Bivariate hypergeometric-trigonometric series and their partial differential equations

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Abstract. In this paper, we introduce two kinds of hypergeometric-trigonometric bivariate series and compute explicit forms of their partial differential equations.

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1. Introduction. Let

$$f^*(z)=\sum_{k=0}^{\infty} a_k^*\, z^k\,, \tag{1}$$

be a convergent series in which $\{a_k^*\}_{k=0}^{\infty}$ are known real numbers. If

$$z=x+iy=r\,e^{i\theta}\ \ (i=\sqrt{-1})\,,$$

is a complex variable with

$$r=\sqrt{x^2+y^2}\quad\text{and}\quad \theta=\arg(x+iy)\,,$$

then it can be verified from (1) that

$$\operatorname{Im}\left(z^{\lambda} f^*(z)+\bar{z}^{\lambda} f^*(\bar{z})\right)=\operatorname{Im}\left(\frac{z^{\lambda} f^*(z)-\bar{z}^{\lambda} f^*(\bar{z})}{i}\right)=0\,,\quad \forall\lambda\in\mathbb{R}. \tag{2}$$

In other words, $z^{\lambda} f^*(z)+\bar{z}^{\lambda} f^*(\bar{z})$ and $(z^{\lambda} f^*(z)-\bar{z}^{\lambda} f^*(\bar{z}))/i$ are always two real functions if (1) holds. Based on the result (2), we have recently introduced two bivariate series in [7] as

$$C_{\alpha}(f^*;r,\theta,m,n)=\sum_{k=0}^{\infty} a^*_{nk+m}\, r^k \cos(\alpha+k)\theta\,, \tag{3}$$

and

$$S_\alpha(f^*;r,\theta,m,n)=\sum_{k=0}^{\infty} a^*_{nk+m}\, r^k \sin(\alpha+k)\theta\,, \tag{4}$$

where r,θ are real variables, $\alpha\in\mathbb{R}$, $n\in\mathbb{N}$ and $m\in\{0,1,...,n-1\}$ and showed that they are convergent if the reduced series $\sum_{k=0}^{\infty} a^*_{nk+m}\, r^k$ is convergent.

Now, let f^* be an analytic function at the point e.g. $z=b\in\mathbb{R}$. By defining the two real functions

$$g_1(r,\theta;\alpha,b)=\frac{1}{2}\Big(e^{\alpha i\theta} f^*(b+r\,e^{i\theta})+e^{-\alpha i\theta} f^*(b+r\,e^{-i\theta})\Big), \tag{5}$$

and

$$g_2(r,\theta;\alpha,b)=\frac{1}{2i}\Big(e^{\alpha i\theta} f^*(b+r\,e^{i\theta})-e^{-\alpha i\theta} f^*(b+r\,e^{-i\theta})\Big), \tag{6}$$

we can directly conclude that

$$\left.\frac{d^k g_1(r,\theta;\alpha,b)}{dr^k}\right|_{r=0}=f^{*(k)}(b)\cos(\alpha+k)\theta,$$

and

$$\left.\frac{d^k g_2(r,\theta;\alpha,b)}{dr^k}\right|_{r=0}=f^{*(k)}(b)\sin(\alpha+k)\theta\,.$$

Hence, the Taylor expansion of the functions (5) and (6) with respect to the variable r at $r=0$ are respectively given as

$$\frac{1}{2}\Big(e^{\alpha i\theta} f^*(b+r\,e^{i\theta})+e^{-\alpha i\theta} f^*(b+r\,e^{-i\theta})\Big)=\sum_{k=0}^{\infty}\frac{f^{*(k)}(b)}{k!}\, r^k\cos(\alpha+k)\theta\,, \tag{7}$$

and

$$\frac{1}{2i}\Big(e^{\alpha i\theta} f^*(b+r\,e^{i\theta})-e^{-\alpha i\theta} f^*(b+r\,e^{-i\theta})\Big)=\sum_{k=0}^{\infty}\frac{f^{*(k)}(b)}{k!}\, r^k\sin(\alpha+k)\theta\,. \tag{8}$$

Clearly equations (7) and (8) are valid only in the convergence region of r for any $\alpha\in\mathbb{R}$ and $\theta\in[-\pi,\pi]$ provided that $\lim_{k\to\infty}\frac{f^{*(k)}(b)}{k!}\, r^k=0$. In this case, we will automatically have

$$\lim_{k\to\infty}\frac{f^{*(k)}(b)}{k!}\, r^k\cos(\alpha+k)\theta=0 \quad \text{and} \quad \lim_{k\to\infty}\frac{f^{*(k)}(b)}{k!}\, r^k\sin(\alpha+k)\theta=0\,. \tag{9}$$

Relations (9) show that the convergence of the two series (7) and (8) directly depends on the convergence radius of the Taylor expansion corresponding to f^* at $r=b$.

