frac{\mathrm{d}}{\mathrm{d}x} \\ & + \frac{1}{4}\{n(2+n)[2\tau' + (n+2r-3)\sigma''][2\tau' + (n+2r-1)\sigma'']\}. \end{aligned} \tag{3}$$

In this letter we want to extend these results to the classical discrete class, i.e., the polynomials P_n of Hahn, Hahn–Eberlein, Krawtchouk, Meixner, and Charlier, which are solutions of the second-order difference equation

$$\sigma(x)\Delta\nabla y(x)+\tau(x)\Delta y(x)+\lambda_n y(x)=0$$

with $\Delta y(x)=y(x+1)-y(x)$ and $\nabla y(x)=y(x)-y(x-1)$. It turns out that the coefficient λ_n is given by $2\lambda_n=-n[(n-1)\sigma''+2\tau']$, see [8]. From the known relations between the recurrence coefficients β_n, γ_n and the polynomials σ and τ [5, 4],

$$\begin{aligned}
\beta_n&=\frac{-\tau(0)\ (\tau'-\sigma'')-n(\tau'+2\sigma'(0))\ (\tau'+(n-1)\sigma''/2)}{((n-1)\sigma''+\tau')(n\sigma''+\tau')},\quad n\geqslant 0,\\
\gamma_n&=-\frac{n(\tau'+(n-2)\sigma''/2)}{(\tau'+(2n-3)\sigma''/2)(\tau'+(2n-1)\sigma''/2)}[\sigma(\eta_{n-1})+\tau(\eta_{n-1})],\quad n\geqslant 1,\\
\eta_n&=-\frac{\tau(0)+n\sigma'(0)-n^2\sigma''/2}{\tau'+n\sigma''},\quad n\geqslant 0,
\end{aligned}\tag{4}$$

it is possible to write the corresponding Eqs. (1)–(3) again in terms of σ and τ, for the generic classical discrete polynomials.

Proposition 1 (Foupouagnigni et al. [3]). *The associated polynomials satisfy*

$$\mathscr{D}_{r,n}[P_n^{(r)}]=\mathscr{N}_{r+1,n-1}[P_{n-1}^{(r+1)}],\tag{5}$$

$$\bar{\mathscr{D}}_{r+1,n-1}[P_{n-1}^{(r+1)}]=\bar{\mathscr{N}}_{r,n}[P_n^{(r)}],\tag{6}$$

where

$$\mathscr{D}_{r,n}=a_2\mathscr{T}^2+a_1\mathscr{T}+a_0\mathscr{T}^0,\qquad \mathscr{N}_{r+1,n-1}=\tilde{a}_1\mathscr{T}+\tilde{a}_0\mathscr{T}^0,\tag{7}$$

$$\bar{\mathscr{D}}_{r+1,n-1}=b_2\mathscr{T}^2+b_1\mathscr{T}+b_0\mathscr{T}^0,\qquad \bar{\mathscr{N}}_{r,n}=\tilde{b}_1\mathscr{T}+\tilde{b}_0\mathscr{T}^0,\tag{8}$$

$$a_2=k_{9,0},\qquad a_1=-k_{2,1}k_{10,0},\qquad a_0=k_{11,0},\qquad \tilde{a}_1=k_{4,0}k_{10,0},\qquad \tilde{a}_0=-k_{4,0}k_{12,0}\tag{9}$$

$$b_2=k_{9,0},\qquad b_1=-k_{5,1}k_{10,0},\qquad b_0=k_{13,0},\qquad \tilde{b}_1=k_{6,0}k_{10,0},\qquad \tilde{b}_0=-k_{6,0}k_{14,0},\tag{10}$$

$\mathscr{T}$ is the shift operator: $\mathscr{T}P(x)=P(x+1)$, and the coefficients $k_{i,j}$ are polynomials given by

$$k_{i,0}(x)=k_i(x)\quad\textit{and}\quad k_{i,j}(x)=k_i(x+j)\tag{11}$$

and (16).

