Stieltjes interlacing of zeros of classical orthogonal sequences

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Abstract

Consider $\{p_n\}_{n=0}^{\infty}$, a sequence of polynomials orthogonal with respect to $w(x) > 0$ on (a, b), and polynomials $\{g_{n,k}\}_{n=0}^{\infty}, k \in \mathbb{N}_0$, orthogonal with respect to $c_k(x)w(x) > 0$ on (a, b), where $c_k(x)$ is a polynomial of degree k in x. We provide conditions necessary for mixed three-term recurrence equations involving the polynomials p_n, p_{n-1} and $g_{n-m,k}, m \in \{2, 3, \ldots, n-1\}$, to exist, where the coefficient of p_{n-1} is a polynomial of degree $m-1$, say G_{m-1}. It is known that, since the zeros of p_n and $G_{m-1}g_{n-m,k}$ interlace, the zeros of G_{m-1} are inner bounds for the extreme zeros of the (classical or q-classical) orthogonal polynomial p_n. A computational approach (Zeilberger's algorithm) is used to obtain the mixed three-term recurrence equations and we give examples where $m = 3$ and/or $m = 4$ to illustrate the accuracy of the bounds obtained from these mixed recurrence equations.

Keywords: Classical orthogonal polynomials, q-classical orthogonal polynomials, mixed three-term recurrence equations, interlacing of zeros, bounds for zeros
2010 MSC: 33C05, 33C45, 33F10, 33D15

1. Introduction

A sequence $\{p_n\}_{n=0}^{\infty}$ of real polynomials, where p_n is of exact degree n, is orthogonal with respect to an absolutely continuous measure that can be represented by a real, positive weight function $w(x)$ on the (finite or infinite) interval (a, b), if

$$\int_a^b p_n(x)p_m(x)w(x)dx = 0, m \neq n.$$

The classical orthogonal polynomials considered in this paper are defined in terms of the generalized hypergeometric series

$${}_pF_q\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!},$$

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where $(a)_m$ denotes the Pochhammer symbol (or shifted factorial) defined by

$$(a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a+1)(a+2)\cdots(a+m-1) & \text{if } m \in \mathbb{N}. \end{cases}$$

Their q-orthogonal analogues, $0 < q < 1$, are given in terms of basic hypergeometric series [19, Section 1.10]

$${}_r\phi_s\left(\begin{matrix} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{matrix} \middle| q; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_k}{(b_1, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^k}{(q;q)_k},$$

where the q-Pochhammer symbol $(a_1, a_2, \ldots, a_k; q)_n$ is defined by

$$(a_1, \ldots, a_r; q)_k := (a_1; q)_k \cdots (a_r; q)_k, \quad \text{with} \quad (a_i; q)_k = \begin{cases} \prod_{j=0}^{k-1}(1 - a_i q^j) & \text{if } k \in \{1, 2, 3, \ldots\} \\ 1 & \text{if } k = 0. \end{cases}$$

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of monic orthogonal polynomials with zeros $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$. It is well known that p_n satisfies a three-term recurrence equation

$$p_n(x) = (x - B_n)p_{n-1}(x) - C_n p_{n-2}(x), \tag{1}$$

where B_n and C_n do not depend on x, $p_{-1} \equiv 0$, $p_0 \equiv 1$ and $C_n > 0$, and that the zeros of p_n and p_{n-1} interlace, i.e.,

$$x_{n,1} < x_{n-1,1} < x_{n,2} < \cdots < x_{n,n-1} < x_{n-1,n-1} < x_{n,n}.$$

Further, the zeros of p_n and $q_m, m \leq n-2$, are interlacing (often called Stieltjes interlacing) if there exist m open intervals with endpoints at successive zeros of p_n, each of which contains exactly one zero of q_m (cf. [28, Theorem 3.3.3]).

It is also known that, if p_n and p_{n-2} do not have a common zero, then the $n-1$ zeros of $(x - B_n)p_{n-2}(x)$ interlace with the n zeros of p_n [4, Theorem 3], therefore $x_{n,1} < B_n < x_{n,n}$ and the point B_n is a natural inner bound for the extreme zeros of p_n. Beardon generalises this result in [4, Theorem 4] and we state it here as a lemma:

Lemma 1. *Suppose $\{p_n\}_{n=0}^{\infty}$ is a sequence of polynomials, satisfying (1). Then, given n, there exists real polynomials S_m of degree m, where $m < n-2$, such that*

$$(-1)^m C_1 C_2 \ldots C_{n+m-1} p_{n-m}(x) = S_{m-1}(x)p_{n-1}(x) + S_{m-2}(x)p_n(x) \tag{2}$$

and if p_{n-m} and p_n do not have any common zeros, their zeros interlace in the Stieltjes sense. More-over, the $n-1$ zeros of $S_{m-1}p_{n-m}$ interlace with the n zeros of p_n.

An important feature of the polynomials S_{m-1} is that they are completely determined by the coefficients in (1) (cf. [27, Theorem 1]). Furthermore, we will refer to the interlacing of the $n-1$ zeros of $S_{m-1}p_{n-m}$ with the n zeros of p_n as completed Stieltjes interlacing. A natural consequence of Lemma 1 is

Corollary 2. *(cf. [10, Corollary 2.2]) Suppose (2) holds for $m, n \in \mathbb{N}$ fixed, $m < n-2$. The smallest and largest zeros of S_{m-1} are inner bounds for the extreme zeros of p_n.*

Equations similar to (2), involving polynomials p_n, p_{n-1}, and $g_{n-m}, m \in \{2, 3, \dots\}$, where the polynomial g_{n-m} belongs to a related orthogonal sequence, obtained by integer shifts of the appropriate parameters, are used to obtain (more accurate) inner bounds for the extreme zeros of orthogonal sequences (cf. [10]). However, as in (2), the coefficient, say G_{m-1}, of $p_{n-1}(x)$ needs to be a polynomial of exact degree $m-1$ in order to have full interlacing between the n zeros of p_n and the $n-1$ zeros of $G_{m-1}(x)g_{n-m}(x)$. In [15, Theorem 2.1], conditions necessary for the existence of such mixed three-term recurrence equations are given for $m = 2$:

