AN EXTENSION OF TAYLOR SERIES EXPANSION USING BELL POLYNOMIALS

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Abstract. By using Bell polynomials, we introduce an extension of Taylor series expansion and consider some of its special cases leading to new series and identities. We also apply the extended expansion for generating functions of some famous sequences of numbers.

1. Introduction

Let $\{x_j\}_{j=0}^n \in [a,b]$ and $\{f_j\}_{j=0}^n$, which may be samples of a function, say $f(x)$, be given. The main aim of interpolation is to find an appropriate model to approximate $f(x)$ at any arbitrary point of $[a,b]$ other than x_j. In other words, if $\Psi(x; a_0, \dots, a_n)$ is a family of functions of a single variable x with $n+1$ free parameters $\{a_j\}_{j=0}^n$, then the interpolation problem for Ψ consists of determining $\{a_j\}_{j=0}^n$ so that for $n+1$ given real or complex pairs of distinct numbers $\{(x_j, f_j)\}_{j=0}^n$ we have

$$\Psi(x_j; a_0, \dots, a_n) = f_j. \tag{1.1}$$

For a polynomial type interpolation problem, various classical methods such as Lagrange, Newton and Hermite interpolations are used. Lagrange's interpolation as a classical method for approximating a continuous function $f : [a,b] \to \mathbb{R}$ at $n+1$ distinct nodes $a \le x_0 < \cdots < x_n \le b$ is applied in several branches of numerical analysis and approximation theory. It is expressed in the form [16, pp. 39–40]

$$L_n(f; x) = \sum_{j=0}^{n} f(x_j)\ell_j^{(n)}(x),$$

for

$$\ell_j^{(n)}(x) = \frac{w_n(x)}{(x-x_j)w_n'(x_j)},$$

where $w_n(x) = \prod_{j=0}^{n}(x - x_j)$ is the node polynomial and $\ell_j^{(n)}(x)$ as the Lagrange polynomials.

$L_n(f; x)$ is a unique element in the space of all polynomials of degree at most n, say $\mathcal{P}_n$, which solves the interpolation problem

$$L_n(f; x_j) = f(x_j) \qquad (j = 0, 1, 2, \dots, n).$$

2010 *Mathematics Subject Classification.* Primary: 65D05, Secondary: 41A05, 42A15.

Key words and phrases. Bell polynomials, Taylor series expansion, interpolation formulae, generating functions, Stirling numbers.

For non-polynomial type interpolation problems, an interpolating function of the form

$$\psi_n(x) = \sum_{j=0}^{n} a_j u_j(x),$$

is usually considered [17] where $\{u_j(x)\}_{j=0}^{n}$ is a set of linearly independent real-valued continuous functions on $[a,b]$ and $\{a_j\}_{j=0}^{n}$ are determined by the initial conditions

$$\psi_n(x_j) = f(x_j) \qquad (j = 0, 1, \ldots, n).$$

The function $\psi_n(x)$ exists and is unique in the space of $\operatorname{span}\{u_j\}_{j=0}^{n}$ for all $f \in C[a,b]$ if and only if the matrix $\{u_j(x_k)\}_{j,k=0}^{n}$ is nonsingular.

The general case of an interpolation problem was proposed by Davis [10] containing all above-mentioned cases. It is indeed concerned with reconstructing functions on a basis of certain functional information, which are linear in many cases. Hence, one can construct new interpolation formulae using linear operators [15]. He also mentioned that the expansion of a function based on a series of predetermined (basis) functions can be interpreted as an interpolation problem with infinite number of conditions. See also [14] in this regard. The problem of the representation of an arbitrary function by means of linear combinations of prescribed functions has received a lot of attention. It is well known that a special case of this problem directly leads to Taylor's series [19].

The main aim of this paper is to introduce a class of interpolation formulas which leads to a generalization of Taylor series expansion. In this direction, we define a certain space of functions as the base of constructing such a class. In the next section, we introduce an extension of Taylor series expansion and consider some interesting cases of it leading to new series and identities. Also, some applications are presented for obtaining generating functions of some famous numbers.

2. A CLASS OF INTERPOLATION FORMULAS

Let $\Lambda_{n-1} := \{\lambda_k\}_{k=1}^{n-1}$ be a sequence of continuous functions defined on $[a,b]$ such that $\lambda_k(x_i) \neq \lambda_k(x_j)$ for any $i \neq j$ and $\{x_k\}_{k=1}^{n}$ are n distinct points in $[a,b]$. We define the functions

$$\Phi(x; x_k, \Lambda_{n-1}) := \prod_{(i,j)\in I^*_{n,k}} \frac{\lambda_i(x) - \lambda_i(x_j)}{\lambda_i(x_k) - \lambda_i(x_j)} \qquad (k = 1, \ldots, n), \tag{2.1}$$

where

$$I^*_{n,k} := \{(r,r)|\, r = 1, \ldots, k-1,\, k > 1\} \cup \{(r, r+1)|\, r = k, \ldots, n-1,\, k < n\}.$$