In this paper, we assume that the coefficients $\{a_k^*\}_{k=0}^{\infty}$ in series (3) and (4) are hypergeometric terms and obtain some of their important properties such as integral representations and several classes of partial differential equations.

2. Bivariate series of hypergeometric-trigonometric type

It is known that there are some differences between power series expansions and trigonometric expansions. For instance, the infinite differentiability of a function does not itself assure that its power series will converge to that function, whereas mere periodicity and a little smoothness are enough to have the Fourier trigonometric series converge uniformly to the function [3]. Further, the terms of the trigonometric series expansion describe simple harmonic motion so that the function may be considered as a linear combination of harmonic motions, whereas the terms of a power series have no such physical interpretation.

In this section, we introduce a mixed type of hypergeometric and trigonometric series that covers many ordinary hypergeometric series in addition to usual trigonometric series. Hypergeometric series as special cases of power series are important tools for investigations in different branches of engineering and mathematical sciences [1, 2, 13]. For instance, there is a large set of hypergeometic-type polynomials whose variable is located in one or more of the parameters of the corresponding hypergeometric series [6]. These polynomials are of great importance in mathematics as well as in some areas of physics. A few samples of their applications are discussed in [10]. See also [8, 9].

The main motivation for introducing and developing hypergeometric series is that many elementary and familiar functions (such as trigonometric, exponential and logarithm functions, classical orthogonal polynomials of Jacobi, Laguerre and Hermite, etc. [9, 12]) can be written in terms of them and therefore their initial properties can be found via the initial properties of hypergeometric functions. They also appear as solutions of many important ordinary differential equations [2, 11].

The generalized hypergeometric series

$$ {}_pF_q\left(\left.\begin{matrix} a_1, a_2, \dots & a_p \\ b_1, b_2, \dots & b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \dots (a_p)_k}{(b_1)_k (b_2)_k \dots (b_q)_k} \frac{z^k}{k!}, \tag{10} $$

in which $(r)_k = \prod_{j=0}^{k-1}(r+j) = \Gamma(r+k)/\Gamma(r)$ and z may be a complex variable is indeed the Taylor series expansion of the function $f^*(z) = \sum_{k=0}^{\infty} a_k^* z^k$ with $a_k^* = f^{*(k)}(0)/k!$ for which the ratio of successive terms can be written as

$$ \frac{a_{k+1}^*}{a_k^*} = \frac{(k+a_1)(k+a_2)\dots(k+a_p)}{(k+b_1)(k+b_2)\dots(k+b_q)(k+1)}. $$

According to the ratio test [5], series (10) is convergent for any $p \le q+1$. In other words, it converges in $|z|<1$ for $p=q+1$, converges everywhere for $p<q+1$ and converges nowhere ($z \ne 0$) for $p>q+1$. Moreover, for $p=q+1$ it absolutely converges for $|z|=1$ if the condition

$$A^* = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) > 0,$$

holds and is conditionally convergent for $|z|=1$ and $z \neq 1$ if $-1 < A^* \leq 0$ and is finally divergent for $|z|=1$ and $z \neq 1$ if $A^* \leq -1$.

One of the important cases of the generalized series (10) arising in many physical problems [9, 13] is the Gauss hypergeometric function, which is convergent in $|z| \leq 1$ and is defined by

$${}_2F_1\left(\begin{matrix} a,\ b \\ c \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\, dt \quad (\mathrm{Re}\, c > \mathrm{Re}\, b > 0), \tag{11}$$

satisfying the differential equation

$$z\big(z-1\big)\, y'' + \big((a+b+1)z - c\big)\, y' + ab\, y = 0. \tag{12}$$

We can now extend the hypergeometric series (10) and define a special case of the series (7) as

$${}_pC_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);\alpha\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{r^k}{k!} \cos(\alpha + k)\theta, \tag{13}$$

and a special case of the series (8) as

$${}_pS_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);\alpha\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_q)_k} \frac{r^k}{k!} \sin(\alpha + k)\theta. \tag{14}$$