If, for $k \in \mathbb{N}_0$ fixed and $\{g_{n,k}\}_{n=0}^{\infty}$ a sequence of polynomials orthogonal with respect to $c_k(x)w(x) > 0$ on (a, b), where $c_k(x)$ is a polynomial of degree k in x, the sequence $\{p_n\}_{n=0}^{\infty}$ satisfies

$$A_n c_k(x) g_{n-2,k}(x) = a_{k-2}(x) p_n(x) - (x - B_n) p_{n-1}(x), \; n \in \{2, 3, \dots\}, \tag{3}$$

with A_n, B_n, a_{-1}, a_{-2} constants and a_{k-2} a polynomial of degree $k-2$ defined on (a, b) whenever $k \in \{2, 3, \dots\}$, then $k \in \{0, 1, 2, 3, 4\}$.

In this paper we generalise the result in [15, Theorem 2.1] and we provide the values of k for which mixed three-term recurrence equations, similar to (3), involving the polynomials $g_{n-m,k}, m \in \{2, 3, \dots, n-1\}$, p_n and p_{n-1}, exist. These equations provide us with associated polynomials of degree $m-1$, say, $G_{m-1}, m \in \{2, 3, \dots, n-1\}$, whose extreme zeros can be used as inner bounds for the extreme zeros of p_n. The bounds obtained in this way are more accurate than the inner bounds obtained using mixed recurrence equations in the specific case when $m = 2$, as was done for the extreme zeros of the Jacobi, Laguerre and Gegenbauer polynomials in [10], Meixner and Krawtchouk polynomials in [15] and Hahn polynomials in [18]. In our applications, the polynomials $g_{n-m,k}, m \in \{2, 3, \dots, n-1\}$ are typically obtained from the polynomials of the orthogonal sequence $\{p_n\}_{n=0}^{\infty}$, by making appropriate parameter shifts of (in total) k units. In this case, the polynomials $G_{m-1}, m \in \{2, 3, \dots, n-1\}$, are completely determined by the coefficients B_n and C_n in the three-term recurrence satisfied by $\{p_n\}_{n=0}^{\infty}$.

The interlacing property of zeros of polynomials is important in numerical quadrature applications. Levit [23] was the first to study the separation of the zeros of different sequences of Hahn polynomials and interlacing results for Jacobi polynomials [3, 7], Krawtchouk polynomials [6, 16] and Meixner and Meixner-Pollaczek polynomials [16] followed. Interlacing results for the zeros of different sequences of q-orthogonal sequences with shifted parameters are given in [12, 17, 24, 29]. Completed Stieltjes interlacing of zeros of different orthogonal sequences was done for the Gegenbauer [11], Laguerre [9] and Jacobi polynomials [8] and apart from the papers cited in the previous paragraph, inner bounds for the extreme zeros of Gegenbauer, Laguerre and Jacobi polynomials were also given in [1, 5, 14, 21, 25, 28]; bounds for the extreme zeros of the discrete orthogonal Charlier, Meixner, Krawtchouk and Hahn polynomials in [2, 22], for the extreme zeros of the q-Jacobi and q-Laguerre polynomials in [14] and for the little q-Jacobi polynomials in [12].

In general it is time-consuming to find the zeros of an expanded polynomial of high degree using a computer program such as Maple, since one has to work with high precision. In such cases it is helpful to have a formula for a bound.

Our main result is stated in section 2, where we also describe the algorithmic approach used to obtain the mixed recurrence equations necessary to prove Stieltjes interlacing between the zeros of polynomials from different orthogonal sequences. In the last three sections we display mixed recurrence equations that provide formulas for inner bounds for the extreme zeros of some specific polynomial systems, merely to show the accuracy of the bounds provided by these recurrence equations. In section 3 we show bounds in the generalized Laguerre case. In section 4 we show the quality of bounds obtained for the extreme zeros of the little q-Jacobi and the alternative q Charlier (or q-Bessel) polynomials, where the equation providing the bounds concerns polynomials $p_n(x;\alpha,\beta)$, $p_{n-1}(x;\alpha,\beta)$ and either $p_{n-m}(x;\alpha+k,\beta)$ or $p_{n-m}(x;\alpha,\beta+k)$, i.e., only one parameter is shifted. Since we observe that the best possible bounds are obtained when one parameter is shifted optimally, polynomial systems depending on 2 or 3 parameters also belong to this group. In section 5, we provide bounds for the extreme zeros of the Stieltjes-Wigert and the discrete q-Hermite II polynomials, as examples of polynomial systems where no parameters are available to be shifted.

In the file http://www.mathematik.uni-kassel.de/~koepf/Publikationen, we provide optimal (upper and/or lower) inner bounds for the extreme zeros of following polynomial systems:

(i) The Gegenbauer, Laguerre, Jacobi, Bessel, Meixner and Hahn polynomials, as well as the big q-Jacobi, q-Hahn, q-Meixner, little q-Laguerre (or Wall), little q-Jacobi, q-Krawtchouk, affine q-Krawtchouk, q-Laguerre, alternative q-Charlier (or q-Bessel) and q-Charlier polynomials. In these cases the bounds were obtained from shifting one parameter optimally. By shifting α to αq^{-6} and γ to γq^{-6}, respectively, we obtain a lower bound for the largest zero of the q-Charlier polynomials and an upper bound for the smallest zero of the q-Meixner polynomial;

(ii) The Hermite, Krawtchouk and Charlier polynomials, as well as the quantum q-Krawtchouk, Stieltjes Wigert, Al-Salam Carlitz I and II and discrete q-Hermite I and II polynomials, where we use equations similar to (2) to provide the bounds. In this way we also find a lower bound for $x_{n,n}$ in the q-Meixner case and an upper bound for the smallest zero of the q-Charlier polynomials.

