On Linearization Coefficients of Jacobi Polynomials

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Abstract

This article deals with the problem of finding closed analytical formulae for generalized linearization coefficients for Jacobi polynomials. By considering some special cases we obtain a reduction formula using for this purpose symbolic computation, in particular Zeilberger's and Petkovsek's algorithms.

Key words. Jacobi polynomials; Linearization coefficients; Reduction formulae

The *general linearization problem* consists in finding the coefficients $L_{ij}(k)$ in the expansion of two polynomials $Q_i(x)$, $R_j(x)$ in terms of an arbitrary sequence $\{P_n\}_{n\geq 0}$ $(\deg P_n = n)$:

$$Q_i(x)R_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{1}$$

Particular case of this problem is the *standard linearization* or *Clebsh-Gordan type problem* ($P_n = Q_n = R_n$),

$$P_i(x)P_j(x) = \sum_{k=0}^{i+j} L_{ij}(k)P_k(x). \tag{2}$$

On the other hand, taking $R_j = 1$ in (1), we are faced with the so-called *connection problem*, which for $Q_i = x^i$ is known as the inversion problem for the family $\{P_n\}_n$.

The literature on linearization and connection problems is extremely vast, and a variety of methods and approaches for computing the coefficients $L_{ij}(k)$ in (1) have been developed. In the standard case (2), when $\{P_n\}_n$ is an orthogonal family (with respect to some positive measure), many results concerning the positivity of the coefficients $L_{ij}(k)$ and the recurrence relation satisfied by these coefficients are known, in some cases (classical orthogonal polynomials) the coefficients $L_{ij}(k)$ are given explicitly, very often in terms of hypergeometric functions.

We recall that ${}_pF_q$ denotes the generalized hypergeometric function with p numerator and q denominator parameters, given by

$${}_pF_q\left(\begin{array}{c}(a_p)\\(b_q)\end{array}\middle|\, x\right) \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{x^k}{k!}, \tag{3}$$

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where the contracted notation (a_p) is used to abbreviate the array of p parameters $a_1, \ldots, a_p$ and $(x)_n := \dfrac{\Gamma(x+n)}{\Gamma(x)}$ denotes the well-known Pochhammer symbol.

In this work, we consider the Jacobi polynomials defined by [14]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, \alpha+\beta+n+1\\ \alpha+1\end{matrix}\,\middle|\,\frac{1-x}{2}\right).$$

The standard linearization problem associated to Jacobi polynomials and to establish the conditions of nonnegativity of the linearization coefficients has been under considerable research for many years. Hyllareas (1962) investigated particular cases [7], Gasper (1970) found the necessary and sufficient conditions for the non-negativity of these coefficients [1, 5] and Koornwinder (1978) approached the same problem from a different point of view [11]. Rahman (1981) gave an explicit representation of the standard linearization coefficients, $L_{ij}(k)$, for the Jacobi polynomials and their continuous q-analogue in terms of ${}_9F_8$ and ${}_{10}\Phi_9$ hypergeometric series respectively, but with distinct explicit representations for even and odd values of k [12, 13].

The main aim of this paper is to give a closed form of the general linearization coefficients for Jacobi polynomials in terms of the Kampé de Fériet function and to prove that in a suitable particular case these coefficients can be expressed as a product of two terminating functions. By using symbolic computation, we show that one of these two hypergeometric functions can be reduced to a simple hypergeometric term. As far as we know, the obtained reduction formula for ${}_3F_2$ is not included in any known reduction formula and appears to be new. At the end of this work, we use known connection and linearization formulae for ultraspherical polynomials to derive a reduction formula associated to a terminating double sum.

The Kampé de Fériet function is the double hypergeometric function defined by: [15, p. 63]

$$F^{p:\,k}_{l:\,n}\left(\begin{matrix}(a_p): & (b_k);\ (c_k); & \\ & & x,\ y\\ (\alpha_l): & (\beta_n);\ (\gamma_n); & \end{matrix}\right) = \sum_{r,s=0}^{\infty}\frac{[a_p]_{r+s}[b_k]_r[c_k]_s}{[\alpha_l]_{r+s}[\beta_n]_r[\gamma_n]_s}\frac{x^r}{r!}\frac{y^s}{s!}, \tag{4}$$

where $[a_p]_r = \prod_{j=1}^{p}(a_j)_r, \ldots.$

To solve the linearization problem for the Jacobi PS, we need the following result which is proved in [4].