It can be directly verified that these functions (2.1) satisfy the biorthogonality relation

$$\Phi(x_l; x_k, \Lambda_{n-1}) = \delta_{k,l}.$$

Moreover, $\{\Phi(x; x_k, \Lambda_{n-1})\}_{k=1}^{n}$ are linearly independent on $[a,b]$. So, by considering the space

$$\Pi_n := \operatorname{span}\{\Phi(x; x_1, \Lambda_{n-1}), \ldots, \Phi(x; x_n, \Lambda_{n-1})\},$$

the explicit solution of the interpolation problems of type (1.1) is as $\sum\limits_{k=1}^{n} f(x_k)\Phi(x;x_k,\Lambda_{n-1})$. Now, suppose that

$$G(x;f;\Lambda_{n-1}) = a_1\Phi(x;x_1,\Lambda_{n-1}) + \ldots + a_n\Phi(x;x_n,\Lambda_{n-1}),$$

denotes an interpolating function. By imposing the initial conditions, we reach a system of linear equations as

$$a_1\Phi(x_k;x_1,\Lambda_{n-1}) + \ldots + a_n\Phi(x_k;x_n,\Lambda_{n-1}) = f(x_k) \qquad k = 1,\ldots,n,$$

which has a unique solution, because

$$\begin{vmatrix} \Phi(x_1;x_1,\Lambda_{n-1}) & \cdots & \Phi(x_1;x_n,\Lambda_{n-1}) \\ \vdots & & \vdots \\ \Phi(x_n;x_1,\Lambda_{n-1}) & \cdots & \Phi(x_n;x_n,\Lambda_{n-1}) \end{vmatrix} = \begin{vmatrix} 1 & & \mathbf{O} \\ & \ddots & \\ \mathbf{O} & & 1 \end{vmatrix} \neq 0.$$

So there exists a unique function $G(x;f;\Lambda_{n-1}) \in \Pi_n$ such that

$$G(x_k;f;\Lambda_{n-1}) = f(x_k) \qquad k = 1,\ldots,n.$$

Also it can be verified that if $f \in C^{n-1}([a,b])$, $f^{(n)}$ exist at any point of (a,b) and $G(\cdot;f;\Lambda_{n-1}) \in C^{n-1}([a,b])$, then there exists a point $\xi_x \in (a,b)$ depending on x such that

$$f(x) - G(x;f;\Lambda_{n-1}) = \frac{f^{(n)}(\xi_x) - G^{(n)}(\xi_x;f;\Lambda_{n-1})}{n!} \prod_{k=1}^{n}(x-x_k).$$

Notice that if $\lambda_l(x) = \lambda(x)$ for every $l = 1,\ldots,n$, with $\lambda(x_i) \neq \lambda(x_j)$ for $i \neq j$, then

$$G(x;f;\{\lambda(x)\}_{l=1}^{n}) = \sum_{k=1}^{n+1} f(x_k)\Phi(x;x_k,\{\lambda(x)\}_{l=1}^{n}),$$

interpolates f at $\{x_k\}_{k=1}^{n+1}$, where

$$\Phi(x;x_k,\{\lambda(x)\}_{l=1}^{n}) = \prod_{\substack{j=1 \\ j\neq k}}^{n+1} \frac{\lambda(x)-\lambda(x_j)}{\lambda(x_k)-\lambda(x_j)}.$$

In this case, we have

$$\operatorname{span}\{\Phi(x;x_k,\{\lambda(x)\}_{l=1}^{n})\}_{k=1}^{n+1} = \operatorname{span}\{1,\lambda(x),\lambda^2(x),\ldots,\lambda^n(x)\},$$

and $G(x;f;\{\lambda(x)\}_{l=1}^{n})$ can be therefore represented as

$$G(x;f;\{\lambda(x)\}_{l=1}^{n}) = A_0 + A_1\lambda(x) + A_2(\lambda(x))^2 + \ldots + A_n(\lambda(x))^n, \tag{2.2}$$

in which $\{A_j\}_{j=0}^{n}$ are obtained by the given interpolation conditions.