2. Stieltjes interlacing of zeros of classical orthogonal sequences

We generalise the result in [15, Theorem 2.1] by providing conditions necessary for equations, similar to (3), involving the polynomials $g_{n-m,k}, m \in \{2,3,\dots,n-1\}$, p_n and p_{n-1}, to exist.

Theorem 3. *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the (finite or infinite) interval (a,b) with respect to the weight function $w(x) > 0$. Let $k \in \mathbb{N}_0$ and $m \in \{2,3,\dots,n-1\}$ be fixed and suppose $\{g_{n,k}\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to $c_k(x)w(x) > 0$ on (a,b), where $c_k(x)$ is a polynomial of degree k, that satisfies*

$$A_n c_k(x) g_{n-m,k}(x) = a_{k-m}(x) p_n(x) - G_{m-1}(x) p_{n-1}(x), \ n \in \{2,3,\dots\}, \tag{4}$$

with A_n, B_n, a_{-1} and a_{-2} constants, a_{k-m} a polynomial defined on (a,b) and of degree $m-2$ when $k-m \in \{-m,-m+1,\dots,m-2\}$ and of degree $k-m$ whenever $k-m \in \{m-1,m,m+1,\dots\}$, and $G_m(x)$ a polynomial of degree m. Then

(i) $k \in \{0, 1, 2, \ldots, 2m\}$;

(ii) *if $g_{n-m,k}$ and p_n are co-prime, the $n-1$ real, simple zeros of $G_{m-1}(x)g_{n-m,k}$ interlace with the zeros of p_n, the smallest zero of G_{m-1} is an upper bound for the smallest zero of p_n, and the largest zero of G_{m-1} is a lower bound for the largest zero of p_n;*

(iii) *if $g_{n-m,k}$ and p_n are not co-prime and have r common zeros counting multiplicity, then*
a) $r \leq \min\{m, n-m-1\}$;
b) these r common zeros are simple zeros of G_{m-1};
c) no two successive zeros of p_n, nor its largest or smallest zero can be a zero of G_{m-1};
d) the $n-2r-1$ zeros of $G_{m-1}g_{n-m,k}(x)$, none of which is a zero of p_n, together with the r common zeros of $g_{n-m,k}$ and p_n, interlace with the $n-r$ non-common zeros of p_n;
e) the smallest zero of G_{m-1} is an upper bound for the smallest zero of p_n, and the largest zero of G_{m-1} is a lower bound for the largest zero of p_n.

Proof. (i) Let $j \in \{0, 1, \ldots, n-m-1\}$. Since $g_{n,k}$ is orthogonal with respect to $c_k(x)w(x)$ we have

$$\int_a^b A_n x^j g_{n-m,k}(x) c_k(x) w(x) dx = 0 \text{ for } n \in \{m+1, m+2, \ldots\}.$$

The maximum degree of $x^j G_{m-1}(x)$ is $n-2$, therefore it follows from (4) and the orthogonality of p_{n-1} with respect to the weight $w(x)$ that

$$\begin{aligned} \int_a^b A_n x^j g_{n-m,k}(x) c_k(x) w(x) dx &= \int_a^b x^j G_{m-1}(x) p_{n-1}(x) w(x) dx + \int_a^b x^j a_{k-m}(x) p_n(x) w(x) dx \\ &= \int_a^b q(x) p_n(x) w(x) dx, \end{aligned} \tag{5}$$

where $q(x) = x^j a_{k-m}(x)$ is a polynomial of degree $j+k-m$ when $k-m \in \{m-1, m, m+1, \ldots\}$ and a polynomial of degree $j+m-2$ when $k-m \in \{0, 1, \ldots, m-2\}$.

Since p_n is orthogonal with respect to $w(x)$ on (a, b) and the integral in (5) vanishes for $n \in \{m+1, m+2, \ldots\}$, it follows that $\deg(q(x)) = j+k-m \in \{0, 1, \ldots, n-1\}$ and this is true for all $j \in \{0, 1, \ldots, n-m-1\}$, which implies that $k-m \in \{m-1, m\}$.

Furthermore, $\deg(q(x)) = j+m-2$ when $k-m \in \{0, 1, \ldots, m-2\}$, and, since $j \in \{0, 1, \ldots, n-m-1\}$, the maximum value of $\deg(q(x))$ is $n-3$, consequently the integral in (5) also vanishes for $k-m \in \{0, 1, \ldots, m-2\}$ and the result follows.

(ii) The proof is similar to the proofs in [10, Theorem 2.1 (i), Corollary 2.2 (i)].

(iii) The proof is similar to the proofs in [10, Theorem 2.1 (ii), Corollary 2.2 (ii)].

□

In order to obtain the specific mixed recurrence equations, we write a procedure that applies the Gosper [13, 20] and Zeilberger [20, 26] algorithms. Gosper's algorithm deals with finding, for

j an integer, an anti-difference s_j, for given a_j, i.e., a sequence s_j for which $a_j = \Delta s_j = s_{j+1} - s_j$, in the particular case that s_j is a hypergeometric term, i.e.,

$$\frac{s_{j+1}}{s_j} \in \mathbb{Q}(j).$$

Given $F(n, j, \alpha)$, a hypergeometric term with respect to the three variables n, j and α, Zeilberger's algorithm provides a recurrence equation (with polynomial coefficients) for

$$s_n = \sum_{j=-\infty}^{\infty} F(n, j, \alpha).$$

Suppose we have the particular case where we let

$$a_j = F(n-m, j, \alpha+k) + \sum_{i=0}^{J} \sigma_i(n) F(n-i, j, \alpha),$$

with undetermined variables $\sigma_i(n)$, $i \in \{0, 1, \ldots, J\}$, and

$$s_n := p_n(x; \alpha) = \sum_{j=-\infty}^{\infty} F(n, j, \alpha).$$

We apply Gosper's algorithm to a_j and, if successful, an anti-difference $g(n, j)$ of a_j is found, i.e., $a_j = g(n, j+1) - g(n, j)$ and at the same time $\sigma_i(n)$, $i \in \{0, 1, \ldots, J\}$. By summation, we obtain