According to (7) and (8), both series (13) and (14) are convergent if the corresponding series (10) is convergent. It is clear for (13) and (14) that we have

$$\begin{aligned} {}_pC_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);\alpha\right) &= \cos\alpha\theta\ {}_pC_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);0\right) \\ &\quad - \sin\alpha\theta\ {}_pS_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);0\right), \end{aligned} \tag{15}$$

and

$$\begin{aligned} {}_pS_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);\alpha\right) &= \sin\alpha\theta\ {}_pC_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);0\right) \\ &\quad + \cos\alpha\theta\ {}_pS_q\left(\begin{matrix} a_1,\ a_2,\ \ldots & a_p \\ b_1,\ b_2,\ \ldots & b_q \end{matrix}\middle| (r,\theta);0\right). \end{aligned} \tag{16}$$

As we pointed out, the main motivation for defining such bivariate series is that many ordinary hypergeometric series (when $\theta = 0$) and Fourier trigonometric series (when r is fixed and pre-assigned) can be represented in terms of them as the following examples show.

Example 1. The fact that a discontinuous function can be represented as an infinite sum of sines and cosines was known already to Euler [4], who obtained the following sum

$$\frac{\pi-\theta}{2} = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k}, \qquad 0 < \theta < 2\pi . \tag{17}$$

But (17) can now be represented as

$$\sum_{k=1}^{\infty} \frac{\sin k\theta}{k} = \sum_{j=0}^{\infty} \frac{\sin(j+1)\theta}{j+1} = \sum_{j=0}^{\infty} \frac{(1)_j (1)_j}{(2)_j} \frac{\sin(j+1)\theta}{j!} = {}_2S_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,(1,\theta);1\right).$$

Example 2. Since

$$\begin{aligned}{}_2F_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,x+iy\right) &= -\frac{\ln(1-x-iy)}{x+iy}\\ &= \frac{-\frac{x}{2}\ln\left((1-x)^2+y^2\right) + y\arctan\frac{y}{1-x}}{x^2+y^2} + i\,\frac{x\arctan\frac{y}{1-x} + \frac{y}{2}\ln\left((1-x)^2+y^2\right)}{x^2+y^2},\end{aligned} \tag{18}$$

replacing $x = r\cos\theta$ and $y = r\sin\theta$ in (18) gives

$$\frac{1}{r}\left(-\frac{\cos\theta}{2}\ln\left(1+r^2-2r\cos\theta\right) + \sin\theta\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right)\right) = {}_2C_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,(r,\theta);0\right),$$

and

$$\frac{1}{r}\left(\cos\theta\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right) + \frac{\sin\theta}{2}\ln\left(1+r^2-2r\cos\theta\right)\right) = {}_2S_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,(r,\theta);0\right).$$

Therefore, by using (15) and (16), we respectively obtain

$$\begin{aligned}{}_2C_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,(r,\theta);\alpha\right) &= \sum_{k=0}^{\infty} \frac{r^k}{k+1}\cos(\alpha+k)\theta\\ &= -\frac{\sin(\alpha-1)\theta}{r}\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \frac{\cos(\alpha-1)\theta}{2r}\ln\left(1+r^2-2r\cos\theta\right),\end{aligned}$$

and

$$\begin{aligned}{}_2S_1\left(\begin{matrix}1,\ 1\\ 2\end{matrix}\,\middle|\,(r,\theta);\alpha\right) &= \sum_{k=0}^{\infty} \frac{r^k}{k+1}\sin(\alpha+k)\theta\\ &= \frac{\cos(\alpha-1)\theta}{r}\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right) - \frac{\sin(\alpha-1)\theta}{2r}\ln\left(1+r^2-2r\cos\theta\right).\end{aligned} \tag{19}$$

It is interesting to note that substituting $r=\alpha=1$ in the right hand side of (19) exactly gives the left hand side of (17).