3. An extension of Taylor Series Expansion

In (2.2), we assumed that $\{x_k\}_{k=1}^{n+1}$ are distinct. Now, let $\{x_k\}_{k=2}^{n+1}$ coincide with x_1 together with the corresponding values $\{f^{(k)}(x_1)\}_{k=0}^{n}$. If $f, \lambda \in C^n[a,b]$ and $f^{(n+1)}, \lambda^{(n+1)}$ exist on (a,b) and $\lambda^{(j)}(x_1) \neq 0$ for any $j = 1, \ldots, n$, then the interpolating function (2.2) changes to

$$G^*(x; f; \{\lambda(x)\}_{l=1}^{n}) = f(x_1) + \sum_{l=1}^{n} \frac{(-1)^{l-1} D_{(l-1,l)} N_l(x)}{\prod_{j=1}^{l} j! \left(\lambda'(x_1)\right)^j}, \tag{3.1}$$

where

$$N_l(x) = \sum_{k=0}^{l} (-1)^{l-k} \binom{l}{k} \left(\lambda(x_1)\right)^{l-k} \lambda^k(x) \qquad l \geq 1, \tag{3.2}$$

and

$$D_{(m,k)} = \begin{vmatrix} f'(x_1) & N_1'(x_1) & 0 & 0 & \ldots & 0 \\ f''(x_1) & N_1''(x_1) & N_2''(x_1) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ f^{(m-1)}(x_1) & N_1^{(m-1)}(x_1) & N_2^{(m-1)}(x_1) & N_3^{(m-1)}(x_1) & \ldots & 0 \\ f^{(m)}(x_1) & N_1^{(m)}(x_1) & N_2^{(m)}(x_1) & N_3^{(m)}(x_1) & \ldots & N_m^{(m)}(x_1) \\ f^{(k)}(x_1) & N_1^{(k)}(x_1) & N_2^{(k)}(x_1) & N_3^{(k)}(x_1) & \ldots & N_m^{(k)}(x_1) \end{vmatrix},$$

for $m \geq 1$, $k > m$ such that

$$D_{(0,k)} = f^{(k)}(x_1) \qquad k \geq 1.$$

According to (3.2), it is clear that (3.1) can be written as

$$G^*(x; f; \{\lambda(x)\}_{l=1}^{n}) = \sum_{j=0}^{n} c_j \left(\lambda(x)\right)^j. \tag{3.3}$$

Notice that (2.2) interpolates f at $\{x_k\}_{k=1}^{n+1}$ while (3.3) interpolates f at x_1 for $n+1$ times, i.e.

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} G^*(x; f; \{\lambda(x)\}_{l=1}^{n}) = f^{(k)}(x_1) \qquad (k = 0, 1, \ldots, n). \tag{3.4}$$

In this sense, the coefficients $\{c_j\}_{j=0}^{n}$ can be directly computed by equating (3.3) with (3.1), or by considering the conditions given in (3.4). However, if $\lambda(x)$ is an invertible function, $\{c_j\}_{j=0}^{n}$ can be derived by an easier method. For this purpose, without loss of generality and just for simplicity in the symbolic representations, we consider $\lambda^{-1}(x)$ instead of $\lambda(x)$ in (3.3), i.e.

$$f(x) \simeq \sum_{j=0}^{n} c_j \left(\lambda^{-1}(x)\right)^j. \tag{3.5}$$

If $\frac{\mathrm{d}^j f(\lambda(x))}{\mathrm{d}x^j}$ exist for any $j = 0, 1, \ldots, n$ at $x = 0$, then from (3.5) one can respectively conclude that

$$f\big(\lambda(x)\big) \simeq \sum_{j=0}^{n} c_j x^j, \tag{3.6}$$

and

$$c_j = \frac{1}{j!} \frac{\mathrm{d}^j f\big(\lambda(x)\big)}{\mathrm{d}x^j}\Big|_{x=0} \qquad (j = 0, 1, \ldots, n). \tag{3.7}$$

Moreover, if $n \to \infty$, relation (3.6) reads as

$$f(x) = \sum_{n=0}^{\infty} c_n \big(\lambda^{-1}(x)\big)^n, \tag{3.8}$$

which is equivalent to

$$f\big(\lambda(x)\big) = \sum_{n=0}^{\infty} c_n x^n,$$

and $\{c_n\}_{n=0}^{\infty}$ are given by (3.7).

Note that the convergence radius of the series (3.8) depends on $\lambda(x)$ and is directly derived by using the ratio test, so that if

$$\rho = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|,$$

then (3.8) converges for all x satisfying the inequality $\left|\lambda^{-1}(x)\right| < \frac{1}{\rho}$.

To explicitly compute the coefficients (3.7), first according to the classical Faá di Bruno formula [8] we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(f\big(\lambda(x)\big)\Big) = \sum_{I_n^k} \frac{n!}{k_1!\, k_2! \,\ldots\, k_{n-k+1}!} f^{(k)}\big(\lambda(x)\big) \prod_{j=1}^{n-k+1} \left(\frac{\lambda^{(j)}(x)}{j!}\right)^{k_j}, \tag{3.9}$$

where

$$I_n^k = \left\{ (k_1, k_2, \ldots, k_{n-k+1}) : \sum_{j=1}^{n-k+1} k_j = k \ \text{ and } \sum_{j=1}^{n-k+1} j k_j = n,\ k_j \in \mathbb{N} \cup \{0\} \right\}. \tag{3.10}$$