$$\begin{aligned}\sum_{j=-\infty}^{\infty} a_j &= \sum_{j=-\infty}^{\infty} \left(F(n-m, j, \alpha+k) + \sum_{i=0}^{J} \sigma_i(n) F(n-i, j, \alpha) \right) \\ &= p_{n-m}(x; \alpha+k) + \sum_{i=0}^{J} \sigma_i(n) p_{n-i}(x; \alpha) = \sum_{j=-\infty}^{\infty} (g(n, j+1) - g(n, j)) = 0,\end{aligned}$$

since the last sum is telescoping. It follows that $p_n(x; \alpha)$ satisfies a recurrence equation of type (4). We refer the reader to [20, Chapters 5–7] and references there-in for more details about the algorithms of Gosper and Zeilberger implemented in the Maple `hsum` package. The q-analogues of Gosper's and Zeilberger's algorithms are implemented in the Maple `qsum` package [20]. We apply an adaption of the `sumdiffeq` [20, p. 210] and `qsumdiffeq` [20, p. 219] procedure of the `hsum` and `qsum` packages, respectively, in order to obtain two procedures to derive recurrence equations of type (4) for the polynomial systems considered in the sequel. The `hsum` and the `qsum` packages can be downloaded from http://www.mathematik.uni-kassel.de/~koepf/Publikationen, as well as the two Maple codes used to obtain our mixed three-term recurrence equations. The first program called `Mixedrec`$(F, k, S(n), s_0, a, s_1, b, s_2)$ finds a recurrence equation of the form

$$S(n - s_0, a + s_1, b + s_2) = \sum_{j=0}^{J} \sigma_j S(n-j, a, b), J \in \{1, 2, \ldots\},$$

where $S(n,a,b) = \sum_{k=-\infty}^{\infty} F$, F is a function of k, n, a and b, and s_0, s_1, s_2 are integers. The second one denoted by $\texttt{qMixedrec}(F, k, S(n), s_0, a, s_1, b, s_2)$ is the q-analogue of the first one and finds a recurrence equation of the form

$$S(n - s_0, aq^{s_1}, bq^{s_2}) = \sum_{j=0}^{J} \sigma_j S(n - j, a, b), J \in \{1, 2, \ldots\}.$$

We use this algebraic method to obtain mixed three-term recurrence equations involving polynomials $p_n(x;\alpha,\beta)$ and $p_{n-1}(x;\alpha,\beta)$, belonging to the same sequence that is orthogonal on an interval (a,b) with respect to a measure $w(x;\alpha,\beta)$, and a polynomial from a related sequence, obtained by integer shifts of the parameters α and β, namely $p_{n-m}(x;\alpha+s,\beta+t), m \in \{2,3,\ldots,n-1\}$, which is orthogonal with respect to

$$w(x;\alpha+s,\beta+t) = c_{s+t}(x;\alpha,\beta)w(x;\alpha,\beta) > 0$$

on (a,b), where $c_k(x;\alpha,\beta)$ is a polynomial of degree k in x. If the sequence is q-orthogonal with respect to the weight $w(x;\alpha,\beta)$, the equations involve the polynomials $p_n(x;\alpha,\beta)$ and $p_{n-1}(x;\alpha,\beta)$, and $p_{n-m}(x;\alpha q^s,\beta q^t), m \in \{2,3,\ldots,n-1\}$ and the latter polynomial is orthogonal with respect to

$$w(x;\alpha q^s,\beta q^t) = c_{s+t}(x;\alpha,\beta)w(x;\alpha,\beta) > 0$$

on (a,b). From Theorem 3(i) it follows that such equations only exist for the values of s and t such that $s+t \in \{0,1,\ldots,2m\}$. We note that the polynomial coefficient of the polynomial $p_{n-1}(x;\alpha,\beta)$ in the mixed recurrence equation involving polynomials $p_n(x;\alpha,\beta)$, $p_{n-1}(x;\alpha,\beta)$ and $p_{n-m}(x;\alpha+s,\beta+t), m \in \{2,3,\ldots,n-1\}$, will be denoted by $G_{m-1}^{(\alpha+s,\,\beta+t)}(x)$.

From Theorem 3(ii) and (iii) we deduce that the smallest and largest zeros of $G_{m-1}(x)$ are (inner) bounds for the extreme zeros of p_n.

3. Bounds for the extreme zeros of the classical orthogonal polynomials

The generalized Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array} \middle| x \right), \ \alpha > -1$$

are orthogonal with respect to $w(x;\alpha) = x^\alpha e^{-x}$ on $(0,\infty)$. Since optimal bounds are obtained (cf. [10]) when we shift α optimally, we use mixed three-term recurrence equations involving the polynomials $L_n^{(\alpha)}(x)$, $L_{n-1}^{(\alpha)}(x)$ and $L_{n-m}^{(\alpha+2m)}(x)$ for $m=3$ and $m=4$, where $L_{n-m}^{(\alpha+2m)}(x)$ is orthogonal on $(0,\infty)$ with respect to $w(x;\alpha+2m)$, where

$$\frac{w(x;\alpha+2m)}{w(x;\alpha)} = x^{2m} = c_{2m}(x),$$

to illustrate the quality of our newly found bounds in the classical case.

