Algorithmic approach for Formal Fourier Series

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Abstract. The study of trigonometric series has started at the beginning of the nineteenth century. Joseph Fourier made the important observation that almost every function of a closed interval can be decomposed into the sum of sine and cosine functions. This technique to develop a function into a trigonometric series was published for the first time in 1822 by Joseph Fourier. The resulting series is nowadays called Fourier series. Since Fourier's time, many different approaches to understand the concept of Fourier series have been discovered, each of which emphasizes different aspects of the topic. Some of the more powerful and elegant approaches are based on mathematical ideas and tools that were not available at the time Fourier completed his original work. Although the original motivation was to solve the heat equation for a metal plate, it later became obvious that the same technique could be applied to a wide variety of mathematical and physical problems and has many applications in electrical engineering, vibration analysis, acoustics, optics, signal treatment, image processing, etc ....

Despite the importance of Fourier series, the method used until now to compute them via computer algebra systems (CAS) is essentially based on the same principle as in Fourier's time, i.e. by the evaluation of certain integrals. Unfortunately this technique is not completely successful for many functions. Although numeric values of the Fourier coefficients might be available, symbolic values are often not accessible. Modern CAS like *Maple* or *Mathematica* can compute such integrals in many cases for a given $n \in \mathbb{Z}$. However if one is interested in the Fourier coefficients for all $n \in \mathbb{Z}$, then n is considered as a given symbolic variable and such integrals can be computed only in few cases.

In this paper we introduce an algorithmic approach to compute those Fourier coefficients, involving differential equations of a particular form, and recurrence equations. This approach extrapolates the computation of the Fourier series for functions for which the computation of Fourier coefficients via the definition is out of reach for current CAS.

A holonomic recurrence equation for a_n, i. e. a recurrence equation which is linear, homogeneous and has polynomial coefficients, can be written in operator notation as $L(a_n) = 0$. The operator L can be interpreted as a non-commutative polynomial via the commutator rule $Nn - nN = N$, N denoting the shift operator $Na_n = a_{n+1}$. In the last section we show how our algorithm can be used to factorize such recurrence operators in certain cases.

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1. Introduction

1.1. Definitions

Definition 1.1. The real and complex Fourier series of an integrable function $f : [a, b] \to \mathbb{R}$ are the expressions

$$\mathcal{F}(f)(t) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,\omega\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,\omega\,t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t} \,, \tag{1.1}$$

where $\omega = \frac{2\pi}{b-a}$ is the circular frequency and the corresponding real Fourier coefficients are given by

$$a_n = \frac{2}{b-a} \int_a^b f(t)\cos(n\omega t)dt \in \mathbb{R} \quad (n \in \mathbb{N} := \{0, 1, 2, 3, \ldots\}) \,, \tag{1.2}$$

$$b_n = \frac{2}{b-a} \int_a^b f(t)\sin(n\omega t)dt \in \mathbb{R} \quad (n \in \mathbb{N}_{\geqq 1}) \,, \tag{1.3}$$

whereas the complex Fourier coefficients are defined as

$$c_n = \frac{1}{b-a} \int_a^b f(t)e^{-in\omega t}dt \in \mathbb{C} \quad (n \in \mathbb{Z}) \,. \tag{1.4}$$

We remark that a finite sum of the form (1.1) is called a *Fourier polynomial*. Of course, by definition we have the relations $c_0 = \frac{a_0}{2}$ as well as

$$c_n = \frac{1}{2}\,(a_n - i\,b_n) \quad \text{and} \quad c_{-n} = \frac{1}{2}\,(a_n + i\,b_n) \quad (n \in \mathbb{N})$$

which—solving for a_n and b_n—gives

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad (n \in \mathbb{N}_{\geqq 1}) \,. \tag{1.5}$$

All above formulas are also valid if $f : [a, b] \to \mathbb{C}$ is a complex function. However if the input function is real, then we get moreover $\bar{c}_n = \frac{1}{2}(a_n + ib_n) = c_{-n}$, which—solving for a_n and b_n—gives

$$a_n = c_n + \bar{c}_n \quad \text{and} \quad b_n = i(c_n - \bar{c}_n) \quad (n \in \mathbb{N}_{\geqq 1}) \,.$$

Under the additional assumption that f is continuous in (a, b), it turns out that pointwise $\mathcal{F}(f)(t) = f(t)$ for all $t \in (a, b)$. As general references for elementary properties of Fourier series see e.g. [2], [3], [20] and [21].

Note that we will not study any convergence issues and therefore we mention that the series (1.1) is a formal series corresponding to $f(t)$. The problem of convergence of Fourier series has been investigated by many authors and one of the first was Dirichlet in [6]. Since we will need to compute successive derivatives in some of our algorithms, all the functions involved in this paper are assumed to be defined and continuous in the interval $I = [a, b]$ and at least N times continuously differentiable on I for suitable $N \geqq 0$. Furthermore the considered function f can be periodically continued to $\mathbb{R}$ with period $T = b - a$. Let's denote the resulting periodic function by $F : \mathbb{R} \to \mathbb{R}$. By definition, the function F is continuous in $\mathbb{R}$ besides the points $a + k\,T$ $(k \in \mathbb{Z})$ which are (possible) discontinuities of step size $\Delta := f(a) - f(b)$.

1.2. Summary of the Main Results

1.2.1. Computation of Fourier Series. The symbolic computation of the Fourier coefficients of a function f using formulas (1.2)–(1.4) is in some cases very complicated, because of the integer parameter n in those formulas. Consider for example the function given as

$$f(t) = \cos(5t)\ln(2 + \cos(5t)) \,.$$

Using the algorithms presented in this paper we get that its Fourier series (in the interval $I = [0, \frac{2\pi}{5}]$) is given by

$$\begin{aligned}\mathcal{F}(f)(t) &= 2(2-\sqrt{3}) + (2\sqrt{3} - \frac{7}{2} + \ln(2+\sqrt{3}) - \ln(2))\cos(5t) \\ &\quad + \sum_{n=2}^{\infty} \frac{2(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)} \cos(5nt) \ .\end{aligned}$$

Note that this algorithmic approach is applicable to a rich family of functions denoted trigonometric holonomic functions, which is defined and characterized in the second chapter. Our algorithm solves the problem of determining the complex Fourier coefficients for those functions in the following way:

1. In the first step we determine a homogeneous linear differential equation DE for $f(t)$ with Fourier polynomial coefficients.
2. In the second step this differential equation is converted towards a homogeneous linear recurrence equation RE with polynomial coefficients—a so-called holonomic recurrence equation—for the complex Fourier coefficients c_n.
3. In the last step the recurrence equation RE is solved using a suitable number of initial values if possible. This is at least possible using known algorithms if the Fourier coefficients constitute a linear combination of hypergeometric terms. A sequence a_n is a hypergeometric term if $\frac{a_{n+1}}{a_n}$ is a rational function w. r. t. n.

Note that this procedure imitates the FPS algorithm that was given by the first author [10] which determines a formal power series development of a (holonomic) function. Note further that if the third step of the above algorithm does not succeed, the result of our procedure is a holonomic recurrence equation for c_n with which the Fourier coefficients can be computed most efficiently.

We would also like to mention that most of the expansions in Fourier series given in the mathematical dictionaries [2] and [20] are algorithmically found by the algorithmic method described in this paper. Furthermore we note that the Fourier coefficients of the previous function f cannot be successfully computed by current computer algebra systems using formulas (1.2)–(1.4).

When browsing the internet for algorithmic methods and recurrence relations for Fourier coefficients, one finds the paper [13] about *Recurrence relations for the coefficients of the Fourier series expansions with respect to q-classical orthogonal polynomials* and the PhD thesis [1] about *Fast semi-numerical algorithms for Chebyshev expansions*. Both papers deal with certain generalizations of Fourier series, namely with questions concerning other bases than the traditional trigonometric one. On the other hand, we could not find algorithmic results on "traditional" Fourier series. Therefore to the best of our knowledge the algorithms in our paper are new and were introduced 2010 in the PhD thesis [15].

1.2.2. Factorization of Holonomic Recurrence Operators. The search for hypergeometric term solutions of *holonomic recurrence equations* is related to the search of first order right factors of *holonomic recurrence operators*, more generally to the factorization of those operators. Marko Petkovšek [16], Mark van Hoeij [8], Peter Horn [9] and others investigated this issue and made important contributions to the factorization of such operators. In the last section we give another approach to factorize such operators, by using Fourier coefficients.