Therefore, the primary form of the coefficients can be represented as

$$c_n = \sum_{I_n^k} \frac{1}{k_1!\, k_2! \,\ldots\, k_{n-k+1}!} f^{(k)}\big(\lambda(0)\big) \prod_{j=1}^{n-k+1} \left(\frac{\lambda^{(j)}(0)}{j!}\right)^{k_j}.$$

However, still there exists an easier way to compute the coefficients. The Bell polynomial version of Faá di Bruno's formula (3.9) is given by [12]

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(f\big(\lambda(x)\big)\Big) = \sum_{k=0}^{n} f^{(k)}\big(\lambda(x)\big) B_{n,k}\Big(\lambda'(x),\, \lambda''(x),\, \ldots,\, \lambda^{(n-k+1)}(x)\Big),$$

where $B_{n,k}(.)$ are known as partial exponential Bell polynomials and are defined by [11, p. 96]

$$B_{n,k}(u_1,u_2,\ldots,u_{n-k+1}) = n!\sum_{I_n^k}\prod_{j=1}^{n-k+1}\frac{1}{k_j!}\Big(\frac{u_j}{j!}\Big)^{k_j}, \tag{3.11}$$

where I_n^k is the same as (3.10). Consequently, we have

$$c_n = \frac{1}{n!}\sum_{k=0}^{n} f^{(k)}\big(\lambda(x)\big)B_{n,k}\left(\lambda'(x),\,\lambda''(x),\,\ldots,\,\lambda^{(n-k+1)}(x)\right)\Big|_{x=0}.$$

Note that since

$$I_n^0 = \begin{cases}\{(0)\} & \text{for } \ n=0,\\ \emptyset & \text{for } \ n\geq 1,\end{cases}$$

we have $B_{0,0}(u)=1$ and $B_{n,0}(u_1,\ldots,u_{n+1})=0$ for $n\geq 1$. Also, $B_{1,1}=u_1$, $B_{2,1}=u_2$, $B_{2,2}=u_1^2$, $B_{3,1}=u_3$, $B_{3,2}=3u_1u_2$, $B_{3,3}=u_1^3$, ... , $B_{n,1}=u_n$ and $B_{n,n}=u_1^n$. A comperehensive table of the $B_{n,k}$ for $k\leq n\leq 12$ is found in [6, p. 307].

$B_{n,k}(.)$ in (3.11) with infinite number of variables can be defined by the double series expansion [6]

$$\begin{aligned}\Phi(t,x) &= \exp\Big(x\sum_{m\geq 1}u_m\frac{t^m}{m!}\Big) = \sum_{n,k\geq 0}B_{n,k}\frac{t^n}{n!}x^k \\ &= 1+\sum_{n\geq 1}\frac{t^n}{n!}\Big(\sum_{k=1}^{n}x^kB_{n,k}(u_1,u_2,\ldots)\Big),\end{aligned} \tag{3.12}$$

and the complete exponential Bell polynomials [2] by

$$\Phi(t,1) = \exp\Big(\sum_{m\geq 1}u_m\frac{t^m}{m!}\Big) = 1+\sum_{n\geq 1}Y_n(u_1,u_2,\ldots)\frac{t^n}{n!}, \tag{3.13}$$

where

$$Y_n(u_1,u_2,\ldots,u_n) = \sum_{k=1}^{n}B_{n,k}(u_1,\ldots,u_{n-k+1}), \qquad (Y_0:=1).$$

Relations (3.12) and (3.13) are widely used in combinatorial analysis with numerous applications in physics and mathematics. It is well-known that many special sequences are constructed from Bell polynomials by an appropriate choice of the variables u_1, u_2, ... For instance, we have

- $B_{n,k}(\underbrace{1,1,\ldots,1}_{n-k+1}) = S(n,k)$ (Stirling number of the second kind [6, p. 50]),
- $B_{n,k}\big(0!,1!,\ldots,(n-k)!\big) = |s(n,k)|$ (signless Stirling number of the first kind [6, p. 50]),
- $B_{n,k}\big(1!,2!,\ldots,(n-k+1)!\big) = \binom{n-1}{k-1}\frac{n!}{k!}$ $n,k\geq 1$, (Lah number [6, p. 156]),
- $B_{n,k}\big(1,2,\ldots,n-k+1\big) = \binom{n}{k}k^{n-k}$ (idempotent number [6, p. 91]).

For more results see [6, pp. 133-137], [9], [18] and [11, pp. 95-98]. Two nice historical surveys have appeared in [8] and [12].