When we take $m = 3$, the mixed three-term recurrence equation

$$x^6 L_{n-3}^{(\alpha+6)}(x) = -n a_3(x) L_n^{(\alpha)}(x) + (n+\alpha) G_2^{(\alpha+6)}(x) L_{n-1}^{(\alpha)}(x), \tag{6}$$

with

$$a_3(x) = (n-1)(n-2)(x^3 + 3\,(\alpha+3)\,x^2) - (\alpha+2)_3\,(\alpha+4\,n-3)\,x + (\alpha+1)_5$$

and

$$G_2^{(\alpha+6)}(x) = (\alpha+3)(3n(n+\alpha+1) + (\alpha+1)_2)x^2 - 2(\alpha+2)_3(\alpha+2n+1)x + (\alpha+1)_5.$$

is clearly in the form of (4) and from Theorem 3 we know that the $n-1$ zeros of $G_2^{(\alpha+6)}(x) L_{n-3}^{(\alpha+6)}(x)$ interlace with the n zeros of $L_n^{(\alpha)}(x)$. The zeros of $G_2^{(\alpha+6)}(x)$ are

$$\begin{aligned} B_3^{(\alpha+6)} &= \frac{(\alpha+2)(\alpha+4)(\alpha+2n+1)}{3n(n+\alpha+1)+(\alpha+1)_2} \\ &\pm \frac{\sqrt{(\alpha+2)(\alpha+4)\Big((\alpha^2+6\,\alpha+17)\,(n^2+\alpha n+n) - (\alpha+2)\,(\alpha+1)^2\Big)}}{3n(n+\alpha+1)+(\alpha+1)_2} \end{aligned} \tag{7}$$

and they are inner bounds for the extreme zeros of $L_n^{(\alpha)}(x)$. By computing the values of these bounds, we find that the smallest value in (7) is an accurate upper bound for $x_{n,1}$, however, by substituting $m = 4, k = 2m = 8$ in (4), we obtain

$$x^8 L_{n-4}^{(\alpha+8)}(x) = a_4(x) L_n^{(\alpha)}(x) + (n+\alpha) G_3^{(\alpha+8)}(x) L_{n-1}^{(\alpha)}(x),$$

where

$$\begin{aligned} a_4(x) &= (n-3)_4\,(x+4\alpha+16)\,x^3 - n\,(\alpha+3)_3\Big(5\,n(2n+\alpha-4) + \alpha^2 - 2\,\alpha + 12\Big)x^2 \\ &+ 2\,n\,(\alpha+2)_5\,(\alpha+3\,n-2)\,x - n\,(\alpha+1)_7 \end{aligned}$$

and

$$\begin{aligned} G_3^{(\alpha+8)}(x) &= -(\alpha+4)\,(2\,n+\alpha+1)\Big(\alpha^2+2\,\alpha\,n+2\,n^2+5\,\alpha+2\,n+6\Big)\,x^3 \\ &+ (\alpha+3)_3\,(3(\alpha+1)_2 + 10\,n\,(n+\alpha+1))\,x^2 - 3\,(\alpha+2)_5\,(2\,n+\alpha+1)\,x + (\alpha+1)_7 \end{aligned} \tag{8}$$

and the smallest zero of $G_3^{(\alpha+8)}(x)$ is an even more accurate bound than the bound obtained from (6). In Table 1 we show examples indicating the accuracy of this value, computed numerically, as upper bound for the lowest zero of $L_n^{(\alpha)}(x)$. To find a lower bound for the largest zero, we let $m = 4$ and $k = 0$ in (4), i.e., we don't consider any parameter shifts, and we obtain the recurrence equation

$$L_{n-4}^{(\alpha)}(x) = -\frac{n\Big(x^2 - 2\,(\alpha+2n-4)\,x + \alpha^2 + 3\,\alpha\,n + 3\,n^2 - 6\,\alpha - 12\,n + 11\Big)\,L_n^{(\alpha)}(x)}{(\alpha-3+n)_3} + \frac{G_3^{(\alpha)}(x) L_{n-1}^{(\alpha)}(x)}{(\alpha-3+n)_3}$$

with

$$G_3^{(\alpha)}(x) = -x^3 + 3\,(\alpha + 2\,n - 3)\,x^2 - \left(3\,\alpha^2 + 10\,\alpha\,n + 10\,n^2 - 15\,\alpha - 30\,n + 18\right)x + (\alpha - 3 + 2\,n)\left(\alpha^2 + 2\,\alpha\,n + 2\,n^2 - 3\,\alpha - 6\,n + 2\right). \tag{9}$$

The largest zero of $G_3^{(\alpha)}(x)$ is a lower bound for the largest zero of $L_n^{(\alpha)}(x)$. Here we use a zero of a third degree polynomial, that can easily be computed numerically, to approximate a zero of a polynomial of much higher degree.

We show the quality of these bounds in Table 1 and we compare our upper bound for $x_{n,1}$ with the upper bound

$$B_{GM} = \frac{(\alpha + 1)\,(\alpha + 2)\,(\alpha + 4)\,(2\,n + \alpha + 1)}{(\alpha + 1)^2\,(\alpha + 2) + (5\,\alpha + 11)\,n\,(n + \alpha + 1)} \tag{10}$$

obtained in [14, Equation 2.11]. We also provide the lower bound for the largest zero given in [21] for the values of $n \geq 30$. This bound is more precise than the bound obtained from (9), however, the recurrence equation involving polynomials $L_n^{(\alpha)}$, $L_{n-1}^{(\alpha)}$ and $L_{n-7}^{(\alpha)}$, which is too big to display, provides us with a polynomial $G_6^{(\alpha)}(x)$ and the largest zero of this polynomial, that can be computed numerically, is a more accurate bound than the bound in [21]. The equation can be found in the accompanying Maple file. Similar results can be obtained for the Jacobi, Gegenbauer, Hermite,

Table 1: Examples to show the quality of the inner bounds for the extreme zeros of $L_n^{(\alpha)}(x)$ for different values of n and α.

n	α	$x_{n,1}$	Bound from (8)	B_{GM} (10)	Bound from (9)	Bound in [21]	$x_{n,n}$
10	-0.5	0.06019206315	0.06019206332	0.060269	28.469	n/a	29.025
10	10.0	3.50805	3.51556	4.0168	44.945	n/a	46.365
100	10.0	0.49692	0.49863	0.5746	353.225	387.960	394.294
100	-0.5	0.00615313229	0.00615313230	0.0061611	335.472	367.816	374.006

Bessel, Meixner and Hahn polynomials and the recurrence equations providing the bounds, as well as the bounds, are also available in the Maple file.