Example 3. Since

$$ {}_2F_1\left(\begin{matrix}1/2,\ 1\\ 3/2\end{matrix}\middle| -(x+iy)^2\right)=\frac{\arctan(x+iy)}{x+iy} $$

$$ =\frac{1}{2(x^2+y^2)}\left(\frac{y}{2}\ln\frac{x^2+(1+y)^2}{x^2+(1-y)^2}+x\arctan\frac{2x}{1-x^2-y^2}\right) $$

$$ +i\,\frac{1}{2(x^2+y^2)}\left(\frac{x}{2}\ln\frac{x^2+(1+y)^2}{x^2+(1-y)^2}-y\arctan\frac{2x}{1-x^2-y^2}\right), $$

so

$$ \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}r^{2k}\cos 2k\theta=\frac{1}{2r}\left(\frac{\sin\theta}{2}\ln\frac{1+r^2+2r\sin\theta}{1+r^2-2r\sin\theta}+\cos\theta\arctan\left(\frac{2r\cos\theta}{1-r^2}\right)\right) $$

$$ = {}_2C_1\left(\begin{matrix}1/2,\ 1\\ 3/2\end{matrix}\middle|(-r^2,\ 2\theta);0\right), $$

and

$$ \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}r^{2k}\sin 2k\theta=\frac{1}{2r}\left(\frac{\cos\theta}{2}\ln\frac{1+r^2+2r\sin\theta}{1+r^2-2r\sin\theta}-\sin\theta\arctan\left(\frac{2r\cos\theta}{1-r^2}\right)\right) $$

$$ = {}_2S_1\left(\begin{matrix}1/2,\ 1\\ 3/2\end{matrix}\middle|(-r^2,\ 2\theta);0\right). $$

Example 4. It is shown in [7] that the equality

$$ \sum_{k=0}^{\infty}\frac{r^{3k+2}}{3k+2}\cos(3k+2)\theta= $$

$$ -\frac{1}{12}\ln\frac{(1+r^2-2r\cos\theta)^2}{4r^2\cos^2\theta+2r(r^2+1)\cos\theta+r^4-r^2+1}-\frac{\sqrt{3}}{6}\arctan\frac{\sqrt{3}\,(r^2+2\,r\cos\theta)}{-r^2+2r\cos\theta+2}, $$

is valid for any $|r|<1$ and $\theta\in[-\pi,\pi]$. Since $\frac{1}{3k+2}=\frac{1}{2}\frac{(2/3)_k}{(5/3)_k}$, the above series can now be represented as

$$ \sum_{k=0}^{\infty}\frac{r^{3k+2}}{3k+2}\cos(3k+2)\theta=\frac{1}{2}r^2\,{}_2C_1\left(\begin{matrix}2/3,\ 1\\ 5/3\end{matrix}\middle|(r^3,\ 3\theta);\frac{2}{3}\right). $$

In the sequel, an integral representation of series (13) and (14) can be derived for $(p,q)=(2,1)$, respectively, as follows: Since

$$ \frac{1}{2}\left((1-tr\,e^{i\theta})^{-a}+(1-tr\,e^{-i\theta})^{-a}\right)=\left(1+t^2r^2-2tr\cos\theta\right)^{-a/2}\cos\left(a\arctan\frac{tr\sin\theta}{tr\cos\theta-1}\right), $$

so via (11) we obtain

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta$$
$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1+t^2r^2-2tr\cos\theta\right)^{-a/2}\cos(\alpha\theta-a\arctan\frac{tr\sin\theta}{tr\cos\theta-1})dt.$$

Similarly, the integral representation of (14) for $(p,q)=(2,1)$ takes the form

$$\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta$$
$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left(1+t^2r^2-2tr\cos\theta\right)^{-a/2}\sin(\alpha\theta-a\arctan\frac{tr\sin\theta}{tr\cos\theta-1})dt.$$

2.1. Partial differential equations of the two introduced series

There are several classes of partial differential equations (PDEs) for each bivariate series (13) and (14). In this section, we first obtain the equations for $(p,q)=(2,1)$ and then extend the results to the general case (p,q). Let us begin with the assumption

$$ {}_2C_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\middle|(r,\theta);\alpha\right)=y(r,\theta)=y. \tag{20}$$

It can be verified that the following equations hold for y defined in (20):

$$(r\frac{\partial}{\partial r}+a)(r\frac{\partial}{\partial r}+b)\,y=\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta, \tag{21}$$

and

$$(-\frac{\partial}{\partial\theta}-\alpha+a)(r\frac{\partial}{\partial r}+b)\,y=(-\frac{\partial}{\partial\theta}-\alpha+b)(r\frac{\partial}{\partial r}+a)\,y=\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta, \tag{22}$$

and

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_{k-1}}\frac{r^{k-1}}{(k-1)!}\cos(\alpha+k)\theta=\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\cos(\alpha+k+1)\theta$$
$$=\cos\theta\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta-\sin\theta\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta. \tag{23}$$