3.1. **Some special cases of the expansion** (3.8)

Example 1. Let $\lambda^{-1}(x)=\cos x$ and consider the expansion

$$f(x)=\sum_{n=0}^{\infty}c_n\,\big(\cos x\big)^n \qquad (0\le x\le\pi), \tag{3.14}$$

which is equivalent to

$$f(\arccos x)=\sum_{n=0}^{\infty}c_n x^n.$$

Since

$$\frac{\mathrm{d}^m\big(\arccos x\big)}{\mathrm{d}x^m}\Big|_{x=0}=\begin{cases}0 & m=2l,\\ -\prod_{j=1}^{l-1}(2j-1)^2 & m=2l-1,\ \prod_{j=1}^{0}(.)=1,\end{cases}$$

the partition set takes the form
(3.15)

$$\hat{I}_n^k=\Big\{(k_1,k_2,\dots,k_{n-k+1})\in I_n^k\ \text{ such that } k_{2l}=0 \text{ for all } l=1,\dots,[\frac{n-k+1}{2}]\Big\},$$

and therefore
(3.16)

$$c_n=\sum_{\hat{I}_n^k}\frac{(-1)^k}{k_1!\,k_2!\,\dots\,k_{n-k+1}!}f^{(k)}\big(\frac{\pi}{2}\big)\prod_{m=1}^{[\frac{n-k}{2}]+1}\Big(\frac{1}{(2m-1)!}\prod_{l=1}^{m-1}(2l-1)^2\Big)^{k_{2m-1}},$$

for $n\ge1$ and $c_0=f(\frac{\pi}{2})$. Note that $\prod_{l=1}^{0}(.)=1$. For instance, for n=5 in (3.14) we obtain

$$f(x)\cong f(\frac{\pi}{2})\;-\;f'(\frac{\pi}{2})\cos x\;+\;\frac{1}{2!}f''(\frac{\pi}{2})\cos^2 x\;-\;\frac{1}{3!}\Big(f'(\frac{\pi}{2})+f'''(\frac{\pi}{2})\Big)\cos^3 x$$
$$+\frac{1}{4!}\Big(4f''(\frac{\pi}{2})+f^{(4)}(\frac{\pi}{2})\Big)\cos^4 x\;-\;\frac{1}{5!}\Big(9f'(\frac{\pi}{2})+10f'''(\frac{\pi}{2})+f^{(5)}(\frac{\pi}{2})\Big)\cos^5 x.$$

A valuable point of example 1 is that there exists a direct relationship between series (3.14) and cosine Fourier series, because

$$f(x)=\sum_{n=0}^{\infty}c_n(\cos x)^n=\frac{\alpha_0}{2}+\sum_{j=1}^{\infty}\alpha_j\cos jx, \tag{3.17}$$

in which, according to the trigonometric formulas [3],

$$\cos^n x=\begin{cases}\dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}}+\dfrac{2}{2^n}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}\dbinom{n}{k}\cos(n-2k)x & (n:\text{even}),\\[2ex] \dfrac{2}{2^n}\displaystyle\sum_{k=0}^{\frac{n-1}{2}}\dbinom{n}{k}\cos(n-2k)x & (n:\text{odd}).\end{cases}$$

Hence

$$\alpha_j = \begin{cases} \displaystyle\sum_{l=\frac{j}{2}}^{\infty} \frac{1}{2^{2l-1}} \binom{2l}{l-\frac{j}{2}} c_{2l} & (j : \text{even}), \\ \displaystyle\sum_{l=\frac{j-1}{2}}^{\infty} \frac{1}{2^{2l}} \binom{2l+1}{l-\frac{j-1}{2}} c_{2l+1} & (j : \text{odd}), \end{cases}$$

or in a unique form we have

$$\alpha_j = \frac{1}{2^{j-1}} \sum_{l=0}^{\infty} \frac{1}{2^{2l}} \binom{2l+j}{l} c_{2l+j} \qquad \text{for any } j = 0, 1, \ldots .$$

On the other hand, (3.17) is a Fourier cosine series of f such that

$$\alpha_j = \frac{2}{\pi} \int_0^{\pi} f(x) \cos jx \, dx \qquad \text{for} \qquad j = 0, 1, \ldots .$$

Therefore

$$\int_0^{\pi} f(x) \cos jx \, dx = \frac{\pi}{2^j} \sum_{l=0}^{\infty} \frac{1}{2^{2l}} \binom{2l+j}{l} c_{2l+j}, \tag{3.18}$$

where $\{c_{2l+j}\}$ are given by (3.16). For instance, if $j = 0$ in (3.18) then

$$\begin{aligned} &\frac{1}{\pi} \int_0^{\pi} f(x) \, dx = \sum_{l=0}^{\infty} \frac{1}{2^{2l}} \binom{2l}{l} c_{2l} \\ &= f(\frac{\pi}{2}) + \big(\frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^2 \times 3^2} + \ldots\big) f''(\frac{\pi}{2}) + \big(\frac{1}{2^6} + \frac{1}{2^6 \times 3^2} + \ldots\big) f^{(4)}(\frac{\pi}{2}) + \ldots . \end{aligned}$$