Remark 4. *(i) The Charlier polynomials*

$$C_n(x; a) = {}_2F_0\left(\begin{array}{c} -n, -x \\ - \end{array}\middle|\, -\frac{1}{a}\right), a > 0,$$

are orthogonal on $(0, \infty)$ *with respect to* $w(x; a) = \frac{a^x}{x!}$ *and when we shift the parameter* a *by* k *units, we obtain the polynomial* $C_n(x; a + k)$*, which is orthogonal with respect to* $w(x; a + k)$ *on* $(0, \infty)$ *and since* $\frac{w(x;a+k)}{w(x;a)} = \frac{(a+k)^x}{a^x}$ *is not a polynomial of degree* k *in* x*, these polynomials do not satisfy the conditions of Theorem 3 and we use Lemma 1 and Corollary 2 to obtain bounds for the extreme zeros of these polynomials.*

(ii) We also use Lemma 1 in the case of the Krawtchouk polynomials

$$K_n(x; p, N) = {}_2F_1\left(\begin{array}{c} -n, -x \\ -N \end{array}\middle|\, \frac{1}{p}\right), n \in \{0, 1, \ldots, N\}, N \in \mathbb{N},$$

that are orthogonal for $0 < p < 1$, since integer shifts of the parameter p will result in the parameter moving outside the interval where orthogonality is guaranteed.

4. Bounds for the extreme zeros of q-orthogonal polynomials obtained by shifting one parameter

In this section we illustrate the method by obtaining new bounds for the extreme zeros of the little q-Jacobi and the alternative q-Charlier (or q-Bessel) polynomials.

4.1. The little q-Jacobi polynomials

The little q-Jacobi polynomials (cf. [19, Section 14.12])

$$p_n(q^x;\alpha,\beta|q) = {}_2\phi_1\left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1} \\ \alpha q \end{array}\middle| q; qq^x\right), 0 < \alpha q < 1,\ \beta q < 1$$

are discrete orthogonal with respect to the weight $w(x;\alpha,\beta) = \dfrac{(\beta q;q)_x}{(q;q)_x}(\alpha q)^x$ on the interval $(0,1)$. The polynomial $p_n(q^x;\alpha q^4,\beta|q)$ is orthogonal on $(0,1)$ with respect to $(q^x)^4 w(x;\alpha,\beta)$. By replacing q^x with x, we obtain

$$\begin{aligned} &\frac{\alpha\,(q^n-q)(\alpha\beta q^{2n}-1)(\beta q^n-q)(\alpha\,\beta\,q^{n+1};q)_2(\alpha\,q^{n+1};q)_2}{(\alpha\,q;q)_4} x^4 p_{n-2}(x;\alpha q^4,\beta|q) \\ = \;& a_2(x)p_n(x;\alpha,\beta|q) + q^{3n-5}\left(1-\alpha q^2\right)G^{(\alpha q^4,\,\beta)}(x)p_{n-1}(x;\alpha,\beta|q), \end{aligned} \tag{11}$$

with

$$a_2(x) = \alpha\,(q^n-q)(\beta q^n-q)\left(\left(\alpha\beta q^{2n}-1\right)q^{2n-3}x + (q+1)\left(\alpha q^2-1\right)q^{3n-5}\right)x + (\alpha q;q)_3 q^{4n-6}$$

and $G^{(\alpha q^4,\,\beta)}(x)$ is a linear function with zero

$$B_2^{(\alpha q^4,\,\beta)} = \frac{\left(\alpha\,q^3-1\right)(\alpha\,q-1)\,q^{n-1}}{(\alpha\,\beta\,q^{2n+1}+1)(\alpha q^2+1)-\alpha\,q^{n+1}(\beta+1)(q+1)}. \tag{12}$$

In Table 2 we show the quality of $B_2^{(\alpha q^4,\,\beta)}$ as bound and we compare it to bounds given in [14, Equation 4.3] when $n = 10, 30$ and in [14, Equation 4.2] for $n = 100$ (see Remark 5). We also observe that the value of q is an upper bound for the zero $x_{n,n-1}$. The accuracy of this upper bound decreases for q in the vicinity of 1. Furthermore, we observe that $\frac{x_{n,j+1}}{x_{n,j}} \approx \frac{1}{q}$ for $j \in \{1,2,\ldots,n-1\}$ and again this is less accurate when $q \to 1$.

Remark 5. *(i) The little q-Laguerre (or Wall) polynomials are obtained from the little q-Jacobi polynomials, by letting $\beta = 0$ and the bound $B_2^{(\alpha q^4,\,\beta)}$ in (12), with $\beta = 0$, can be used as an upper bound for the smallest zero of the little q-Laguerre polynomial.*

(ii) In [14], Gupta and Muldoon provide bounds for the smallest zeros of the little q-Jacobi polynomial $p_n((1-q)x;\alpha,\beta|q)$ and the q-Laguerre polynomial $L_n^{(\alpha)}((1-q)x;q), \alpha > -1$. Using a suitable comparison we observe that, for both these systems, the upper bounds for the smallest zeros obtained by our method are more accurate than the upper bounds obtained in [14].

Table 2: Examples to show the quality of the inner bounds for the extreme zeros of $p_n(x; 0.5, \beta|q)$ for different values of n, β and q.

n	β	q	$x_{n,1}$	$B_2^{(\alpha q^4,\, \beta)}$ in (12)	Bound in [14]	$x_{n,n-1}$	$x_{n,n}$
10	1	0.6	0.005216	0.005359	0.021776	0.600000	1.000000
30	-10	0.6	$1.8642 \cdot 10^{-7}$	$1.9497 \cdot 10^{-7}$	$7.9382 \cdot 10^{-7}$	0.600000	1.000000
100	-10	0.9	0.0000059	0.0000073	0.0000131	0.8999999	1.000000