Theorem 1. *Let $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be three polynomial sets generated, respectively, by*

$$\begin{aligned} A_1(t)B_1\left(xC_1(t)\right) &= \sum_{n=0}^{\infty}\lambda_n^{(1)}P_n(x)t^n,\\ A_2(t)B_2\left(xC_2(t)\right) &= \sum_{n=0}^{\infty}\lambda_n^{(2)}Q_n(x)t^n, \qquad (5)\\ A_3(t)B_3\left(xC_3(t)\right) &= \sum_{n=0}^{\infty}\lambda_n^{(3)}R_n(x)t^n, \end{aligned}$$

where $A_p,\ B_p$ and C_p, are three formal power series satisfying $A_p(0)\neq 0,\ C_p(0)=0,\ C_p'(0)\neq 0,\ B_p^{(k)}(0)\neq 0\ \forall\, k\neq 0$ and $\lambda_n^{(p)}\neq 0;\ p=1,2,3.$
Then the associated linearization coefficients in (1) are given by

$$L_{ij}(k) = \frac{\lambda_k^{(1)}}{\lambda_i^{(2)}\lambda_j^{(3)}}\sum_{r=0}^{i}\sum_{s=0}^{j}\frac{\gamma_r^{(2)}\gamma_s^{(3)}}{\gamma_{r+s}^{(1)}}a_r^{(2)}(i)a_s^{(3)}(j)\psi_{r+s}(k),\quad k=0,1,..,i+j, \tag{6}$$

where

$$A_p(t)C_p^m(t) = \sum_{i=m}^{\infty} a_m^{(p)}(i)t^i, \quad B_p(t) = \sum_{k=0}^{\infty} \gamma_k^{(p)} t^k; \; p = 1, 2, 3; \;\; \textit{and} \;\; \frac{C_1^{-k}(t)}{A_1(C_1^{-1}(t))} = \sum_{n=k}^{\infty} \psi_n(k) t^n. \tag{7}$$

Recall here that a polynomial set defined by a generating function like in (5) is said to be of Boas-Buck type [3].

The Jacobi polynomial set is generated by [8]

$$(1-t)^{-\tau} {}_2F_1\left(\begin{matrix} \frac{\tau}{2}, \frac{\tau+1}{2} \\ \alpha+1 \end{matrix} \middle| \frac{-2(x-1)t}{(1-t)^2} \right) = \sum_{n=0}^{\infty} \frac{(\tau)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n} t^n,$$

where $\tau = \alpha + \beta + 1$.

It follows that the shifted Jacobi polynomial set is of Boas-Buck type with

$$A(t) = (1-t)^{-\tau}, \; C(t) = \frac{-t}{(1-t)^2} \;\; \text{and} \;\; B(t) = {}_2F_1\left(\begin{matrix} \frac{\tau}{2}, \frac{\tau+1}{2} \\ \alpha+1 \end{matrix} \middle| t \right). \tag{8}$$

For this case, and to get the development of the formal power series in (7), we need the following lemma.

Lemma 2 (Lagrange's inversion formula [15]). *Let ξ be a function of t implicitly defined by*

$$\xi = t(1+\xi)^{s+1}, \quad \xi(0) = 0. \tag{9}$$

Then we have

$$(1+\xi(t))^r = \sum_{n=0}^{\infty} \frac{r}{r+(s+1)n} \binom{r+(s+1)n}{n} t^n, \tag{10}$$

where r and s are complex numbers independent of n.

In our case we have

$$A(t) = (1-t)^{-\tau} \text{ and } \quad C(t) = \frac{-t}{(1-t)^2} \; .$$

C^{-1} is defined, implicitly, by

$$(1 - C^{-1}(t))^2 t = -C^{-1}(t) \; .$$

Using (10), with $\xi = -C^{-1}$, $s = 1$ and $r = \tau + 2k$, we obtain

$$\begin{aligned} \frac{(C^{-1})^k(t)}{A(C^{-1}(t))} &= (-1)^k (1 - C^{-1}(t))^{2k+\tau} t^k \\ &= (-1)^k \sum_{n=0}^{\infty} \frac{\tau+2k}{\tau+2n+2k} \binom{2n+2k+\tau}{n} t^{n+k} \\ &= (-1)^k \sum_{n=k}^{\infty} \frac{\tau+2k}{\tau+2n} \frac{(\tau+1+n+k)_{n-k}}{(n-k)!} t^n \; . \end{aligned}$$

On the other hand, it is easy to check that

$$A(t)C^m(t) = (-1)^m \frac{t^m}{(1-t)^{2m+\tau}} = (-1)^m \sum_{n=m}^{\infty} \frac{(2m+\tau)_{n-m}}{(n-m)!} t^n. \tag{11}$$