Example 2. Let $\lambda^{-1}(x) = \sin x$ and consider the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n (\sin x)^n \qquad (-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}), \tag{3.19}$$

which is equivalent to

$$f(\arcsin x) = \sum_{n=0}^{\infty} c_n x^n .$$

Since

$$\frac{\mathrm{d}^m \big(\arcsin x\big)}{\mathrm{d}x^m}\Big|_{x=0} = \begin{cases} 0 & m = 2l, \\ \displaystyle\prod_{j=1}^{l-1} (2j-1)^2 & m = 2l-1, \ \displaystyle\prod_{j=1}^{0} (.) = 1, \end{cases}$$

the partition set is as the same form as (3.15) and therefore

$$c_n = \sum_{\hat{I}_n^k} \frac{1}{k_1! \, k_2! \, \ldots \, k_{n-k+1}!} f^{(k)}(0) \prod_{m=1}^{[\frac{n-k}{2}]+1} \Big(\frac{1}{(2m-1)!} \prod_{l=1}^{m-1} (2l-1)^2\Big)^{k_{2m-1}}, \tag{3.20}$$

for $n \geq 1$ and $c_0 = f(0)$. For instance, for n=5 in (3.19) we obtain

$$f(x) \cong f(0) + f'(0)\sin x + \frac{1}{2!}f''(0)\sin^2 x + \frac{1}{3!}\big(f'(0)+f'''(0)\big)\sin^3 x$$

$$+\frac{1}{4!}\big(4f''(0)+f^{(4)}(0)\big)\sin^4 x + \frac{1}{5!}\big(9f'(0)+10f'''(0)+f^{(5)}(0)\big)\sin^5 x.$$

Once again, since

$$f(x) = \sum_{n=0}^{\infty} c_n(\sin x)^n = \frac{\beta_0}{2} + \sum_{j=1}^{\infty}\beta_{2j}\cos 2jx + \sum_{j=0}^{\infty}\beta_{2j+1}\sin(2j+1)x, \tag{3.21}$$

in which, according to the trigonometric formulas [3],

$$\sin^n x = \begin{cases} \dfrac{1}{2^n}\dbinom{n}{\frac{n}{2}} + \dfrac{(-1)^{\frac{n}{2}}}{2^{n-1}}\displaystyle\sum_{k=0}^{\frac{n}{2}-1}(-1)^k\binom{n}{k}\cos(n-2k)x & (n: \text{even}), \\ \dfrac{2}{2^n}(-1)^{\frac{n-1}{2}}\displaystyle\sum_{k=0}^{\frac{n-1}{2}}(-1)^k\binom{n}{k}\sin(n-2k)x & (n: \text{odd}), \end{cases}$$

we obtain

$$\beta_k = \begin{cases} (-1)^{\frac{k}{2}}\displaystyle\sum_{l=\frac{k}{2}}^{\infty}\frac{1}{2^{2l-1}}\binom{2l}{l-\frac{k}{2}}c_{2l} & (k: \text{even}), \\ (-1)^{\frac{k-1}{2}}\displaystyle\sum_{l=\frac{k-1}{2}}^{\infty}\frac{1}{2^{2l}}\binom{2l+1}{l-\frac{k-1}{2}}c_{2l+1} & (k: \text{odd}), \end{cases}$$

or in a unique form

$$\beta_k = \frac{(-1)^{[\frac{k}{2}]}}{2^{k-1}}\sum_{l=0}^{\infty}\frac{1}{2^{2l}}\binom{2l+k}{l}c_{2l+k} \qquad \text{for any } k = 0,1,\ldots.$$

Now, consider a special form of the Fourier series of a periodic function $f(x)$ on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as follows

$$f(x) = \frac{b_0}{2} + \sum_{j=1}^{\infty} b_{2j}\cos 2jx + \sum_{j=0}^{\infty} b_{2j+1}\sin(2j+1)x, \tag{3.22}$$

where

$$b_k = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos kx\,dx & (k: \text{ even}), \\ \dfrac{2}{\pi}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\sin kx\,dx & (k: \text{ odd}). \end{cases}$$

By equating the right hand sides of (3.21) and (3.22) we respectively obtain

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos jx\,dx = \frac{(-1)^{\frac{j}{2}}}{2^j}\sum_{l=0}^{\infty}\frac{1}{2^{2l}}\binom{2l+j}{l}c_{2l+j} \qquad (j:\ \text{even}),$$

and

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\sin jx\,dx = \frac{(-1)^{\frac{j-1}{2}}}{2^j}\sum_{l=0}^{\infty}\frac{1}{2^{2l}}\binom{2l+j}{l}c_{2l+j} \qquad (j:\ \text{odd}),$$

where $\{c_{2l+j}\}$ are given by (3.20).