1.3. Outline of the Paper

In the second section we derive a connection between the complex Fourier coefficients of a function f and those of its first derivative (Theorem 2.1). This connection implies a more general statement, this time between the complex Fourier coefficients of f and those of its successive derivatives (Theorem 2.2), from which an explicit formula for the complex Fourier coefficients of polynomials involving successive derivatives is deduced. This theorem will also be used in the third section to convert the differential equations for f obtained in the second section into recurrence equations for c_n (Theorem 4.5).

In the third section we introduce the set of trigonometric holonomic functions and we give some of its characteristics and properties. This family contains not only many elementary functions, but also many functions whose Fourier coefficients cannot be successfully computed in the classical way. We give some example types of trigonometric holonomic functions. We present an algorithm for the computation of the trigonometric holonomic differential equations that those functions satisfy.

The fourth section is devoted to the algorithmic computation of the Fourier coefficients.

The fifth section deals with the factorization of holonomic recurrence operators. We present an algorithm to convert a holonomic recurrence equation into a differential equation with side conditions. This algorithm can be used to compute right factors of given holonomic recurrence operators.

2. Some Particular Cases

In this section we derive an identity for the Fourier coefficients of a differentiable function $f(t)$ in terms of the Fourier coefficients of its derivative $f'(t)$. This yields an algorithm to compute the Fourier coefficients of $f(t)$ whenever the Fourier coefficients of $f'(t)$ are known, and vice versa. Furthermore this generates an iterative scheme for N times differentiable functions complementing the direct computation of Fourier coefficients via the defining integrals which can also be treated automatically in certain cases, see [22] using *Maple* [14] and [5] using *Mathematica* [23]. As direct consequence of that scheme we deduce an explicit formula for the computation of the complex Fourier coefficients of polynomials. In the third section we will use that scheme to present an algorithm for the computation of the complex Fourier coefficients of the set of trigonometric holonomic functions which will be introduced in the next section.

2.1. Notation

Let $f:[a,b]\to\mathbb{R}$ be a continuous function in the interval $I=[a,b]$ which is continuously differentiable in (a,b). Then f' is continuous and has a Fourier series itself, for which we use the following notations

$$\mathcal{F}(f')(t)=\frac{a_0'}{2}+\sum_{n=1}^{\infty}a_n'\,\cos(n\,\omega\,t)+\sum_{n=1}^{\infty}b_n'\,\sin(n\,\omega\,t)=\sum_{n=-\infty}^{\infty}c_n'\,e^{in\omega t}\;,$$

i.e., the Fourier coefficients of the derivative function are denoted by dashes. If $f\in C^N[a,b]$, then we can continue taking derivatives, and for the kth derivative ($k\leqq N$) we use the notation

$$\mathcal{F}(f^{(k)})(t)=\frac{a_0^{(k)}}{2}+\sum_{n=1}^{\infty}a_n^{(k)}\,\cos(n\,\omega\,t)+\sum_{n=1}^{\infty}b_n^{(k)}\,\sin(n\,\omega\,t)=\sum_{n=-\infty}^{\infty}c_n^{(k)}\,e^{in\omega t}\;.$$

2.2. An Identity for Fourier Coefficients

Let us recall that the complex Fourier coefficients c_n of a continuous function f in an interval $[a,b]$ are given by the equation

$$c_n=\frac{1}{b-a}\int_a^b f(t)e^{-in\frac{2\pi}{b-a}t}dt=\frac{1}{T}\int_a^{a+T}f(t)e^{-in\omega t}dt\;,$$

where $b-a=T=\frac{2\pi}{\omega}$. Integrating by parts, using $u(t)=f(t), v'(t)=e^{-in\omega t}$, hence $u'(t)=f'(t)$ and $v(t)=\frac{1}{-in\omega}e^{-in\omega t}$, we get

$$\begin{aligned}
c_n &= \left[\frac{f(t)e^{-in\omega t}}{-in\omega T}\right]_a^{a+T}+\frac{1}{T}\int_a^{a+T}\frac{f'(t)e^{-in\omega t}}{in\omega}dt\\
&= \frac{f(a+T)e^{-in\omega(a+T)}-f(a)e^{-in\omega a}}{-in\omega T}+\frac{1}{in\omega}\left(\frac{1}{T}\int_a^{a+T}f'(t)e^{-in\omega t}dt\right)\\
&= \frac{f(a+T)-f(a)}{-2\pi in}e^{-in\omega a}-\frac{i}{n\omega}c_n'\;.
\end{aligned}$$

In particular, we have derived the identity

$$c_n + \frac{i}{n\omega} c_n' = \frac{i}{2\pi n}(f(b) - f(a))e^{-in\omega a} \quad (n \neq 0) . \tag{2.1}$$

As we shall discuss later, this easy-to-derive relation has interesting applications, and can be used to compute the complex Fourier coefficients recursively under certain conditions, and therefore a_n and b_n via (1.5). We summarize the above identity in the following

Theorem 2.1 (Fourier coefficients and derivatives). *Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and continuously differentiable in (a, b). Then the real and complex Fourier coefficients of $f(t)$ and of $f'(t)$ satisfy the identities*

$$c_n + \frac{i}{n\omega} c_n' = \frac{i}{2\pi n}(f(b) - f(a))e^{-in\omega a} \quad (n \in \mathbb{Z}, n \neq 0)$$

and

$$\begin{aligned} a_n + \frac{1}{n\omega} b_n' &= \frac{1}{\pi n}(f(b) - f(a)) \sin(n\omega a) \qquad \text{and} \\ -b_n + \frac{1}{n\omega} a_n' &= \frac{1}{\pi n}(f(b) - f(a)) \cos(n\omega a) \quad (n \in \mathbb{N}_{\geqq 1}) , \end{aligned}$$

respectively.

2.3. Iterated Derivatives

In this section, we assume that $f \in C^N[a, b]$ for some $N \in \mathbb{N}_{\geqq 1}$. Then by Theorem 2.1 we can write down a series of identities for successive derivatives of f. Using (2.1), we get for the complex Fourier coefficients and for $n \in \mathbb{Z}, n \neq 0$

$$\begin{aligned} c_n + \frac{i}{n\omega} c_n' &= \frac{i}{2\pi n}(f(b) - f(a))e^{-in\omega a} \\ c_n' + \frac{i}{n\omega} c_n'' &= \frac{i}{2\pi n}(f'(b) - f'(a))e^{-in\omega a} \\ c_n'' + \frac{i}{n\omega} c_n^{(3)} &= \frac{i}{2\pi n}(f''(b) - f''(a))e^{-in\omega a} \\ &\vdots \\ c_n^{(N-1)} + \frac{i}{n\omega} c_n^{(N)} &= \frac{i}{2\pi n}(f^{(N-1)}(b) - f^{(N-1)}(a))e^{-in\omega a} . \end{aligned}$$

In order to manipulate the previous relations easier, let us rewrite them in the following way:

$$\begin{aligned} c_n + \tau\, c_n' &= \alpha_0 \\ c_n' + \tau\, c_n'' &= \alpha_1 \\ c_n'' + \tau\, c_n^{(3)} &= \alpha_2 \\ &\vdots \\ c_n^{(N-1)} + \tau\, c_n^{(N)} &= \alpha_{N-1} \end{aligned}$$

with the abbreviations $\tau = \frac{i}{n\omega}$ and $\alpha_k = \frac{i}{2\pi n}(f^{(k)}(b) - f^{(k)}(a))e^{-in\omega a}$.