By using Theorem 1, we deduce that the linearization coefficients in

$$P_i^{(\lambda,\delta)}(x) P_j^{(\mu,\gamma)}(x) = \sum_{k=0}^{i+j} L_{ij}(k) P_k^{(\alpha,\beta)}(x),$$

are given by

$$\begin{aligned} L_{ij}(i+j-k) = {} & \frac{(\alpha+\beta+1)_{i+j-k}(\alpha+1)_{i+j}(2(i+j-k)+\alpha+\beta+1)}{(\alpha+1)_{i+j-k}(\alpha+\beta+1)_{2(i+j)-k+1}} \\ & \times \frac{(-1)^k (i+j)!}{i!j!k!} \frac{(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{(\lambda+\delta+1)_i(\mu+\gamma+1)_j} \\ & \times F^{2:\,2}_{2:\,1}\left(\begin{matrix} -k,\ -\alpha-\beta-1-2(i+j)+k:\ -i,\ -\lambda-i;\ -j,\ -\mu-j; \\ \\ -(i+j),\ -\alpha-(i+j):\ -2i-\lambda-\delta;\ -2j-\mu-\gamma; \end{matrix} \ 1,\ 1 \right). \end{aligned} \tag{12}$$

In the special case $\alpha = \mu + \lambda$, $\beta = \delta + \gamma$, we get

$$\begin{aligned} L_{ij}(i+j-k) = {} & \frac{(\mu+\lambda+\delta+\gamma+1)_{i+j-k}(\mu+\lambda+1)_{i+j}(2(i+j-k)+\mu+\lambda+\delta+\gamma+1)}{(\mu+\lambda+1)_{i+j-k}(\mu+\lambda+\delta+\gamma+1)_{2(i+j)-k+1}} \\ & \times \frac{(-1)^k (i+j)!}{i!j!k!} \frac{(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{(\lambda+\delta+1)_i(\mu+\gamma+1)_j} \\ & \times F^{2:\,2}_{2:\,1}\left(\begin{matrix} -k,\ -\lambda-\mu-\delta-\gamma-1-2(i+j)+k:\ -i,\ -\lambda-i;\ -j,\ -\mu-j; \\ \\ -(i+j),\ -\lambda-\mu-(i+j):\ -2i-\lambda-\delta;\ -2j-\mu-\gamma; \end{matrix} \ 1,\ 1 \right). \end{aligned} \tag{13}$$

In view of the Gasper's reduction formula [6] for the product of two terminating hypergeometric functions in terms of a Kampé de Fériet function

$$\begin{aligned} {}_3F_2\left(\begin{matrix} -n, n+a, b \\ c, d \end{matrix} \middle| 1 \right) {}_3F_2\left(\begin{matrix} -n, n+a, e \\ c, f \end{matrix} \middle| 1 \right) = {} & \frac{(-1)^n (a-c+1)_n}{(c)_n} \\ \times F^{2:\,2}_{2:\,1}\left(\begin{matrix} -n,\ n+a:\ b,\ e;\ d-b,\ f-e; \\ \\ d,\ f:\ c;\ a-c+1; \end{matrix} \ 1,\ 1 \right), & \end{aligned} \tag{14}$$

the linearization coefficient in (13) can be written as

$$\begin{aligned} L_{ij}(i+j-k) &= \frac{(\alpha+\beta+1)_{i+j-k}(\alpha+1)_{i+j}(2(i+j-k)+\alpha+\beta+1)}{(\alpha+1)_{i+j-k}(\alpha+\beta+1)_{2(i+j)-k+1}} \\ &\times \frac{(i+j)!}{i!j!k!} \frac{(\lambda+\delta+1)_{2i}(\mu+\gamma+1)_{2j}}{(\lambda+\delta+1)_i(\mu+\gamma+1)_j} \frac{(-2i-\lambda-\delta)_k}{(-2j-\mu-\gamma)_k} \\ &\times {}_3F_2\left(\begin{matrix} -k, -\lambda-\mu-\delta-\gamma-1-2(i+j)+k, -i \\ -2i-\lambda-\delta, -i-j \end{matrix} \middle| 1 \right) \\ &\times {}_3F_2\left(\begin{matrix} -k, -\lambda-\mu-\delta-\gamma-1-2(i+j)+k, -\lambda-i \\ -2i-\lambda-\delta, -\lambda-\mu-i-j \end{matrix} \middle| 1 \right) \end{aligned} \tag{15}$$

Next, we consider the particular case $\lambda = \delta = \mu = \gamma$ and we will prove that one of the above terminating ${}_3F_2$ can be summed using, for this purpose, computer algebra.