2. Fourth-order difference equation for associated polynomials

Replacing $\mathscr{T}^2P_{n-1}^{(r+1)}$ given by (6) in the shifted Eq. (5), we obtain

$$[c_3\mathscr{T}^3+c_2\mathscr{T}^2+c_1\mathscr{T}+c_0](P_n^{(r)})=[\tilde{c}_1\mathscr{T}+\tilde{c}_0](P_{n-1}^{(r+1)}).\tag{12}$$

By the same process, using again $\mathscr{T}^2 P_{n-1}^{(r+1)}$ given by (6) in the shifted Eq. (12), we obtain

$$[d_4\mathscr{T}^4 + d_3\mathscr{T}^3 + d_2\mathscr{T}^2 + d_1\mathscr{T} + d_0](P_n^{(r)}) = [\tilde{d}_1\mathscr{T} + \tilde{d}_0](P_{n-1}^{(r+1)}), \tag{13}$$

where the polynomial coefficients $c_i, \tilde{c}_i, d_i$ and $\tilde{d}_i$ are easily computed from the coefficients $a_i, \tilde{a}_i, b_i$ and $\tilde{b}_i$.

Now, use of Eqs. (5), (12) and (13) gives the expected fourth-order difference equation satisfied by each $P_n^{(r)}$

$$\begin{vmatrix} a_2\mathscr{T}^2P_n^{(r)} + a_1\mathscr{T}P_n^{(r)} + a_0P_n^{(r)} & \tilde{a}_1 & \tilde{a}_0 \\ c_3\mathscr{T}^3P_n^{(r)} + c_2\mathscr{T}^2P_n^{(r)} + c_1\mathscr{T}P_n^{(r)} + c_0P_n^{(r)} & \tilde{c}_1 & \tilde{c}_0 \\ d_4\mathscr{T}^4P_n^{(r)} + d_3\mathscr{T}^3P_n^{(r)} + d_2\mathscr{T}^2P_n^{(r)} + d_1\mathscr{T}P_n^{(r)} + d_0P_n^{(r)} & \tilde{d}_1 & \tilde{d}_0 \end{vmatrix} = 0, \tag{14}$$

which can be written in the form

$$\left[\sum_{j=0}^{4} I_j(r,n,x)\mathscr{T}^j\right] P_n^{(r)}(x) = 0.$$

We have used Maple V Release 4 to compute the coefficients I_j depending on r, n and x, and after cancelling common factors, we obtain

$$\begin{aligned}
I_4 &= k_{9,2}(k_{10,0}k_{10,1} - k_{12,0}k_{12,1}),\\
I_3 &= k_{10,2}(k_{12,0}(k_{2,3}k_{12,1} + k_{13,1}) - k_{10,0}k_{10,1}(k_{2,3} + k_{5,2})) + k_{9,1}k_{10,0}k_{12,2},\\
I_2 &= k_{10,1}(k_{10,2}(k_{10,0}k_{10,1} + k_{13,0} - k_{5,1}k_{12,0}) - k_{9,1}k_{10,0}) - k_{12,1}(k_{12,2}k_{13,0} + k_{11,2}k_{12,0}),\\
I_1 &= k_{10,0}k_{12,2}(k_{2,2}k_{12,0} + k_{13,0}) + k_{10,2}k_{12,0}(k_{9,0} - k_{10,0}k_{10,1}),\\
I_0 &= k_{9,-1}(k_{10,1}k_{10,2} - k_{12,1}k_{12,2}),
\end{aligned} \tag{15}$$

where the polynomials $k_{i,j}$ are given by (11) and

$$\begin{aligned}
E_r(x) &= \tau(x) - \frac{\tau(\beta_r)}{2} + r\frac{\tau'}{2} + (r^2 - r(1+2\beta_r) - 2)\frac{\sigma''}{4} + (r-2)\frac{\sigma'(x)}{2} - \sigma'(0)\frac{r}{2},\\
F_r(x) &= -\frac{\tau(\beta_r)}{2} - r\frac{\tau'}{2} - (r^2 - r(3-2\beta_r))\frac{\sigma''}{4} + (\sigma'(x) - \sigma'(0))\frac{r}{2},
\end{aligned} \tag{16}$$

$$\zeta_n = (2n-1)\frac{\sigma''}{2} + \tau', \qquad k_1(x) = \sigma(x+1) + E_{n+r+1}(x),$$

$$k_2(x) = \sigma(x+1) - F_r(x), \qquad k_3(x) = \zeta_{n+r}, \qquad k_4(x) = \begin{cases} \gamma_r\zeta_{r-1} & \text{if } r \geqslant 1,\\ 0 & \text{if } r=0,\end{cases}$$