Multiplying the kth equation by $(-1)^k \tau^k$ and summing up obviously yields a telescoping sum with the result

$$c_n = \alpha_0 - \tau\alpha_1 + \tau^2\alpha_2 - \tau^3\alpha_3 + \cdots + (-1)^{N-1}\tau^{N-1}\alpha_{N-1} + (-1)^N \tau^N c_n^{(N)}$$

which finally leads to the following

Theorem 2.2 (Fourier coefficients and iterated derivatives). *For $f \in C^N[a,b]$ and $\omega = \frac{2\pi}{b-a}$, the following identities for the complex Fourier coefficients are valid ($n \in \mathbb{N}$):*

$$\begin{aligned} c_n - \left(\frac{-i}{n\omega}\right)^N c_n^{(N)} &= \sum_{j=0}^{N-1} (-1)^j (b-a)^j \left(\frac{i}{2n\pi}\right)^{j+1} \left(f^{(j)}(b) - f^{(j)}(a)\right) e^{-in\omega a} \,, \\ c_0^{(N)} &= \frac{1}{b-a}\left(f^{(N-1)}(b) - f^{(N-1)}(a)\right) . \end{aligned} \tag{2.2}$$

Since a polynomial $f(t)$ of degree N satisfies $f^{(N)}(t) = \text{constant}$, and therefore $c_n^{(N)} = 0$ for $n \in \mathbb{N}_{\geqq 1}$, Theorem 2.2 implies the following

Corollary 2.3 (Fourier coefficients of polynomials). *Let $f : [0,T] \to \mathbb{R}$ resp. $[-\frac{T}{2}, \frac{T}{2}] \to \mathbb{R}$ be a polynomial of degree N. Then the complex Fourier coefficients of f can be written in the form ($n \in \mathbb{Z}, n \neq 0$)*

$$c_n = \sum_{k=0}^{N-1} (-1)^k \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n}\left(f^{(k)}(T) - f^{(k)}(0)\right) .$$

resp.

$$c_n = \sum_{k=0}^{N-1} (-1)^{k+n} \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n}\left(f^{(k)}\left(\frac{T}{2}\right) - f^{(k)}\left(-\frac{T}{2}\right)\right) . \tag{2.3}$$

Note that the computation of the Fourier coefficients of t^m, e.g., using (2.3) is much more efficient than the computation using the definition.

Remark 2.4. For many functions, the computation of their Fourier coefficients in an interval $[a,b]$ via definitions (1.2)–(1.4) is not successful using current CAS. However, if one can compute the Fourier coefficients of one of their successive derivatives, then Theorem 2.2 outlines the computation of the Fourier coefficients of the foresaid function. We summarize this process in Algorithm 1. In this algorithm we apply the above method up to derivative order N.

Algorithm 1: Computation of the Fourier coefficients c_n of a function from those of one of its successive derivatives $c_n^{(m)}$.

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input : A function $f \in C^N[a,b]$ such that the computation of its Fourier coefficients in an interval $[a,b]$ is not successful, but one can compute those of one of its successive derivatives.

output : The complex Fourier coefficients of f in the interval $[a,b]$ or the message "the complex Fourier coefficients of f cannot be computed using this algorithm".

begin
  $m \leftarrow 1$.
  while $m \leqq N$ **do**
    Compute the complex Fourier coefficients of $f^{(m)}$.
    if *the computation is successful* **then**
      use relation (2.2) to achieve the computation of the Fourier coefficients of f. The coefficient c_0 must be computed independently.
      return *the complex Fourier coefficients of f in the interval* $[a,b]$.
    end
    $m \leftarrow m+1$.
  end
  The complex Fourier coefficients of f cannot be computed using this algorithm.
end

Example 2.5. Consider the function defined by

$$f(t) = \arctan(2 + \cos(t)e^{it}) \ .$$

The complex Fourier coefficients of f cannot be successfully computed using formulas (1.2)–(1.4).[1] However we remark that the function under consideration belongs to the set of trigonometric holonomic functions which will be defined in the next section. Moreover, its first derivative f' is a rational trigonometric function. Using Algorithm 3 which will be presented in the fourth section, we can compute the Fourier coefficients of f' and finally using Algorithm 1, we deduce those of f and get

$$c_n = \begin{cases} \frac{-i}{n}\left(\frac{1+(-1)^n}{2}\right)\left((\frac{1}{29}\sqrt{-145+58i})^n - (\frac{1}{29}\sqrt{-145-58i})^n\right) & n \geqq 1 \\ \arctan\left(\frac{5}{2}\right) & \text{if} \quad n = 0 \\ 0 & \text{otherwise} \ . \end{cases}$$

3. Trigonometric Holonomic Functions

In this section we introduce the set of trigonometric holonomic functions TH for which we will present an algorithm to compute their Fourier coefficients in the next section. We will present some particular subsets of TH. Then we will give some algebraic properties of TH, focussing on the aspects concerning the aims of this paper.

3.1. Notations and Definitions

Let $\mathbb{K}$ denote a field of characteristic zero, and $\mathbb{K}^\star = \mathbb{K} \setminus \{0\}$. For simplification purposes we write $\mathbb{K}[\cos(t), \sin(t)]$ for the set of trigonometric polynomials instead of $\mathbb{K}[x,y]/\langle x^2+y^2-1\rangle$ and analogously in similar cases. We will also understand $\cos(2t)$, e. g., by the relation $\cos(2t) = \cos(t)^2 - \sin(t)^2$ as a member of $\mathbb{Q}[\cos(t), \sin(t)]$. It is well-known that every trigonometric polynomial $p = \sum_{i=0}^{I}\sum_{j=0}^{J} a_{ij}\cos^i(t)\sin^j(t) \in \mathbb{K}[\cos(t), \sin(t)]$ can be written as a Fourier polynomial in the form $\sum_{k=0}^{K}(a_k\cos(kt) + b_k\sin(kt))$ and vice versa via the following addition theorems

$$\cos(t \pm u) = \cos(t)\cos(u) \mp \sin(t)\sin(u)$$
$$\sin(t \pm u) = \sin(t)\cos(u) \pm \cos(t)\sin(u)$$

and for recursive use

$$\cos(kt) = \cos((k-1)t)\cos(t) - \sin((k-1)t)\sin(t)$$

and

$$\sin(kt) = \sin((k-1)t)\cos(t) + \cos((k-1)t)\sin(t) \ ,$$

and for the backward process the substitution rules

$$\cos(t)\cos(u) = \frac{1}{2}\cos(t-u) + \frac{1}{2}\cos(t+u)$$
$$\sin(t)\cos(u) = \frac{1}{2}\sin(t-u) + \frac{1}{2}\sin(t+u)$$
$$\sin(t)\sin(u) = \frac{1}{2}\cos(t-u) - \frac{1}{2}\cos(t+u)$$

and for recursive use

$$\cos^k(t) = \left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right)\cos^{k-2}(t) \quad (k \geqq 2)$$

and

$$\sin^k(t) = \left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right)\sin^{k-2}(t) \quad (k \geqq 2) \ .$$

[1] *Maple* is not even able to compute c_0 when using f as defined. The value c_0 was therefore computed using a suitable rewriting.

For more details about this conversion see e. g. ([11], Section 9).

We recall also that the sum $f = f_1 + f_2 + \cdots + f_n$ of a finite sequence of periodic functions $\{f_1, f_2, \ldots, f_n\}$ is periodic if and only if their periods T_1, T_2, ..., T_n, respectively are commensurable.

The commensurability of $T_1, T_2, \ldots, T_n$ means that there exist n integers $N_1, \ldots, N_n$ such that

$$N_1 T_1 = N_2 T_2 = \cdots = N_n T_N \ . \tag{3.1}$$

It follows from (3.1) that a period of f is $T = N_1 T_1 = N_2 T_2 = \cdots = N_n T_N$.

Definition 3.1 (Trigonometric holonomic functions). Let $\omega \neq 0$ be a given real number. By $\mathrm{TH}(\omega)$ we denote the set of ω-trigonometric holonomic functions, i. e. the set of functions satisfying a differential equation of the form

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t)\right) f^{(p)}(t) = 0 \tag{3.2}$$

for appropriate integers $P \geqq 1$ and $L \geqq 0$, where α_{pl} and β_{pl} are constants.
A function f is said to be trigonometric holonomic if there exist $\omega \in \mathbb{R}^\star$ such that $f \in \mathrm{TH}(\omega)$. By TH we denote the set of those functions, i.e. $\bigcup \mathrm{TH}(\omega) = \mathrm{TH}$.
Differential equations of the form (3.2) are called trigonometric holonomic differential equations.

Definition 3.2 (Type). A function $f \in \mathrm{TH}(\omega)$ is said to be of type $L(\omega)$ if L is the smallest non-negative integer for which it satisfies a differential equation of the form (3.2).

Note that the type is important since it gives a priory information about the resulting holonomic recurrence equation for the corresponding Fourier coefficients as we shall see in Theorem 4.5.

Definition 3.3 (Degree). A trigonometric holonomic differential equation is said to be of degree P if P is the smallest positive integer appearing in its representation (3.2). Let f be a function of $\mathrm{TH}(\omega)$. The smallest integer P for which f satisfies a differential equation of the form (3.2) is called the degree of f in $\mathrm{TH}(\omega)$.