Hence, substituting (21) and (22) into (23) respectively gives

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=(\cos\theta\,(r\frac{\partial}{\partial r}+a)(r\frac{\partial}{\partial r}+b)-\sin\theta\,(-\frac{\partial}{\partial\theta}-\alpha+a)(r\frac{\partial}{\partial r}+b))y, \tag{24}$$

and

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=(\cos\theta\,(r\frac{\partial}{\partial r}+a)(r\frac{\partial}{\partial r}+b)-\sin\theta\,(-\frac{\partial}{\partial\theta}-\alpha+b)(r\frac{\partial}{\partial r}+a))y. \tag{25}$$

Another method is to use the variable θ to get

$$(\frac{\partial}{\partial\theta}-\alpha+a)(-\frac{\partial}{\partial\theta}-\alpha+b)\,y=(\frac{\partial}{\partial\theta}-\alpha+b)(-\frac{\partial}{\partial\theta}-\alpha+a)\,y=\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta, \tag{26}$$

and

$$(r\frac{\partial}{\partial r}+a)(-\frac{\partial}{\partial\theta}-\alpha+b)\,y=(r\frac{\partial}{\partial r}+b)(-\frac{\partial}{\partial\theta}-\alpha+a)\,y=\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta, \tag{27}$$

and

$$\begin{aligned}(\frac{\partial}{\partial\theta}-\alpha)(-\frac{\partial}{\partial\theta}-\alpha+c-1)\,y&=\sum_{k=1}^{\infty}\frac{(a)_k(b)_k}{(c)_{k-1}}\frac{r^k}{(k-1)!}\cos(\alpha+k)\theta\\&=\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^{k+1}}{k!}\cos(\alpha+k+1)\theta\\&=r\cos\theta\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta-r\sin\theta\sum_{k=0}^{\infty}\frac{(a)_{k+1}(b)_{k+1}}{(c)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta.\end{aligned} \tag{28}$$

Again, substituting (26) and (27) into (28) respectively gives

$$(\frac{\partial}{\partial\theta}-\alpha)(-\frac{\partial}{\partial\theta}-\alpha+c-1)\,y=(r\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+a)(-\frac{\partial}{\partial\theta}-\alpha+b)-r\sin\theta\,(r\frac{\partial}{\partial r}+a)(-\frac{\partial}{\partial\theta}-\alpha+b))y, \tag{29}$$

and

$$(\frac{\partial}{\partial\theta}-\alpha)(-\frac{\partial}{\partial\theta}-\alpha+c-1)\,y=(r\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+a)(-\frac{\partial}{\partial\theta}-\alpha+b)-r\sin\theta\,(r\frac{\partial}{\partial r}+b)(-\frac{\partial}{\partial\theta}-\alpha+a))y, \tag{30}$$

and

$$(\frac{\partial}{\partial\theta}-\alpha)(-\frac{\partial}{\partial\theta}-\alpha+c-1)\,y=(r\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+b)(-\frac{\partial}{\partial\theta}-\alpha+a)-r\sin\theta\,(r\frac{\partial}{\partial r}+a)(-\frac{\partial}{\partial\theta}-\alpha+b))y, \tag{31}$$

and

$$(\frac{\partial}{\partial\theta}-\alpha)(-\frac{\partial}{\partial\theta}-\alpha+c-1)\,y=(r\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+b)(-\frac{\partial}{\partial\theta}-\alpha+a)-r\sin\theta\,(r\frac{\partial}{\partial r}+b)(-\frac{\partial}{\partial\theta}-\alpha+a))y. \tag{32}$$

The third method may be a simultaneous combination of relations (21) with (26) and (22) with (27) so that we have