Put

$$S(k) = {}_3F_2\left(\begin{matrix} -k, -4\lambda - 1 - 2(i+j) + k, -i \\ -2i - 2\lambda, -i - j \end{matrix} \middle| 1\right),$$

with Zeilberger's algorithm (see e. g. [9], Chapter 7) via the Maple `sumrecursion` command, we obtain:

$$\begin{aligned} 0 &= (1+k)(2j - k + 2\lambda)(j + i + 4\lambda - k)(-1 + j - k + 2\lambda + i)S(k) \\ &-(1 - 2i - 2\lambda + k)(-k + i + j - 1)(-k + j + i + 2\lambda)(4\lambda + 2i + 2j - k)S(2+k) \\ &-2\lambda(-i + j)(j + 2\lambda + 1 + i)(2j - 1 - 2k + 4\lambda + 2i)S(1+k) \end{aligned} \tag{16}$$

With the `rechyper` Maple command, which is an implementation of Petkovsek's algorithm detecting all hypergeometric term solutions of a holonomic recurrence equation ([9], Chapter 9)[1] we obtain that 0 is the only hypergeometric solution of the recurrence relation (16), hence the first ${}_3F_2$ in the r.h.s. of relation (15) cannot be reduced to any hypergeometric term.

For the second ${}_3F_2$ of (15), consider

$$T(k) = {}_3F_2\left(\begin{matrix} -k, -4\lambda - 1 - 2(i+j) + k, -\lambda - i \\ -2i - 2\lambda, -2\lambda - i - j \end{matrix} \middle| 1\right). \tag{17}$$

Again by Zeilberger's algorithm we obtain

$$(2j - k + 2\lambda)(1+k)T(k) - (1 - 2i - 2\lambda + k)(4\lambda + 2i + 2j - k)T(2+k) = 0, \tag{18}$$

with initial conditions $T(0) = 1$ and $T(1) = 0$.

¿From this recurrence it follows with Petkovsek's algorithm that $T(k)$ is 0 for odd k which is also the only hypergeometric solution of relation (18).

For even values $k = 2m$, we get

$$(j + \lambda - m)(2m+1)T(m) + (2i - 1 - 2m + 2\lambda)(2\lambda + i + j - m)T(m+1) = 0, \tag{19}$$

which admits the hypergeometric solution

$$T(k) = T(2m) = \frac{(-\lambda - j)_m (2m)!}{4^m (1/2 - \lambda - i)_m (-i - j - 2\lambda)_m m!}. \tag{20}$$

Therefore, for integer m we obtain the following reduction formula

$${}_3F_2\left(\begin{matrix} -2m, -4\lambda - 1 - 2(i+j) + 2m, -\lambda - i \\ -2i - 2\lambda, -2\lambda - i - j \end{matrix} \middle| 1\right) = \frac{(-\lambda - j)_m (2m)!}{4^m (1/2 - \lambda - i)_m (-i - j - 2\lambda)_m m!}. \tag{21}$$

It follows that the linearization coefficients in

$$P_i^{(\lambda,\lambda)}(x) P_j^{(\lambda,\lambda)}(x) = \sum_{k=0}^{i+j} L_{ij}(i + j - k) P_{i+j-k}^{(2\lambda,2\lambda)}(x), \tag{22}$$

[1] This computation can in principle also be handled by Mark van Hoeij's faster algorithm [16] implemented in Maple's `LREtools[hypergeomsols]` command.

are given by 0 if $k=2m+1$, which can be also proven directly by the symmetry property of the ultraspherical polynomials $\{P_n^{(\lambda,\lambda)}\}_n$, and

$$\begin{aligned} L_{ij}(i+j-2m) &= \binom{i+j}{i}\frac{(4\lambda+1)_{i+j-2m}(2\lambda+1)_{i+j}(2(i+j-2m)+4\lambda+1)}{(2\lambda+1)_{i+j-2m}(4\lambda+1)_{2(i+j)-2m+1}} \\ &\times \frac{(2\lambda+1)_{2i}(2\lambda+1)_{2j}}{(2\lambda+1)_i(2\lambda+1)_j}\frac{(-2i-2\lambda)_{2m}}{(-2j-2\lambda)_{2m}} \\ &\times {}_3F_2\left(\left.\begin{matrix}-2m,-4\lambda-1-2(i+j)+2m,-i\\ -2i-2\lambda,-i-j\end{matrix}\right| 1\right) \\ &\times \frac{(-\lambda-j)_m}{4^m(1/2-\lambda-i)_m(-i-j-2\lambda)_m m!}. \end{aligned} \tag{23}$$

Next, we use the above results to obtain a reduction formula for a finite sum of a terminating hypergeometric function, using for this purpose the well-known connection and linearization formulae for Gegenbauer polynomials.