Example 3. Let $\lambda(x)=\ln(x+1)$ and consider the expansion

$$f(x)=\sum_{n=0}^{\infty} c_n\left(e^x-1\right)^n, \tag{3.23}$$

which is equivalent to

$$f\big(\ln(x+1)\big)=\sum_{n=0}^{\infty} c_n\,x^n,$$

where

$$c_n=\frac{1}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}f\big(\ln(x+1)\big)_{|x=0}.$$

To compute c_n, first notice that for $n\geq 1$, we have

$$\begin{aligned}
&B_{n,k}\left(\lambda'(x),\,\lambda''(x),\,\dots,\,\lambda^{(n-k+1)}(x)\right)\\
&=B_{n,k}\big(0!\,(x+1)^{-1},\,-(1!)\,(x+1)^{-2},\dots,\,(-1)^{n-k}(n-k)!\,(x+1)^{-(n-k+1)}\big)\\
&=(x+1)^{-n}B_{n,k}\big(0!,\,-(1!),\dots,\,(-1)^{n-k}(n-k)!\big),
\end{aligned}$$

where we have used the identity [6]

$$B_{n,k}\big(abu_1,\,ab^2u_2,\,ab^3u_3,\,\dots,\,ab^{n-k+1}u_{n-k+1}\big)=a^kb^nB_{n,k}\big(u_1,\,u_2,\,u_3,\,\dots,\,u_{n-k+1}\big),$$

for $a=1$ and $b=(x+1)^{-1}$.

Now, replacing $u_m=(-1)^{m-1}(m-1)!$ in (3.12) gives

$$\Phi(t,x)=\exp\left(x\sum_{m\geq 1}\frac{(-1)^{m-1}}{m}\,t^m\right)=\exp\Big(x\ln(1+t)\Big)=(1+t)^x,$$

which is indeed the generating function of the first kind of Stirling numbers, i.e.

$$B_{n,k}\big(0!,\,-(1!),\dots,\,(-1)^{n-k}(n-k)!\big)=s(n,k).$$

In other words, we have

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}f\big(\ln(x+1)\big)=(x+1)^{-n}\sum_{k=1}^{n}s(n,k)\,f^{(k)}\big(\ln(x+1)\big),$$

and

$$c_n=\frac{1}{n!}\Big((x+1)^{-n}\sum_{k=1}^{n}s(n,k)f^{(k)}\big(\ln(x+1)\big)\Big)_{|x=0}=\frac{1}{n!}\sum_{k=1}^{n}s(n,k)f^{(k)}(0).$$

By noting that $s(0,0)=1$ and $s(n,0)=0$ for $n\geq 1$, (3.23) can be finally written as

$$f(x)=\sum_{n=0}^{\infty}\frac{(e^x-1)^n}{n!}\Big(\sum_{k=0}^{n}s(n,k)f^{(k)}(0)\Big),$$

generating many new identities for the first kind of Stirling numbers.

Example 4. Let $\lambda(x)=\frac{1}{1-x},\ x\neq 0,1$ and consider the expansion

$$f(x)=\sum_{n=0}^{\infty}c_n\,\big(1-\frac{1}{x}\big)^n,$$

leading to

$$f\big(\frac{1}{1-x}\big)=\sum_{n=0}^{\infty}c_n\,x^n.$$

Since

$$\frac{\mathrm{d}^j}{\mathrm{d}x^j}\lambda(x)\big|_{x=0}=j!\qquad\text{for any}\quad j=0,1,\ldots,n-k+1,$$

the coefficients are given by

$$\begin{aligned}c_n &= \frac{1}{n!}\sum_{k=1}^{n}f^{(k)}(1)\,B_{n,k}\big(1!,\,2!,\,\ldots\,,(n-k+1)!\big)\\ &= \frac{1}{n!}\sum_{k=1}^{n}f^{(k)}(1)\binom{n-1}{k-1}\frac{n!}{k!}=\frac{1}{n!}\sum_{k=1}^{n}f^{(k)}(1)\,L(n,k),\end{aligned}$$

for $n\geq 1$ and $c_0=f(1)$ where $L(n,k)$ are known as Lah numbers [5].
Consequently, we have

$$f(x)=f(1)+\sum_{n=1}^{\infty}\big(1-\frac{1}{x}\big)^n\sum_{k=1}^{n}\frac{f^{(k)}(1)}{k!}\binom{n-1}{k-1}.$$

3.2. On generating functions of some famous numbers

Following the same approach, we can consider some cases of the expansion (3.8) leading to generating functions of some well-known numbers such as Bell, Stirling and idempotent numbers.

Example 1. Let $f(x)=\sum_{n=0}^{\infty}c_n\,\big(\ln x\big)^n$. So

$$f\big(e^x\big)=\sum_{n=0}^{\infty}c_n\,x^n, \tag{3.24}$$

and the coefficients are computed as

$$c_n=\frac{1}{n!}\sum_{k=0}^{n}f^{(k)}(1)\,B_{n,k}(\underbrace{1,1,\ldots,1}_{n-k+1})=\frac{1}{n!}\sum_{k=0}^{n}f^{(k)}(1)\,S(n,k),$$

where $S(n,k)$ as the second kind of Stirling numbers is explicitly denoted by [1]

$$S(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n.$$

As the first sample, replacing $f(x)=e^x$ in (3.24) eventually gives

$$e^{e^x-1}=\sum_{n=0}^{\infty}\frac{x^n}{n!}\Big(\sum_{k=0}^{n}S(n,k)\Big),$$

which is a generating function for the Bell numbers $B_n=\sum\limits_{k=0}^{n}S(n,k)$.