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=(\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+a)(-\frac{\partial}{\partial\theta}-\alpha+b)-\sin\theta\,(-\frac{\partial}{\partial\theta}-\alpha+a)(r\frac{\partial}{\partial r}+b))y, \tag{33}$$

and

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=(\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+a)(-\frac{\partial}{\partial\theta}-\alpha+b)-\sin\theta\,(-\frac{\partial}{\partial\theta}-\alpha+b)(r\frac{\partial}{\partial r}+a))y, \tag{34}$$

and

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=(\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+b)(-\frac{\partial}{\partial\theta}-\alpha+a)-\sin\theta\,(-\frac{\partial}{\partial\theta}-\alpha+a)(r\frac{\partial}{\partial r}+b))y, \tag{35}$$

and

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}+c-1)\,y=(\cos\theta\,(\frac{\partial}{\partial\theta}-\alpha+b)(-\frac{\partial}{\partial\theta}-\alpha+a)-\sin\theta\,(-\frac{\partial}{\partial\theta}-\alpha+b)(r\frac{\partial}{\partial r}+a))y. \tag{36}$$

Corollary 1. After simplifying and computing each ten equations in (24), (25) and (29) to (36), the explicit forms of PDEs for series (20) are, respectively, as follows:

$$r\big((\cos\theta)r-1\big)\frac{\partial^2 y}{\partial r^2}+r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}+\big(((a+b+1)\cos\theta+(\alpha-a)\sin\theta\big)r-c\big)\frac{\partial y}{\partial r}$$
$$+b\sin\theta\frac{\partial y}{\partial\theta}+b(a\cos\theta+(\alpha-a)\sin\theta)y=0, \tag{24a}$$

and

$$r\big((\cos\theta)r-1\big)\frac{\partial^2 y}{\partial r^2}+r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}+\big(((a+b+1)\cos\theta+(\alpha-b)\sin\theta\big)r-c\big)\frac{\partial y}{\partial r}$$
$$+a\sin\theta\frac{\partial y}{\partial\theta}+a(b\cos\theta+(\alpha-b)\sin\theta)y=0, \tag{25a}$$

and

$$-\big(r\cos\theta-1\big)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(\alpha-b)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\big(((b-a)\cos\theta+a\sin\theta\big)r-c+1\big)\frac{\partial y}{\partial\theta}+\big((\alpha-b)(a\sin\theta+(\alpha-a)\cos\theta)r+\alpha(c-1-\alpha)\big)y=0, \tag{29a}$$

and

$$-\big(r\cos\theta-1\big)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(\alpha-a)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\big(((b-a)\cos\theta+b\sin\theta\big)r-c+1\big)\frac{\partial y}{\partial\theta}+\big((\alpha-a)(b\sin\theta+(\alpha-b)\cos\theta)r+\alpha(c-1-\alpha)\big)y=0, \tag{30a}$$

and

$$-\big(r\cos\theta-1\big)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(\alpha-b)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\big((-(b-a)\cos\theta+a\sin\theta\big)r-c+1\big)\frac{\partial y}{\partial\theta}+\big((\alpha-b)(a\sin\theta+(\alpha-a)\cos\theta)r+\alpha(c-1-\alpha)\big)y=0, \tag{31a}$$

and

$$-\big(r\cos\theta-1\big)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(\alpha-a)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\big((-(b-a)\cos\theta+b\sin\theta\big)r-c+1\big)\frac{\partial y}{\partial\theta}+\big((\alpha-a)(b\sin\theta+(\alpha-b)\cos\theta)r+\alpha(c-1-\alpha)\big)y=0. \tag{32a}$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}+\big((a-\alpha)r\sin\theta+c\big)\frac{\partial y}{\partial r}$$
$$-\big((b-a)\cos\theta+b\sin\theta\big)\frac{\partial y}{\partial\theta}-(a-\alpha)\big((b-\alpha)\cos\theta-b\sin\theta\big)y=0, \tag{33a}$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}+\big((b-\alpha)r\sin\theta+c\big)\frac{\partial y}{\partial r}$$
$$-\big((b-a)\cos\theta+a\sin\theta\big)\frac{\partial y}{\partial\theta}-(b-\alpha)\big((a-\alpha)\cos\theta-a\sin\theta\big)y=0, \tag{34a}$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}+\big((a-\alpha)r\sin\theta+c\big)\frac{\partial y}{\partial r}$$
$$-\big(-(b-a)\cos\theta+b\sin\theta\big)\frac{\partial y}{\partial\theta}-(a-\alpha)\big((b-\alpha)\cos\theta-b\sin\theta\big)y=0, \tag{35a}$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}+\big((b-\alpha)r\sin\theta+c\big)\frac{\partial y}{\partial r}$$
$$-\big(-(b-a)\cos\theta+a\sin\theta\big)\frac{\partial y}{\partial\theta}-(b-\alpha)\big((a-\alpha)\cos\theta-a\sin\theta\big)y=0. \tag{36a}$$

It is interesting to note that replacing $\theta=0$ in (24a) and (25a) exactly gives the Gauss differential equation (12).