4.2. *The alternative q-Charlier or q-Bessel polynomials*

The alternative q-Charlier polynomials (cf. [19, Section 14.22])

$$y_n(q^x; \alpha, q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, -\alpha q^n \\ 0 \end{matrix} \middle| q; q\, q^x \right), \alpha > 0,$$

are orthogonal with respect to the weight function $w(x; \alpha) = \dfrac{\alpha^x}{(q; q)_x} q^{\binom{x+1}{2}}$ on $(0, 1)$ and $y_n(q^x; \alpha q^4, q)$ are orthogonal on $(0, 1)$ with respect to $(q^x)^4 w(x; \alpha)$. By replacing q^x with x, these polynomials satisfy

$$\begin{aligned} & (-\alpha\, q^n, q)_2\, (q^n - q) \left(\alpha q^{2\,n} + q\right) (\alpha q^4) x^4 y_{n-2}(x; \alpha q^4, q) \\ = \; & q^{2\,n} \left(\alpha \left(\alpha q^{3\,n+1} - \alpha q^{2\,n+2} + q^{n+2} - q^3\right) x^2 - \alpha q^n \left(q^2 - q^{n+1} - q^n + q\right) x + q^n\right) y_n(x; \alpha, q) \\ - \; & q^{2\,n} \left(\left(\alpha q^{2\,n+1} - \alpha q^{n+2} - \alpha q^{n+1} - q\right) x + q^n\right) y_{n-1}(x; \alpha, q). \end{aligned}$$

From the coefficient of y_{n-1}, we obtain

$$B_2^{(\alpha q^4)} = \frac{q^{n-1}}{1 - \alpha q^n (q^n + q + 1)}, \tag{13}$$

which is a relatively sharp upper bound for the lowest zero of y_n, as shown in Table 3. In the recurrence equation involving $y_{n-3}(x; \alpha q^6, q), y_{n-1}(x; \alpha, q)$ and $y_n(x; \alpha, q)$, the coefficient of y_{n-1} is the quadratic polynomial

$$\begin{aligned} G_2^{(\alpha q^6)}(x) \; = \; & q^3 \left(1 + \left(q^{4\,n} + (q^{2\,n+2} - q^{3\,n})(q^2 + q + 1)\right) \alpha^2 + \left(q^n + q^{n+2} - q^{2\,n} + q^{n+1}\right) \alpha\right) x^2 \\ + \; & q^{n+1} (q + 1) \left(\alpha q^{2\,n} - \alpha q^{n+2} - \alpha\, q^n - 1\right) x + q^{2\,n} \end{aligned}$$

and the lowest zero of this polynomial is

$$B_3^{(\alpha q^6)} = \frac{-b - \sqrt{b^2 - 4ac}}{2a}, \tag{14}$$

with $c = G_2(0)$, $b = G_2'(0)$ and $a = \frac{G_2''(0)}{2}$, which is a more accurate upper bound for $x_{n,1}$, as shown in Table 3. In Table 3 we also illustrate the quality of the bound

$$B_n = -\frac{q^{n+1} \left(q^{2\,n} \alpha - \alpha\, q^{n+1} - \alpha\, q^n - q^2\right)}{(q^{2\,n} \alpha + q)\, (q^{2\,n} \alpha + q^3)}, \tag{15}$$

Table 3: Examples to show the quality of the upper bound for the lowest zero of $y_n(x;\alpha,q)$ for different values of n, α and q.

n	q	α	$x_{n,1}$	$B_3^{(\alpha q^6)}$ in (14)	$B_2^{(\alpha q^4)}$ in (13)	B_n in (15)
10	0.55	0.5	0.0045925	0.0045925	0.0045964	0.004635
10	0.99	10	0.06207968	0.06796546	0.08444314	0.115245
10	0.45	100	0.00071318	0.00071318	0.00072109	0.000941
70	0.45	10	$1.179406 \cdot 10^{-24}$	$1.179406 \cdot 10^{-24}$	$1.179406 \cdot 10^{-24}$	$1.179406 \cdot 10^{-24}$
70	0.8	100	$2.05671 \cdot 10^{-7}$	$2.05671 \cdot 10^{-7}$	$2.05682 \cdot 10^{-7}$	$2.05783 \cdot 10^{-7}$

obtained from the three-term recurrence equation satisfied by these polynomials, which increases for large values of n.

The value of q is an upper bound for the zero $x_{n,n-1}$ of $y_n(x;\alpha,q)$, as shown in Table 4 below and the accuracy of this upper bound decreases for q in the vicinity of 1. Furthermore, $\frac{x_{n,j+1}}{x_{n,j}} \approx \frac{1}{q}$ for $j \in \{1,2,\dots,n-1\}$ and again this is less accurate for the values of q in the vicinity of 1.

Table 4: Examples to show the quality of q as upper bound for $x_{n,n-1}$ for different values of n, α and q.

n	q	α	$x_{n,n-1}$	$x_{n,n}$
10	0.55	0.50	0.54999999	1.00000000
10	0.99	0.10	0.9735504	0.9935188
10	0.10	1000	0.099999999	1.00000000
70	0.70	1000	0.700000000	0.99999999
70	0.70	0.10	0.700000000	0.99999999

Remark 6. *Finding inner bounds by using Theorem 3 is not possible for the following polynomial systems, since*

(i) the quantum q-Krawtchouk polynomials $K_n^{(qtm)}(q^{-x};p,N;q)$ are orthogonal on $[0,N]$ for $p > q^{-N}$ and shifting p causes a change in the interval of orthogonality;

(ii) the Al-Salam Carlitz I polynomials $U_n^{(\alpha)}(x;q)$ are orthogonal for $\alpha < 0$ on $(\alpha,1)$ and shifting α to αq^k will result in a change in the interval of orthogonality;

(iii) the Al-Salam Carlitz II polynomials $V_n^{(\alpha)}(q^{-x};q)$ are orthogonal for $0 < \alpha q < 1$ on $(0,1)$ with respect to

$$w(x;\alpha) = \frac{q^{x^2}\alpha^x}{(q;q)_k(\alpha q;q)_k} \quad \text{and} \quad \frac{w(x;\alpha q^{-k})}{w(x;\alpha)} = \frac{\left(\frac{1}{\alpha}q^{-x};q\right)_k}{\left(\frac{1}{\alpha};q\right)_k} = c_k(q^{-x};\alpha)$$

is a polynomial of degree k in the variable q^{-x}. However, when we substitute α with αq^{-k}, $k \in \{1,2,\dots\}$, the condition $0 < \alpha q < 1$ is not satisfied.