Example 3.4. The following differential equation

$$DE1 : (\sin(2\sqrt{7}t) + 2) f(t) + 5f'(t) + (3 + \cos(4\sqrt{7}t)) f''(t) = 0$$

is a trigonometric holonomic differential equation, since its non-constant coefficients $\sin(2\sqrt{7}t) + 2$ and $3 + \cos(4\sqrt{7}t)$ are periodic with commensurable periods. It is of degree 2, and of type 4 w. r. t. $\omega = \sqrt{7}$, but of type 2 w. r. t. $\omega = 2\sqrt{7}$

But

$$DE2 : (\sin(8t) + \cos(16t)) f(t) + (\cos(\sqrt{3}t) + 3) f'(t) + 5f'''(t) = 0$$

is not a trigonometric holonomic differential equation, since the periods of its non-constant coefficients $\sin(8t) + \cos(16t)$ and $\cos(\sqrt{3}t) + 3$ are not commensurable.

Algorithm 2—given on p. 10—which is based on linear algebra, is decisive since it shows how a trigonometric holonomic differential equation of a trigonometric holonomic function can be computed. This algorithm is an adaption of an algorithm given in ([10], see also [11]) for the computation of a holonomic differential equation. After substituting all derivatives, the left hand side of (3.2) is zero if and only if the coefficients of the linearly independent summands have all zero coefficients. This leads to a system of linear equations which can be solved for the unknowns $\alpha_{pl}, \beta_{pl} \in \mathbb{C}$ $(l = 0, \ldots, L, p = 0, \ldots, P)$. If P and L are chosen large enough such that the number of variables is $\geqq$ than the number of equations, the corresponding homogeneous linear system has a solution.

Of course the algorithm can be executed in such a way that either the type or the degree of the resulting differential equation is minimized. For the most efficient computation of Fourier coefficients it will be best to minimize the type, not the degree, see Theorem 4.5. Note that if the input function is not trigonometric holonomic, then Algorithm 2 does not terminate.

Example 3.5. We would like to give an example for the computation of a trigonometric differential equation. Let

$$f(t) = \frac{1}{\cos t + 2} \,.$$

Then

$$f'(t) = \frac{\sin t}{(\cos t + 2)^2}$$

so that with $P = L = 1$ we get

$$\begin{aligned}
0 &= \sum_{p=0}^{1}\sum_{l=0}^{1} (\alpha_{pl}\cos(lt) + \beta_{pl}\sin(lt))\, f^{(p)}(t) \\
&= (\alpha_{00} + \alpha_{01}\cos t + \beta_{01}\sin t)\, f(t) + (\alpha_{10} + \alpha_{11}\cos t + \beta_{11}\sin t)\, f'(t) \\
&= (\alpha_{00}+\alpha_{01}\cos t+\beta_{01}\sin t)\,\frac{1}{\cos t+2} + (\alpha_{10}+\alpha_{11}\cos t+\beta_{11}\sin t)\,\frac{\sin t}{(\cos t+2)^2} \\
&= \frac{1}{(\cos t + 2)^2}\Big(\alpha_{00}(\cos t+2) + \alpha_{01}(\cos^2 t+2\cos t) + \beta_{01}(\sin t\cos t+2\sin t) + \\
&\qquad \alpha_{10}\sin t + \alpha_{11}\sin t\cos t + \beta_{11}\sin^2 t\Big) \\
&= \frac{1}{(\cos t + 2)^2}\Bigg(\left(2\alpha_{00} + \frac{1}{2}\alpha_{01} + \frac{1}{2}\beta_{11}\right) + (\alpha_{00} + 2\alpha_{01})\cos t + \\
&\qquad \left(\frac{1}{2}\alpha_{01} - \frac{1}{2}\beta_{11}\right)\cos(2t) + (2\beta_{01} + \alpha_{10})\sin t + \left(\frac{1}{2}\beta_{01} + \frac{1}{2}\alpha_{11}\right)\sin(2t)\Bigg) \,.
\end{aligned}$$

Note that in the last step conversion towards a Fourier polynomial took place. This is essential to guarantee linear independence (stemming from the fact that the representation of a trigonometric polynomial as a Fourier polynomial is a canonical form, see e. g. [11]). Solving the linear system by equating the coefficients of this expression yields the differential equation

$$(2 + \cos t)f'(t) - \sin t\, f(t) = 0$$

which is unique up to a constant factor.

Example 3.6. We would like to give some more examples of trigonometric differential equations. For $f(t) = \frac{\sin t}{\cos t+2}$, we get the differential equation

$$(\cos(t) + 2)\, f''(t) - 2\sin(t)\, f'(t) + 2\, f(t) = 0$$

of type 1, as well as the differential equation

$$(4\sin(t) + \sin(2\,t))\, f'(t) - 2\,(2\cos(t) + 1)f(t) = 0$$

of degree 1.

As a more complicated example, we consider the function

$$f(t) = \arctan(2 + \cos(t)e^{it}) = \arctan(2 + \cos(t)(\cos(t) + i\sin(t)))$$

from Example 2.5, again. It turns out that f satisfies the trigonometric holonomic differential equation

$$(14\,i\sin(2\,t) - 15\cos(2\,t) - 5)\, f''(t) + (28\,i\cos(2\,t) + 30\sin(2\,t))\, f'(t) = 0\,.$$

The type of $f(t) = \sum\limits_{k=1}^{5}\sin(kt)$ is 0 with the corresponding differential equation

$$f^{(10)}(t) + 55\, f^{(8)}(t) + 1023\, f^{(6)}(t) + 7645\, f^{(4)}(t) + 21076\, f''(t) + 14400\, f(t) = 0$$

and its degree is 3 with

$$\sin(t)f'''(t) + (\cos(t) + 2)f''(t) + 30\sin(t)f'(t) + 30f(t) = 0\,.$$

3.2. Functions Satisfying Differential Equations with Coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$

For a given $\omega \in \mathbb{R}^\star$ one can convert a trigonometric holonomic differential equation (as we have defined until now) into a differential equation with coefficients in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ and conversely. In this section we will emphasize on differential equations whose coefficients are *linear* polynomials of either $e^{-i\omega t}$ or $e^{i\omega t}$ exclusively. As we will see in the next section, the particular importance of that type of differential equations is that they lead to first order holonomic recurrence equations for the Fourier coefficients of the considered function in an interval of length T, where ω is chosen according to $\omega = \frac{2\pi}{T}$. We remark that such recurrence equations (of first order) do not result once the coefficients of the considered differential equation are not of the foresaid form.

Theorem 3.7. *For a given $\omega \in \mathbb{R}^\star$, functions satisfying a differential equation of the form*

$$\sum_{p=0}^{P}\sum_{l=0}^{L}\left(\gamma_{pl}e^{-il\omega t} + \delta_{pl}e^{il\omega t}\right) f^{(p)}(t) = 0 \tag{3.3}$$

for appropriate integers $P \geqq 1, L \geqq 0$, where γ_{pl} and $\delta_{pl} \in \mathbb{C}$ are ω-trigonometric holonomic functions.

Proof. Using Euler's identity the proof is obvious. □

Algorithm 2: Determination of a trigonometric holonomic differential equation for a trigonometric holonomic function

---

input : A real number $\omega = \frac{2\pi}{b-a}$, a function $f \in \mathrm{TH}(\omega)$ such that $f \in C^{(N)}[a,b]$ for N large enough, a value L_{max} as maximal type, and a value P_{max} as maximal degree.

output : A differential equation satisfied by f in the form either (3.2) or (3.3).

begin
  for $L = 0$ **to** L_{max} **do**
    for $P = 1$ **to** P_{max} **do**
      if *f contains expressions of the form $e^{\pm i\omega t}$ where $\omega = \frac{2\pi}{b-a}$* **then**
        search for coefficients γ_{pl} and δ_{pl} such that the equation
        $\sum\limits_{p=0}^{P}\sum\limits_{l=0}^{L}\left(\gamma_{pl}e^{-il\omega t} + \delta_{pl}e^{il\omega t}\right) f^{(p)}(t) = 0$ is valid.
      else
        search for coefficients α_{pl} and β_{pl} such that the equation
        $\sum\limits_{p=0}^{P}\sum\limits_{l=0}^{L}\left(\alpha_{pl}\cos(l\omega t) + \beta_{pl}\sin(l\omega t)\right) f^{(p)}(t) = 0$ is valid
      end
      if *the search is successful* **then**
        return *the differential equation of f in the form either (3.2) or (3.3).*
      end
    end
  end
end

Definition 3.8. The set of functions satisfying trigonometric holonomic differential equations leading to first order recurrence equations for their complex Fourier coefficients are called *simple trigonometric holonomic functions* (sTH). The coefficients of such trigonometric holonomic differential equations are linear polynomials of either $e^{i\omega t}$ or $e^{-i\omega t}$, but not of both.