$$k_5(x) = \sigma(x+1) + E_r(x), \qquad k_6(x) = -\zeta_r, \qquad k_7(x) = \sigma(x+1) - F_{n+r+1}(x),$$

$$k_8(x) = -\gamma_{n+r+1}\zeta_{n+r+1}, \qquad k_9(x) = k_7(x+1)k_1(x+1) - k_3(x)k_8(x),$$

$$k_{10}(x)=k_7(x+1)+k_1(x),\qquad k_{11}(x)=k_2(x+1)k_2(x)+k_4(x)k_6(x),$$

$$k_{12}(x)=k_2(x+1)+k_5(x),\qquad k_{13}(x)=k_5(x+1)k_5(x)+k_4(x)k_6(x),$$

$$k_{14}(x)=k_5(x+1)+k_2(x).$$

The polynomials k_3, k_4, k_6, k_8 are constant with respect to the variable x and β_r, γ_r are given by (4).

If $r=0$, from (2), and (16) we have $k_4=K_0=0$. Then, $\mathcal{N}_{1,n-1}$ is equal to zero, thus the fourth-order difference equation for the first associated $P_n^{(1)}$ factorizes in the form [1, 3, 10] $(\bar{A}_1\mathcal{T}^2+\bar{B}_1\mathcal{T}+\bar{C}_1\mathcal{T}^0)(A_1\mathcal{T}^2+B_1\mathcal{T}+C_1\mathcal{T}^0)[P_n^{(1)}]=0$.

For $r=0$, if we are inside the Hahn class with $\alpha+\beta+1=0$ (discrete Grosjean polynomials), from (2), (16), and [2, 9] we have $K_0^*=2k_6=0$. Then $\bar{\mathcal{N}}_{1,n}$ is equal to zero and the difference equation in this case reduces to the second-order difference equation $\bar{\mathcal{D}}_{1,n}[P_n^{(1)}]=0$.

Using the result of this letter, we have computed the coefficients I_j for all classical polynomials of a discrete variable, generalizing the results given in [6, 10].

For the Krawtchouk case for example, ($\sigma(x)=x$, $\tau(x)=(1/q)((1-q)N-x)$), the rth Krawtchouk associated $P_n^{(r)}$ with $n+r\leqslant N$ is annihilated by the following difference operator, where t is given by $t=r+x-2xq+qN-5q-N+2$:

$$\begin{aligned}
&q(4+x)(x+3-N)(q-1)(n-2+4q+2t)\mathcal{T}^4-(10xq+nq-6Nq\\
&\quad-42q^2-4xq^3N-2xNq+2x^2q-12q^3N+20xq^3+4x^2q^3+28q^3\\
&\quad-3nq^2+14q-2t-n^2q+18q^2N+6xq^2N-30xq^2-6x^2q^2+3nt^2\\
&\quad+2t^3-6tq^2+n^2t-nt+6tq)\mathcal{T}^3-(10xq-8nq-6Nq-42q^2\\
&\quad-5nNq^2+8xq^2n-4xq^3N+2x^2nq^2+5nNq+2xNqn+n^2\\
&\quad-2xNq+2x^2q-12q^3N+20xq^3+4x^2q^3-2xNq^2n+28q^3+6nq^2\\
&\quad+14q-2t-8xqn-n^3-4n^2q+18q^2N+6xq^2N-30xq^2-2x^2nq\\
&\quad-6x^2q^2-12nqt-4x^2qt-6nt^2-4t^3+12tq^2-12qt^2-4n^2t\\
&\quad+4nt-12tq+6t^2-10q^2Nt-16xqt+16xq^2t-4xq^2Nt+4xNqt+10Nqt+4x^2q^2t)\mathcal{T}^2\\
&\quad+(6xq+9nq-4Nq-12q^2-4xq^3N+2n^2-2xNq+2x^2q-8q^3N+12xq^3+4x^2q^3+8q^3\\
&\quad-9nq^2+4q-2n-4t-3n^2q+12q^2N+6xq^2N-18xq^2-6x^2q^2\\
&\quad-12nqt-3nt^2-2t^3-18tq^2-12qt^2-n^2t+7nt+18tq+6t^2)\mathcal{T}\\
&\quad+q(1+x)(x-N)(q-1)(2t+n).
\end{aligned}$$

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