We find bounds for the zeros of these systems by using Corollary 2.

5. Bounds for the extreme zeros of q-orthogonal polynomials obtained without any parameter shifts

In this section we obtain inner bounds for the extreme zeros of the Stieltjes-Wigert and discrete q-Hermite II polynomials, where no parameter shifts are involved, in order to illustrate the accuracy of the bounds obtained by Corollary 2, for $m = 3$ and/or $m = 4$.

5.1. *The Stieltjes-Wigert polynomials*

The Stieltjes-Wigert polynomials (cf. [19, Section 14.27])

$$S_n(x;q) = \frac{1}{(q;q)_n} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix} \middle| q; -q^{n+1}x \right)$$

are orthogonal on $(0,\infty)$. When we let $m = 3, k = 0$ in (4), we obtain the equation

$$\begin{aligned} q^6 S_{n-3}(x;q) &= -q\,(q^n-1)\left(xq^{2n}+q^{n+2}-q^3-q^4\right)S_n(x;q) \\ &+ \left(q^{4n}x^2+q^{2n+1}(q+1)\left(q^n-q^2-1\right)x+q^3\left(q^{2n}+(q-q^n)(q^2+q+1)\right)\right)S_{n-1}(x;q). \end{aligned} \tag{16}$$

The coefficient of S_{n-1} provides us with the two inner bounds

$$B_3 = \frac{(q^2-q^n+1)(q+1) \pm \sqrt{q^6-2\,q^{n+4}+2\,q^5+q^{2n+2}-q^4-2\,q^{2n+1}+q^{2n}-q^2-2\,q^n+2\,q+1}}{2q^{2n-1}}.$$

In Table 5 we show the quality of the largest of these two bounds, which is a good lower bound for the largest zero of $S_n(x;q)$. A more accurate lower bound is obtained from

$$\begin{aligned} q^{12}S_{n-4}(\alpha,q) &= q\,(q^n-1)\,(q^{4n}x^2+(q^{3n+2}-q^{2n+5}-q^{2n+3})(q+1)x \\ &+ q^{2n+5}+(q^8-q^{n+6})(q^2+q+1))S_n(\alpha,q)-A_3(x)S_{n-1}(\alpha,q), \end{aligned}$$

with

$$\begin{aligned} A_3(x) &= q^{6n}x^3+\left(q^{5n+1}-q^{4n+4}-q^{4n+1}\right)\left(q^2+q+1\right)x^2 \\ &+ \left(\left(q^{2n+8}+q^{2n+6}+q^{2n+4}+q^{4n+3}-q^{3n+6}-q^{3n+5}-q^{3n+4}\right)\left(q^2+q+1\right)-q^{3n+3}(q+1)\right)x \\ &- q^9(q^3+q^2+q+1)-q^{2n+6}(q^3+q^2+q+1)+q^{3n+6}+q^{n+7}(q^2+1)(q^2+q+1). \end{aligned} \tag{17}$$

The values of the zeros of $A_3(x)$ can be found numerically and we also show the value of the largest zero of $A_3(x)$ in Table 5.

5.2. *The discrete q-Hermite II polynomials*

The discrete q-Hermite II polynomials (cf. [19, Section 14.29])

$$\tilde{h}_n(x;q) = i^{-n}q^{-\binom{n}{2}} {}_2\phi_0\left(\begin{matrix} q^{-n}, ix \\ - \end{matrix} \middle| q; -q^n \right),$$

Table 5: Examples to show the quality of the inner bound for the extreme zeros of $S_n(x;q)$ for different values of n and q.

n	q	B_3 from (16)	Lowest zero of A_3 (17)	$x_{n,n}$
10	0.5	$8.3925988 \cdot 10^5$	$8.3946795 \cdot 10^5$	$8.3946799 \cdot 10^5$
10	0.9	15.3689887	16.0730951	16.1508699
70	0.5	$1.116350296 \cdot 10^{42}$	$1.1166280 \cdot 10^{42}$	$1.1166281 \cdot 10^{42}$
70	0.9	$5.9402911 \cdot 10^6$	$6.2658907 \cdot 10^6$	$6.3132591 \cdot 10^6$

are orthogonal on the real line. The zeros of these polynomials are symmetric about the origin with a simple zero at the origin when n is odd. We get the recurrence equation

$$\left(q^{-n+1};q\right)_4 q^{14}\tilde{h}_{n-5}(x;q) = -\left(q^{n+3}+q^{n+4}-q^5-q^7+q^{2n}x^2\right)q^{2n}x\tilde{h}_n(x;q) + G_4(x)\tilde{h}_{n-1}(x;q)$$

with

$$G_4(x) = q^{4n}x^4 + \left(q^2+q^n-q-q^3\right)\left(q^2+q+1\right)x^2q^{2n+2} + q^6\left(q^n-q\right)\left(q^n-q^3\right).$$

The largest zero of $G_4(x)$ is

$$B_5 = \left(\frac{(q+q^3-q^2-q^n)(q^2+q+1)+\sqrt{(q^2+q^n-q-q^3)^2(q^2+q+1)^2-4(q^n-q^3)(q^n-q)q^2}}{2q^{2n-2}}\right)^{\frac{1}{2}}. \tag{18}$$

We show the quality of this bound in Table 6.

Table 6: Examples to show the quality of the inner bound for the extreme zeros of $\tilde{h}_n(x;q)$ for different values of n and q.

n	q	B_5 in (18)	$x_{n,n}$
10	0.5	406.3811	406.4079
10	0.98	0.72968	0.76655
100	0.5	$5.0368 \cdot 10^{29}$	$5.0372 \cdot 10^{29}$
100	0.98	10.7735	12.1420

6. Acknowledgments

This work has been supported by a TWAS / DFG fellowship for A. Jooste and the Institute of Mathematics of the University of Kassel (Germany) for D.D. Tcheutia. All these institutions receive our sincere thanks.

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