Example 3.9. Set $\omega = 1$ and consider the function defined by

$$f(t) = e^{ie^{it}} .$$

f satisfies the differential equation

$$DE : e^{it}f(t) + f'(t) = 0 ,$$

from which it follows that f is of degree 1 in TH(1). Moreover, $f \in$ sTH, see Example 4.8.

3.3. Example Types of Trigonometric Holonomic Functions

The set of trigonometric holonomic functions is rich and contains, in particular, the following types.

3.3.1. Polynomials. The following theorem is trivial:

Theorem 3.10. *For all $\omega \in \mathbb{R}^\star$, the set $\mathbb{K}[t]$ of polynomials is a subset of* $\mathrm{TH}(\omega)$ *and each polynomial f of degree N is of type 0 and of degree $N+1$ in* $\mathrm{TH}(\omega)$*.*

3.3.2. Exp-like Functions. We call a function *exp-like* if it satisfies a differential equation with constant coefficients, see [10], i.e. a differential equation of the form

$$a_n f^{(n)}(t) + a_{n-1} f^{(n-1)}(t) + \cdots + a_2 f''(t) + a_1 f'(t) + a_0 f(t) = 0, \tag{3.4}$$
$$a_n \neq 0, \quad a_k \in \mathbb{K} \quad (k = 0, \ldots, n)\ .$$

According to the solution theory of ordinary differential equations exp-like functions can be classified and are products of polynomials, exponentials, sines and cosines, and linear combinations of such functions. Of course we have

Theorem 3.11. *Every exp-like function f lies in* $\mathrm{TH}(\omega)$ *for every $\omega \in \mathbb{K}^*$ and has type 0.*

3.3.3. Rational Trigonometric Functions. We get

Theorem 3.12. *For all $\omega \in \mathbb{R}^\star$ the set of rational trigonometric functions* $\mathbb{K}(\cos(\omega t), \sin(\omega t))$ *is a subset of* $\mathrm{TH}(\omega)$ *and each function of* $\mathbb{K}(\cos(\omega t), \sin(\omega t))$ *is of degree at most* 1 *in* $\mathrm{TH}(\omega)$*.*

Proof. Let $f \in \mathrm{TH}(\omega)$. Hence $f(t) = \frac{p(t)}{q(t)}$ with $p(t), q(t) \in \mathbb{K}[\cos(t), \sin(t)]$. Differentiation yields

$$\begin{aligned} p(t) &= q(t)\, f(t) \\ p'(t) &= q'(t)\, f(t) + q(t)\, f'(t)\ . \end{aligned}$$

Multiplying the first line by $p'(t)$ and the second line by $p(t)$ and subtracting yields a first order trigonometric holonomic differential equation. □

Example 3.13. Note that every rational trigonometric function $R(\cos(\omega t), \sin(\omega t))$ not only satisfies a trigonometric differential equation, but its complex Fourier coefficients satisfy a linear recurrence equation with constant coefficients. This can be seen as follows:[2] W. l. o. g. we assume that $\omega = 1$. Using Euler's formula and the variable $z = e^{it}$ we can write

$$x = \cos t = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad y = \sin t = \frac{1}{2i}\left(z - \frac{1}{z}\right)\ .$$

Then

$$r(z) := R(x, y) = R\left(\frac{1}{2}\left(z + \frac{1}{z}\right), \frac{1}{2i}\left(z - \frac{1}{z}\right)\right) \in \mathbb{K}[i](z)$$

is a rational function of the variable z. To compute the complex Fourier coefficient c_n of R we write

$$r(z) = \frac{p(z)}{q(z)}$$

with $\gcd(p(z), q(z)) = 1$. Assume $p(z) = \sum_{k=0}^{P} p_k z^k$ and $q(z) = \sum_{k=0}^{Q} q_k z^k$, then we can write

$$\sum_{k=0}^{Q} q_k e^{ikt} r(e^{it}) = \sum_{k=0}^{P} p_k z^k\ ,$$

and multiplying by $\frac{1}{2\pi} e^{-itn}$ and integrating from $t = 0, \ldots, 2\pi$, we get the recurrence equations

$$\sum_{k=0}^{Q} q_k c_{n-k} = 0 \quad \text{for } n < 0 \text{ and } n > P \tag{3.5}$$

[2]We would like to thank an anonymous reviewer of a preliminary version of this paper for these important comments.

and

$$\sum_{k=0}^{Q} q_k c_{n-k} = p_n \quad \text{for } 0 \leqq n \leqq P \,. \tag{3.6}$$

That makes the rational trigonometric functions very special. Of course the above algorithm can be implemented, and this can yield a formula for the complex Fourier coefficients. We would like to give the following comments on this approach. In the next section we will show that the Fourier coefficients of all functions in TH satisfy a holonomic recurrence equation, i. e. a homogeneous linear recurrence equation with polynomial coefficients. Let's call the latter recurrence RE_{hol}, and the recurrence of type (3.5)–(3.6) RE_{const}. It turns out that

- In most cases the recurrence RE_{const} has much higher order than the holonomic recurrence equation RE_{hol}. This is the bottleneck since many more initial values have to be computed.
- To find a formula for the complex Fourier coefficients by solving RE_{const}, one must factorize the polynomial $\sum_{k=0}^{Q} q_k\, z^k \in \mathbb{K}[z]$. If this polynomial does not have a complete factorization into linear factors in $\mathbb{K}$, we have to work in algebraic extension fields which makes the algorithm typically very slow.
- The holonomic recurrence equation RE_{hol} often has low order and can be (independently of the order) algorithmically solved very efficiently using van Hoeij's algorithm ([12, Chapter 9], [8] and [4]), often without introducing algebraic numbers.

That's why we don't give further details about the above algorithm. However we will give a typical example for the recurrence RE_{const} together with an efficiency consideration in Example 4.10.

We finish this subsection with

Theorem 3.14. *The set of functions of the form $g(t)\cdot h(t)$ where $g(t)$ is exp-like and $h(t) \in \mathbb{K}(\cos(\omega t), \sin(\omega t))$ is a subset of* $\mathrm{TH}(\omega)$.

3.4. Some More Example Types of Trigonometric Holonomic Functions

By Euler's identity, from Theorem 3.11 we may deduce the following theorem.

Theorem 3.15. *The set $\mathbb{K}[t, e^{-\alpha t}, e^{\beta t}, e^{-i\gamma t}, e^{i\delta t}]$ where α, β, γ, $\delta \in \mathbb{R}$ is a subset of $\mathrm{TH}(\omega)$ for all $\omega \in \mathbb{R}^\star$. Each of its functions is of type 0 in $\mathrm{TH}(\omega)$.*

Example 3.16. Consider the function defined by

$$f(t) = t\sin(t)e^{3it} + 3e^{it}\cos(t)$$

f is a solution of the following complex differential equation

$$-64if'(t) + 96f''(t) + 52if'''(t) - 12f^{(4)}(t) - if^{(5)}(t) = 0 \,.$$

From Theorem 3.12 we may also deduce the following.

Theorem 3.17. *The set $\mathbb{K}(e^{-i\omega t}, e^{i\omega t})$ is a subset of $\mathrm{TH}(\omega)$ and each function of $\mathbb{K}(e^{-i\omega t}, e^{i\omega t})$ is of degree 1 in $\mathrm{TH}(\omega)$.*

Furthermore, we have

Theorem 3.18. $\mathbb{K}(\cos(\omega t), \sin(\omega t), e^{-i\omega t}, e^{i\omega t})$ *is a subset of* $\mathrm{TH}(\omega)$.

We deduce from Theorem 3.14 that

Theorem 3.19. *Functions of the form $g(t) \cdot h(t)$ where $g(t)$ is exp-like and $h(t) \in \mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ are contained in $\mathrm{TH}(\omega)$.*

Remark 3.20. We cannot give a complete list of families of functions satisfying a differential equation either of the form (3.2) or (3.3). In the following example we list some functions whose form has not been mentioned previously.

Example 3.21. 1. Consider the function

$$f(t) = \arctan(2 + e^{it}) \in \mathrm{TH}(1)\ .$$

f is solution of the trigonometric holonomic differential equation

$$DE : (-e^{it} + 5e^{-it})f'(t) + i(4 + 5e^{-it} + e^{it})f''(t) = 0\ .$$

We deduce that f is of degree 2 in TH(1).