Corollary 1 shows that there are various classes of PDEs for the general case $(p,q)>(2,1)$. However, due to pages limitation, we here obtain only one class. Let us assume that

$${}_pC_q\left(\begin{matrix}a_1, a_2, \dots & a_p\\ b_1, b_2, \dots & b_q\end{matrix}\middle|(r,\theta);\alpha\right)=y(r,\theta)=y. \tag{37}$$

The following equations hold for y defined in (37):

$$\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y=\sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_k\dots(b_q)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta,$$

and

$$\left\{(-\frac{\partial}{\partial\theta}-\alpha+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y=\left\{(-\frac{\partial}{\partial\theta}-\alpha+a_2)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_3)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y$$
$$=\dots=\left\{(-\frac{\partial}{\partial\theta}-\alpha+a_p)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_{p-1})\right\}y=\sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_k\dots(b_q)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta.$$

Consequently we have

$$\frac{\partial}{\partial r}\left\{(r\frac{\partial}{\partial r}+b_1-1)(r\frac{\partial}{\partial r}+b_2-1)\dots(r\frac{\partial}{\partial r}+b_q-1)\right\}y=\sum_{k=1}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_{k-1}\dots(b_q)_{k-1}}\frac{r^{k-1}}{(k-1)!}\cos(\alpha+k)\theta$$
$$=\sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_k\dots(b_q)_k}\frac{r^k}{k!}\cos(\alpha+k+1)\theta$$
$$=\cos\theta\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y-\sin\theta\left\{(-\frac{\partial}{\partial\theta}-\alpha+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y$$
$$=\cos\theta\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y-\sin\theta\left\{(-\frac{\partial}{\partial\theta}-\alpha+a_2)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_3)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$=\cos\theta\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y-\sin\theta\left\{(-\frac{\partial}{\partial\theta}-\alpha+a_p)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_{p-1})\right\}y.$$

Similarly, if we take

$$ {}_pS_q\left(\begin{matrix} a_1,\, a_2,\, \dots & a_p \\ b_1,\, b_2,\, \dots & b_q \end{matrix}\middle| (r,\theta);\alpha\right) = y(r,\theta) = y, \tag{38}$$

then we obtain

$$\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y = \sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_k\dots(b_q)_k}\frac{r^k}{k!}\sin(\alpha+k)\theta, \tag{39}$$

and

$$\left\{(\frac{\partial}{\partial\theta}-\beta+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y = \left\{(\frac{\partial}{\partial\theta}-\alpha+a_2)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_3)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y$$
$$= \dots = \left\{(\frac{\partial}{\partial\theta}-\alpha+a_p)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_{p-1})\right\}y = \sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_k\dots(b_q)_k}\frac{r^k}{k!}\cos(\alpha+k)\theta. \tag{40}$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial r}\left\{(r\frac{\partial}{\partial r}+b_1-1)(r\frac{\partial}{\partial r}+b_2-1)\dots(r\frac{\partial}{\partial r}+b_q-1)\right\}y &= \sum_{k=1}^{\infty}\frac{(a_1)_k\dots(a_p)_k}{(b_1)_{k-1}\dots(b_q)_{k-1}}\frac{r^{k-1}}{(k-1)!}\sin(\alpha+k)\theta \\
&= \sum_{k=0}^{\infty}\frac{(a_1)_{k+1}\dots(a_p)_{k+1}}{(b_1)_k\dots(b_q)_k}\frac{r^k}{k!}\sin(\alpha+k+1)\theta \\
= \cos\theta\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y &+ \sin\theta\left\{(\frac{\partial}{\partial\theta}-\alpha+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y \\
= \cos\theta\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y &+ \sin\theta\left\{(\frac{\partial}{\partial\theta}-\alpha+a_2)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_3)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y \\
\vdots \qquad\qquad & \qquad \vdots \qquad\qquad \vdots \\
= \cos\theta\left\{(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_p)\right\}y &+ \sin\theta\left\{(\frac{\partial}{\partial\theta}-\alpha+a_p)(r\frac{\partial}{\partial r}+a_1)(r\frac{\partial}{\partial r}+a_2)\dots(r\frac{\partial}{\partial r}+a_{p-1})\right\}y.
\end{aligned} \tag{41}$$