The Gegenbauer polynomials are Jacobi polynomials with $\alpha=\beta=\mu-\dfrac{1}{2}$ and another standardization:

$$C_n^\mu(x)=\frac{(2\mu)_n}{(\mu+\frac{1}{2})_n}P_n^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(x). \tag{24}$$

The connection and linearization formulae are respectively given by the formulae ([2, p. 39], compare [10])

$$C_n^\omega(x)=\sum_{k=0}^{[\frac{n}{2}]}\frac{(\mu+n-2k)(\omega-\mu)_k(\omega)_{n-k}}{k!(\mu)_{n+1-k}}C_{n-2k}^\mu(x), \tag{25}$$

and,

$$C_i^\mu(x)C_j^\mu(x)=\sum_{k=0}^{\min(i,j)}\frac{(i+j+\mu-2k)}{(i+j+\mu-k)}\frac{(\mu)_k(\mu)_{i-k}(\mu)_{j-k}(2\mu)_{i+j-k}}{k!(i-k)!(j-k)!(\mu)_{i+j-k}}\frac{(i+j-2k)!}{(2\mu)_{i+j-2k}}C_{i+j-2k}^\mu(x). \tag{26}$$

That leads, by virtue of (24), to the following connection and linearization formulae for the ultraspherical polynomials

$$\begin{aligned} P_{i+j-2k}^{(2\lambda,2\lambda)}(x) &= \frac{(2\lambda+1)_{i+j-2k}}{(4\lambda+1)_{i+j-2k}}\sum_{p=0}^{[\frac{i+j}{2}]-k}\frac{(\lambda+i+j-2k-2p+\frac{1}{2})(\lambda)_p(2\lambda+\frac{1}{2})_{i+j-2k-p}}{p!(\lambda+\frac{1}{2})_{i+j-2k-p+1}} \\ &\times\frac{(2\lambda+1)_{i+j-2k-2p}}{(\lambda+1)_{i+j-2k-2p}}P_{i+j-2k-2p}^{(\lambda,\lambda)}, \end{aligned} \tag{27}$$

and

$$\begin{aligned} P_i^{(\lambda,\lambda)}(x)P_j^{(\lambda,\lambda)}(x) &= \frac{(\lambda+1)_i(\lambda+1)_j}{(2\lambda+1)_i(2\lambda+1)_j}\sum_{k=0}^{\min(i,j)}\frac{(\lambda+i+j-2k+\frac{1}{2})(i+j-2k)!}{(\lambda+i+j-k+\frac{1}{2})k!(i-k)!(j-k)!} \\ &\times\frac{(2\lambda+1)_{i+j-k}(\lambda+\frac{1}{2})_k(\lambda+\frac{1}{2})_{i-k}(\lambda+\frac{1}{2})_{j-k}}{(\lambda+\frac{1}{2})_{i+j-k}(\lambda+1)_{i+j-2k}}P_{i+j-2k}^{(\lambda,\lambda)}(x). \end{aligned} \tag{28}$$

Substituting (27) in (22), using (23) and comparing with (28), we get the following reduction formula, for $0 \le k \le \min(i,j)$,

$$\sum_{p=0}^{k} \frac{(\lambda)_{k-p}(2\lambda+\frac{1}{2})_{i+j-k-p}}{(4\lambda+1)_{2i+2j-2p+1}(\frac{1}{2}-\lambda-j)_p} \frac{[2(i+j-2p)+4\lambda+1)]}{p!(k-p)!2^{2p}(\lambda+\frac{1}{2})_{i+j-p-k+1}} \frac{\binom{\lambda+i}{p}}{\binom{2\lambda+i+j}{p}}$$
$$\times {}_3F_2\left(\begin{matrix}-2p, -4\lambda-1-2(i+j)+2p, -i \\ -2i-2\lambda, -i-j\end{matrix}\,\middle|\, 1\right) = \tag{29}$$
$$\frac{\binom{i}{k}\binom{j}{k}}{\binom{i+j}{2k}} \frac{k!}{(2k)!} \frac{(2\lambda+1+i+j-2k)_k(\lambda+1)_i(\lambda+1)_j}{(2\lambda+1)_{i+j}(2\lambda+1)_{2i}(2\lambda+1)_{2j}} \frac{(\lambda+\frac{1}{2})_k(\lambda+\frac{1}{2})_{i-k}(\lambda+\frac{1}{2})_{j-k}}{(\lambda+\frac{1}{2})_{i+j+1-k}}.$$

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