2. Consider now the function

$$f(t) = \frac{\sqrt{e^{it} + 3}}{e^{it} + 2} \in \mathrm{TH}(1)\ .$$

f satisfies the trigonometric holonomic differential equation

$$i(4 + e^{it})f(t) + (10 + 12e^{-it} + 2e^{it})f'(t) = 0\ .$$

f is of type 1 and of degree 1 in TH(1).

3. Set $\omega = \sqrt{7}$ and consider the function

$$f(t) = \cos(\sqrt{7}t)\ln(2 + \sin(\sqrt{7}t))\ .$$

f satisfies the trigonometric holonomic differential equation

$$\begin{aligned}98\sqrt{7}(4 + 3\sin(\sqrt{7}t))f(t) + 343\cos(\sqrt{7}t)f'(t) + 7\sqrt{7}(10 + 3\sin(\sqrt{7}t))f''(t)\\ +21\cos(\sqrt{7}t)f'''(t) + \sqrt{7}(2 + \sin(\sqrt{7}t))f^{(4)}(t) = 0\end{aligned}$$

from which we deduce that $f \in \mathrm{TH}(\sqrt{7})$.

3.5. A Function Which is not Trigonometric Holonomic

In this section we give an example of a function which is not trigonometric holonomic. W.l.o.g. we may assume that $\omega = 1$. We will show that the rational function $f(t) = \frac{1}{t}$ is not a trigonometric holonomic function.

Proof. We use a proof by contradiction. Let us assume that f is a trigonometric holonomic function. Then there exist integers $P \geqq 1$ and $L \geqq 0$ and coefficients α_{pl} and β_{pl} for which f satisfies a differential equation of the form (3.2). At least one of α_{pl} and β_{pl} is non-vanishing. The successive derivatives of f are given by $f^{(p)}(t) = \frac{(-1)^p p!}{t^{p+1}}$. The substitution of those derivatives in (3.2) leads to

$$\sum_{p=0}^{P}\sum_{l=0}^{L}\left(\alpha_{pl}\cos(lt) + \beta_{pl}\sin(lt)\right)\frac{(-1)^p p!}{t^{p+1}} = 0 \Longleftrightarrow$$

$$\begin{aligned}\frac{1}{t}\sum_{l=0}^{L}(\alpha_{0l}\cos(lt) + \beta_{0l}\sin(lt)) - \frac{1}{t^2}\sum_{l=0}^{L}(\alpha_{1l}\cos(lt) + \beta_{1l}\sin(lt)) + \cdots\\ +\frac{(-1)^P P!}{t^{P+1}}\sum_{l=0}^{L}(\alpha_{Pl}\cos(lt) + \beta_{Pl}\sin(lt)) = 0\ .\end{aligned}$$

Multiplying the previous equation by t^{P+1} we get

$$\begin{aligned}t^P\sum_{l=0}^{L}(\alpha_{0l}\cos(lt) + \beta_{0l}\sin(lt)) - t^{P-1}\sum_{l=0}^{L}(\alpha_{1l}\cos(lt) + \beta_{1l}\sin(lt)) + \cdots\\ +(-1)^P P!\sum_{l=0}^{L}(\alpha_{Pl}\cos(lt) + \beta_{Pl}\sin(lt)) = 0\ .\end{aligned} \tag{3.7}$$

Collecting the previous equation with respect to the expressions $\cos(lt), \sin(lt)$, $l = 0, \ldots, L$, we get:

$$\begin{aligned}\left(\alpha_{00}t^P - \alpha_{10}t^{P-1} + \cdots + (-1)^P P!\alpha_{P0}\right) + \left(\alpha_{01}t^P - \alpha_{11}t^{P-1} + \cdots + (-1)^P P!\alpha_{P1}\right)\cos(t) +\\ \left(\beta_{01}t^P - \beta_{11}t^{P-1} + \cdots + (-1)^P P!\beta_{P1}\right)\sin(t) + \cdots + \left(\alpha_{0L}t^P - \alpha_{1L}t^{P-1} + \cdots\right.\\ \left.+(-1)^P P!\alpha_{PL}\right)\cos(Lt) + \left(\beta_{0L}t^P - \beta_{1L}t^{P-1} + \cdots + (-1)^P P!\beta_{PL}\right)\sin(Lt) = 0\ .\end{aligned}$$

The previous equation is satisfied for every t if and only if all the polynomial coefficients vanish, i.e.

$$\alpha_{00} = \alpha_{10} = \cdots = \alpha_{PL} = \beta_{00} = \beta_{10} = \cdots = \beta_{PL} = 0\ ,$$

which is in contradiction with our initial assumption. □

3.6. Algebraic Properties of Trigonometric Holonomic Functions

The following results can be easily obtained:

Theorem 3.22. *If f is a ω-trigonometric holonomic function, then its derivative and anti-derivative are also trigonometric holonomic functions.*

Theorem 3.23. *(a) $\mathrm{TH}(\omega)$ is closed under addition and multiplication.*

(b) If $f(t)$ and $g(t)$ are two functions of degree P and Q in $\mathrm{TH}(\omega)$, respectively, then $f(t) + g(t)$ is of degree $\leqq P + Q$ in $\mathrm{TH}(\omega)$ and $f(t) \cdot g(t)$ is of degree $\leqq P \cdot Q$ in $\mathrm{TH}(\omega)$.

Note that this theorem follows in a similar way as for holonomic functions, see e. g. ([19], [18] or [15]).

Remark 3.24. From the above theorem we may deduce that $(\mathrm{TH}, +, \cdot)$ is a commutative unitary ring.

4. Fourier Coefficients of Trigonometric Holonomic Functions

In the previous section we defined the set of trigonometric holonomic functions, and we gave some of their properties. In this section we present a general algorithm for the computation of the complex Fourier coefficients of trigonometric holonomic functions.

4.1. Hypergeometric Terms and Closed Forms

In this section, we will deal with recurrence equations (for Fourier coefficients), having interest in their solutions. The type of solution in which we are mainly interested is a special type of "closed form" which was given in [17]. We will make this notion more precise in a moment. Mark van Hoeij presented in [8] a very efficient algorithm to solve recurrence equations in closed form when such solutions exist. That algorithm is a reviewed and improved version of Petkovšek's algorithm [16], see also [4]. Nevertheless in the cases where a closed form solution does not exist, we may return the Fourier coefficient in any other form, if possible, rather than not to give an output.

Definition 4.1 (Hypergeometric term). An expression a_n is called hypergeometric term if the ratio $\frac{a_{n+1}}{a_n}$ represents a rational function in n.

Definition 4.2 (Closed form). An expression a_n is said to be of closed form if it is a linear combination of a fixed number of hypergeometric terms.

Example 4.3. The sum

$$s_n = \sum_{k=0}^{[\frac{n}{3}]} \binom{n-2k}{k} \left(-\frac{4}{27}\right)^k = \frac{1}{9}\left(-\frac{1}{3}\right)^n + \frac{2(3n+4)}{9}\left(\frac{2}{3}\right)^n$$

is in closed form, see e. g. [12], Example 7.2.

4.2. Holonomic Recurrence Equation for Trigonometric Holonomic Functions

Definition 4.4 (Holonomic recurrence equation). *A recurrence equation is holonomic* if it is homogeneous and linear and has polynomial coefficients $\in \mathbb{K}[n]$.

4.2.1. Conversion of a Trigonometric Holonomic Differential Equation Into a Recurrence Equation. We recall that the set of the trigonometric holonomic functions consists of those functions satisfying a relation of type (3.2) resp. (3.3), namely

$$\sum_{p=0}^{P}\sum_{l=0}^{L}\left(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\right)f^{(p)}(t)=0 \tag{4.1}$$

$$\text{resp.}\quad \sum_{p=0}^{P}\sum_{l=0}^{L}\left(\gamma_{pl}e^{-il\omega t}+\delta_{pl}e^{il\omega t}\right)f^{(p)}(t)=0 \tag{4.2}$$

for appropriate integers $P \geqq 1, L \geqq 0$ and $\omega \in \mathbb{R}^{\star}$, where α_{pl} and β_{pl} resp. γ_{pl} and $\delta_{pl} \in \mathbb{K}$.