Corollary 2. Let $(p,q)=(2,1)$ in (38). After applying the relations (39), (40) and (41), ten different PDEs would appear for ${}_2S_1\left(\begin{matrix} a,\, b \\ c \end{matrix}\middle| (r,\theta);\alpha\right) = y$ as follows:

$$r\big((\cos\theta)r-1\big)\frac{\partial^2 y}{\partial r^2} + r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta} + \Big(\big((a+b+1)\cos\theta+(a-\alpha)\sin\theta\big)r-c\Big)\frac{\partial y}{\partial r}$$
$$+ b\sin\theta\frac{\partial y}{\partial\theta} + b(a\cos\theta+(a-\alpha)\sin\theta)y = 0,$$

and

$$r\big((\cos\theta)r-1\big)\frac{\partial^2 y}{\partial r^2} + r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta} + \Big(\big((a+b+1)\cos\theta+(b-\alpha)\sin\theta\big)r-c\Big)\frac{\partial y}{\partial r}$$
$$+ a\sin\theta\frac{\partial y}{\partial\theta} + a(b\cos\theta+(b-\alpha)\sin\theta)y = 0,$$

and

$$-\left(r\cos\theta-1\right)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(b-\alpha)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\Big(\big((b-a)\cos\theta+a\sin\theta\big)r-c+1\Big)\frac{\partial y}{\partial\theta}+\Big((b-\alpha)((a-\alpha)\cos\theta+a\sin\theta)r-\alpha(\alpha+1-c)\Big)y=0,$$

and

$$-\left(r\cos\theta-1\right)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(a-\alpha)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\Big(\big((b-a)\cos\theta+b\sin\theta\big)r-c+1\Big)\frac{\partial y}{\partial\theta}+\Big((a-\alpha)((b-\alpha)\cos\theta+b\sin\theta)r-\alpha(\alpha+1-c)\Big)y=0,$$

and

$$-\left(r\cos\theta-1\right)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(b-\alpha)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\Big(\big((a-b)\cos\theta+a\sin\theta\big)r-c+1\Big)\frac{\partial y}{\partial\theta}+\Big((b-\alpha)((a-\alpha)\cos\theta+a\sin\theta)r-\alpha(\alpha+1-c)\Big)y=0,$$

and

$$-\left(r\cos\theta-1\right)\frac{\partial^2 y}{\partial\theta^2}+r^2\sin\theta\frac{\partial^2 y}{\partial\theta\,\partial r}+(a-\alpha)r^2\sin\theta\frac{\partial y}{\partial r}$$
$$+\Big(\big((a-b)\cos\theta+b\sin\theta\big)r-c+1\Big)\frac{\partial y}{\partial\theta}+\Big((a-\alpha)((b-\alpha)\cos\theta+b\sin\theta)r-\alpha(\alpha+1-c)\Big)y=0,$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}-\big((b-\alpha)r\sin\theta-c\big)\frac{\partial y}{\partial r}$$
$$-\big((b-a)\cos\theta+a\sin\theta\big)\frac{\partial y}{\partial\theta}-(b-\alpha)\big((a-\alpha)\cos\theta+a\sin\theta\big)y=0,$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}-\big((a-\alpha)r\sin\theta-c\big)\frac{\partial y}{\partial r}$$
$$-\big((b-a)\cos\theta+b\sin\theta\big)\frac{\partial y}{\partial\theta}-(a-\alpha)\big((b-\alpha)\cos\theta+b\sin\theta\big)y=0,$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}-\big((b-\alpha)r\sin\theta-c\big)\frac{\partial y}{\partial r}$$
$$-\big((a-b)\cos\theta+a\sin\theta\big)\frac{\partial y}{\partial\theta}-(b-\alpha)\big((a-\alpha)\cos\theta+a\sin\theta\big)y=0,$$

and

$$r\frac{\partial^2 y}{\partial r^2}+\cos\theta\frac{\partial^2 y}{\partial\theta^2}-r\sin\theta\frac{\partial^2 y}{\partial r\partial\theta}-\big((a-\alpha)r\sin\theta-c\big)\frac{\partial y}{\partial r}$$
$$-\big((a-b)\cos\theta+b\sin\theta\big)\frac{\partial y}{\partial\theta}-(a-\alpha)\big((b-\alpha)\cos\theta+b\sin\theta\big)y=0.$$

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