Another interesting case is when $f(x)=\frac{1}{2-x}$, because $f^{(k)}(1)=k!$ and the coeffecients in this case are computed as

$$c_n=\frac{1}{n!}\sum_{k=0}^{n}k!\,S(n,k)=\frac{1}{n!}\,\tilde{b}(n),$$

where $\tilde{b}(n)$ is the nth ordered Bell number and can be also represented by the important infinite series [20]

$$\tilde{b}(n)=\frac{1}{2}\sum_{m=0}^{\infty}\frac{m^n}{2^m}.$$

Therefore, by noting (3.24), we obtain

$$\frac{1}{2-e^x}=\sum_{n=0}^{\infty}\tilde{b}(n)\,\frac{x^n}{n!},$$

which is convergent in $(\frac{1}{2},2)$, because for large n we have [20]

$$\tilde{b}(n)\approx\frac{n!}{2(\ln 2)^{n+1}},$$

and therefore

$$\lim_{n\to\infty}\frac{1}{n+1}\frac{\tilde{b}(n+1)}{\tilde{b}(n)}=\frac{1}{\ln 2},$$

which yields $x\in(e^{-\ln 2},e^{\ln 2})=(\frac{1}{2},2)$.

The third interesting case is when $f(x)=-\ln x$. Since

$$-\ln x=\sum_{n=1}^{\infty}\frac{(\ln x)^n}{n!}\Big(\sum_{k=1}^{n}(-1)^k(k-1)!\,S(n,k)\Big),$$

by equating both sides of the above equality we conclude that

$$\sum_{k=1}^{n}(-1)^k(k-1)!\,S(n,k)=0 \qquad n\geq 2,$$

which is a direct consequence of the following result [13] for $z=-1$,

$$\sum_{k=1}^{n}S(n,k)\,(k-1)!\,z^k=(-1)^n\,\mathrm{Li}_{1-n}(1+\frac{1}{z}) \qquad n\geq 2,$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function defined by

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n} \qquad |z|<1.$$

Example 2. Assume that $\lambda(x) = xe^x$ for $x \geq -1$. The inverse function $\lambda^{-1}(x) = W_0(x)$, which satisfies the equation

$$x = W_0(x)e^{W_0(x)},$$

is the principal branch of the Lambert function [7]. In fact, the additional constraint $W_0(x) \geq -1$ defines the single-valued function $W_0(x)$ for $x \geq -e^{-1}$. Although $W_0(x)$ cannot be expressed in terms of elementary functions, one can use Lagrange inversion theorem (see e.g. [4]) for computing the Taylor series of $W_0(x)$ at $x = 0$ such that we have

$$W_0(x) = \sum_{n=1}^{\infty} \lim_{\omega \to 0} \Big(\frac{\mathrm{d}^{n-1}}{\mathrm{d}\omega^{n-1}} \, e^{-n\omega} \Big) \frac{x^n}{n!} \; = \sum_{n=1}^{\infty} (-n)^{n-1} \, \frac{x^n}{n!},$$

which is convergent in $x \in (-e^{-1}, e^{-1})$. Now, consider the expansion

$$f(x) = \sum_{n=0}^{\infty} c_n \big(W_0(x) \big)^n,$$

which is equivalent to

$$f\big(xe^x\big) = \sum_{n=0}^{\infty} c_n \, x^n.$$

Since $\dfrac{\mathrm{d}^k}{\mathrm{d}x^k} \, (xe^x)_{|x=0} = k$, the coefficients are given by

$$c_n = \frac{1}{n!} \sum_{k=1}^{n} f^{(k)}(0) \, B_{n,k}\big(1, \, 2, \, \ldots \, , n-k+1\big) = \frac{1}{n!} \sum_{k=1}^{n} f^{(k)}(0) \binom{n}{k} \, k^{n-k},$$

for $n \geq 1$ and $c_0 = f(0)$. For instance, replacing $f(x) = e^x$ leads to the relation

$$\exp(xe^x) = 1 + \sum_{n=1}^{\infty} t(n) \, \frac{x^n}{n!},$$

where $t(n) = \sum\limits_{k=1}^{n} \binom{n}{k} \, k^{n-k}$ is known as the number of idempotent maps. See e.g. [6, p. 91] for more details.

Acknowledgments

This work has been supported by the Alexander von Humboldt Foundation under the grant number: Ref 3.4 - IRN - 1128637 - GF-E.

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