Theorem 4.5 (DE to RE). *Let $f : [a, b] \to \mathbb{R}$ satisfy a differential equation of the form (3.2) resp. (3.3). Then the complex Fourier coefficients $c_n^{(p)}$ of the derivatives $f^{(p)}(t)$ satisfy the recurrence equation*

$$\sum_{p=0}^{P}2\alpha_{p0}c_n^{(p)}+\sum_{p=0}^{P}\sum_{l=1}^{L}\left(c_{n+l}^{(p)}(\alpha_{pl}+i\beta_{pl})+c_{n-l}^{(p)}(\alpha_{pl}-i\beta_{pl})\right)=0 \tag{4.3}$$

$$\textit{resp.}\quad \sum_{p=0}^{P}\sum_{l=0}^{L}\left(\delta_{pl}c_{n-l}^{(p)}+\gamma_{pl}c_{n+l}^{(p)}\right)=0 \tag{4.4}$$

After replacing $c_{n\pm l}^{(p)}$ for $p>0$ according to Equation (2.2) in terms of $c_{n\pm l}$ this yields a holonomic recurrence equation for the complex Fourier coefficients c_n of $f(t)$, given as

$$\begin{aligned}
0 \;=\; & \sum_{p=0}^{P}2\alpha_{p0}\left(-\frac{2n\pi}{iT}\right)^{p}\left(c_n-\sum_{j=0}^{p-1}(-T)^j\left(\frac{i}{2n\pi}\right)^{j+1}(f^{(j)}(b)-f^{(j)}(a))e^{-in\omega a}\right)\\
+\; & \sum_{p=0}^{P}\sum_{l=1}^{L}\Bigg[(\alpha_{pl}+i\beta_{pl})\left(\frac{2(n+l)\pi i}{T}\right)^{p}\\
& \cdot\left(c_{n+l}-\sum_{j=0}^{p-1}(-T)^j\left(\frac{i}{2(n+l)\pi}\right)^{j+1}(f^{(j)}(b)-f^{(j)}(a))e^{-i(n+l)\omega a}\right)\\
+\; & (\alpha_{pl}-i\beta_{pl})\left(\frac{2(n-l)\pi i}{T}\right)^{p}\\
& \cdot\left(c_{n-l}-\sum_{j=0}^{p-1}(-T)^j\left(\frac{i}{2(n-l)\pi}\right)^{j+1}(f^{(j)}(b)-f^{(j)}(a))e^{-i(n-l)\omega a}\right)\Bigg]
\end{aligned} \tag{4.5}$$

resp.

$$\begin{aligned}
0 \;=\; & \sum_{p=0}^{P}\sum_{l=1}^{L}\Bigg[\delta_{pl}\left(\frac{2(n-l)\pi i}{T}\right)^{p}\\
& \cdot\left(c_{n-l}-\sum_{j=0}^{p-1}(-T)^j\left(\frac{i}{2(n-l)\pi}\right)^{j+1}(f^{(j)}(b)-f^{(j)}(a))e^{-i(n-l)\omega a}\right)\\
+\; & \gamma_{pl}\left(\frac{2(n+l)\pi i}{T}\right)^{p}\\
& \cdot\left(c_{n+l}-\sum_{j=0}^{p-1}(-T)^j\left(\frac{i}{2(n+l)\pi}\right)^{j+1}(f^{(j)}(b)-f^{(j)}(a))e^{-i(n+l)\omega \alpha}\right)\Bigg]
\end{aligned} \tag{4.6}$$

Proof. Starting from the differential equation (4.1), we will construct two relations which are linear combinations of the real Fourier coefficients of the successive derivatives of the function f. Then we will combine both relations to get a linear combination between the complex Fourier coefficients of the successive derivatives of f, which will be converted into a recurrence equation for the complex Fourier coefficients of f itself.
Multiplying (4.1) by $\frac{2}{T}\cos(n\omega t)$ and integrating over the interval $[a,b]$ we get

$$\frac{2}{T}\int_a^b\left(\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t)+\sum_{p=0}^{P}\sum_{l=1}^{L}\left(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\right)f^{(p)}(t)\right)\cos(n\omega t)dt=0$$

$$\Longleftrightarrow \frac{2}{T}\int_a^b\Bigg[\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t)\cos(n\omega t)+\sum_{p=0}^{P}\sum_{l=1}^{L}(\alpha_{pl}\cos(l\omega t)\cos(n\omega t)+$$
$$\beta_{pl}\sin(l\omega t)\cos(n\omega t))\,f^{(p)}(t)\Bigg]\,dt=0\;.$$

Using the trigonometric addition theorems the previous equation becomes

$$\frac{2}{T}\int_a^b\Bigg[\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t)\cos(n\omega t)+\sum_{p=0}^{P}\sum_{l=1}^{L}\Bigg(\frac{1}{2}\alpha_{pl}\left(\cos((n+l)\omega t)+\cos((n-l)\omega t)\right)+$$
$$\frac{1}{2}\beta_{pl}\left(\sin((n+l)\omega t)-\sin((n-l)\omega t)\right)\Bigg)\,f^{(p)}(t)\Bigg]\,dt=0\;,$$

which leads to the relation

$$\sum_{p=0}^{P}\alpha_{p0}\frac{2}{T}\int_a^b f^{(p)}(t)\cos(n\omega t)dt+$$
$$\sum_{p=0}^{P}\sum_{l=1}^{L}\Bigg[\frac{1}{2}\alpha_{pl}\left(\frac{2}{T}\int_a^b f^{(p)}(t)\cos((n+l)\omega t)dt+\frac{2}{T}\int_a^b f^{(p)}(t)\cos((n-l)\omega t)dt\right)+$$
$$\frac{1}{2}\beta_{pl}\left(\frac{2}{T}\int_a^b f^{(p)}(t)\sin((n+l)\omega t)dt-\frac{2}{T}\int_a^b f^{(p)}(t)\sin((n-l)\omega t)dt\right)\Bigg]=0\;.$$

Finally, we therefore obtain

$$\sum_{p=0}^{P}\alpha_{p0}a_n^{(p)}+\sum_{p=0}^{P}\sum_{l=1}^{L}\left[\frac{1}{2}\alpha_{pl}\left(a_{n+l}^{(p)}+a_{n-l}^{(p)}\right)+\frac{1}{2}\beta_{pl}\left(b_{n+l}^{(p)}-b_{n-l}^{(p)}\right)\right]=0\;. \tag{4.7}$$

Hence we reveived an identity between the sine Fourier coefficients $b_{n+l}^{(p)}$ and the cosine Fourier coefficients $a_{n+l}^{(p)}$ of the successive derivatives of a function f.

Starting from the same relation (4.1) and using the same process as previously, but multiplying this time by $\frac{2}{b-a}\sin(n\omega t)$ instead of $\frac{2}{b-a}\cos(n\omega t)$, we get

$$\frac{2}{b-a}\int_a^b\left(\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t)+\sum_{p=0}^{P}\sum_{l=1}^{L}(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t))f^{(p)}(t)\right)\sin(n\omega t)dt=0\;.$$

This leads to the following second relation which is again a linear combination of the sine Fourier coefficients $b_{n+l}^{(p)}$ and the cosine Fourier coefficients $a_{n+l}^{(p)}$ of the successive derivatives of the function f

$$\sum_{p=0}^{P}\alpha_{p0}b_n^{(p)}+\sum_{p=0}^{P}\sum_{l=1}^{L}\left[\frac{1}{2}\alpha_{pl}\left(b_{n+l}^{(p)}+b_{n-l}^{(p)}\right)+\frac{1}{2}\beta_{pl}\left(a_{n-l}^{(p)}-a_{n+l}^{(p)}\right)\right]=0\;. \tag{4.8}$$

Aiming to find a recurrence equation for the complex Fourier coefficients

$$c_n=\frac{1}{2}(a_n-ib_n)$$

of the function f, we combine the relations (4.7) and (4.8) in the form (4.7)$-i$(4.8) which (after simplification) leads to (4.3) finishing the first part of the proof. Equation (4.4) can be proved in a similar way.

Finally we convert (4.3) into a recurrence equation for the complex Fourier coefficients of f. From (2.2) we deduce that

$$\begin{aligned} c_{n-l}^{(p)} &= \left(\frac{2(n-l)\pi i}{T}\right)^p \Bigg(c_{n-l} - \\ &\sum_{j=0}^{p-1} (-T)^j \left(\frac{i}{2(n-l)\pi}\right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-i(n-l)\omega a} \Bigg) \end{aligned} \tag{4.9}$$

and that

$$\begin{aligned} c_{n+l}^{(p)} &= \left(\frac{2(n+l)\pi i}{T}\right)^p \Bigg(c_{n+l} - \\ &\sum_{j=0}^{p-1} (-T)^j \left(\frac{i}{2(n+l)\pi}\right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-i(n+l)\omega a} \Bigg) \end{aligned} \tag{4.10}$$

Substitution of (4.9) and (4.10) in (4.3) leads to (4.5) which is a holonomic recurrence equation satisfied by the complex Fourier coefficients of f. By the same process we obtain the relation (4.6). □

Remark 4.6. Note that Theorem 4.5 shows, in particular, that the order of the resulting recurrence equation for the Fourier coefficients is at most $2L$. Therefore the type $L(\omega)$ should be kept as small as possible.

Remark 4.7. In practice for the conversion of a differential equation of the form (3.3) into a recurrence equation, we will retrieve from the differential equation the coefficients γ_{pl} and δ_{pl} and substitute them in (4.6). This process is summarized in Algorithm 3.

4.3. Some examples of Fourier coefficients

Example 4.8. As we saw in Example 3.9, the function $f : [0, 2\pi] \to \mathbb{C}$ defined by

$$f(t) = e^{ie^{it}} .$$

satisfies the differential equation

$$f'(t) + e^{it} f(t) = 0 . \tag{4.11}$$

The retrieval of the coefficients γ_{pl} and δ_{pl} from the previous differential equation gives

$$\gamma_{00} = 0, \quad \gamma_{01} = 0, \quad \gamma_{10} = 0, \quad \gamma_{11} = 0$$

and

$$\delta_{00} = 0, \quad \delta_{01} = 1, \quad \delta_{10} = 1, \quad \delta_{11} = 0 .$$

Substituting the previous γ_{pl} and δ_{pl} in (4.6) we obtain the recurrence equation

$$c_{n-1} + inc_n = 0 .$$

Solving with one initial value we get that the Fourier coefficients of f are given as

$$c_n = \begin{cases} \frac{i^n}{n!} & \forall n \geqq 0 \\ 0 & \text{otherwise} . \end{cases}$$

Algorithm 3: Fourier coefficients in the general case.

input : A function $f \in C^{(N)}[a,b]$ for N large enough.

output : Either the complex Fourier coefficients of f on I, or the recurrence equation satisfied by the complex Fourier coefficients of f and enough initial values, or "This algorithm is not applicable for the computation of the Fourier coefficients of f".

```
1  begin
2      Compute the complex Fourier coefficients of f by the defining integral.
3      if the computation is successful then
4          return The complex Fourier coefficients of f in the interval I.
5      end
6      Compute the complex Fourier coefficients of one of the successive derivatives or anti-derivatives
       of f on I.
7      if the computation is successful then
8          Apply Algorithm 1 to f in the interval I.
9      end
10     if f is a trigonometric holonomic function then
11         if f contains expressions of the form e^{±iωt} where ω = 2π/(b−a) then
12             Find a differential equation in the form (4.2) and convert it into a recurrence equation of
               the form RE (4.6)
13             if Sufficiently many initial values can be computed then
14                 Solve RE.
15                 if A closed form solution can be found then
16                     return that solution
17                 else
18                     return RE and the initial values
19                 end
20             else
21                 Apply the steps 12–19 to one of the successive derivatives of f.
22                 if Step 13 and 15 is successful then
23                     Use Algorithm 1 to deduce the sought Fourier coefficients.
24                 else
25                     return RE
26                 end
27             end
28         end
29     end
30     return This algorithm is not applicable for the computation of the complex Fourier coefficients of
       f.
31 end
```

Example 4.9. Consider now the function defined on $I = [0, \frac{2\pi}{5}]$ by

$$f(t) = \cos(5t)\ln(2+\cos(5t))\ .$$

$\omega = \frac{2\pi}{\frac{2\pi}{5}-0} = 5$. f satisfies the differential equation

$$\begin{gathered} DE : (-500000 + 843750\cos(5t))f'(t) + 28125\sin(5t)f''(t) \\ +(54375\cos(5t) - 45000)f^{(3)}(t) + 4625\sin(5t)f^{(4)}(t) + (825\cos(5t) - 1200)f^{(5)}(t) \\ +120\sin(5t)f^{(6)}(t) + (-4\cos(5t) - 8)f^{(7)}(t) = 0\ . \end{gathered}$$

The conversion of DE into a recurrence equation for the complex Fourier coefficients c_n of f gives

$$\begin{gathered} RE : (n-1)(n+2)(n+1)(n-2)^2(2n+1)^2 c_{n-1} + 16(n-2)(n+2)n(n+1)^2(n-1)^2 c_n \\ +(n-1)(n-2)(n+1)(n+2)^2(2n-1)^2 c_{n+1} = 0\ . \end{gathered}$$

Solving RE using 2 initial values, we get

$$c_n = \begin{cases} \frac{(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)} & \forall n \geqq 2 \\ \sqrt{3} - \frac{7}{4} + \frac{1}{2}\ln(2+\sqrt{3}) - \frac{1}{2}\ln(2) & \text{if} \quad n = 1 \\ 2-\sqrt{3} & \text{if} \quad n = 0 \\ \sqrt{3} - \frac{7}{4} + \frac{1}{2}\ln(2+\sqrt{3}) - \frac{1}{2}\ln(2) & \text{if} \quad n = -1 \\ \frac{(-2+\sqrt{3})^{-n}(\sqrt{3}-2n)}{(-n+1)(-n-1)} & \forall n \leqq -2\ . \end{cases}$$

Since f is even, $b_n = 0$ and $a_n = 2c_n$. Hence the Fourier series of f on I is given as

$$\begin{aligned} f(t) \quad &= \quad 2(2-\sqrt{3}) + \left(2\sqrt{3} - \frac{7}{2} + \ln(2+\sqrt{3}) - \ln(2)\right)\cos(5t) \\ &+ \sum_{n=2}^{\infty} \frac{2(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)}\cos(5nt)\ . \end{aligned}$$

Example 4.10. Let us aim to compute the complex Fourier coefficients of the function

$$f(t) = \frac{1}{(2+\cos(t))^{20}}$$

in the interval $[0, 2\pi]$. f satisfies the following trigonometric differential equation

$$DE : (2+\cos(t))f'(t) - 20\sin(t)f(t) = 0$$

and its complex Fourier coefficients satisfy the holonomic recurrence equation RE

$$RE : i(n+19)c_{n-1} + 4inc_n + i(-19+n)c_{n+1} = 0\ . \tag{4.12}$$

Solving RE we get $\forall n \geqq 0$

$$\begin{aligned} c_n = \frac{(-2+\sqrt{3})^n}{21549064602123362304000} \cdot \\ \Big(3\,n^{19} + 380\,\sqrt{3}\,n^{18} + 68400\,n^{17} + 2587230\,\sqrt{3}\,n^{16} \\ + 207784608\,n^{15} + 4189258320\,\sqrt{3}\,n^{14} + 198221547000\,n^{13} \\ + 2507083241260\,\sqrt{3}\,n^{12} + 77548709950608\,n^{11} \\ + 658724628579160\,\sqrt{3}\,n^{10} + 13910290044027000\,n^{9} \\ + 81271048798518540\,\sqrt{3}\,n^{8} + 117924653590824 2448\,n^{7} \\ + 4686690914390935200\,\sqrt{3}\,n^{6} + 45298720378942521900\,n^{5} \\ + 115724365700595819470\,\sqrt{3}\,n^{4} + 678681872501747249208\,n^{3} \\ + 956086266762532871940\,\sqrt{3}\,n^{2} + 2598373260585253340700\,n \\ + 1139118803030468009750\,\sqrt{3}\Big) \end{aligned}$$

and since $f(t)$ is even of course $c_{-n} = c_n$ and $a_n = 2\,c_n$ for all $n \in \mathbb{N}$.

The whole computation for this example took about 6 seconds.[3] Note that this example can also be successfully solved by the algorithm given in Example 3.13. The detection of the constant-coefficient recurrence RE_{const} for this example is very fast. However, this recurrence equation has order 40. Its characteristic polynomial $\chi(z)$ has a simple factorization

$$\chi(z) = (z^2 + 4z + 1)^{20}\ .$$

Nevertheless, the bottleneck is the fact that 40 initial values have to be computed. The computation time for this part was 300 seconds. Of course the two results agree.

[3]Computations were done with *Maple* 17 on a PC with 64-bit Intel processor i7-3820QM with 2.7 GHz and 8 GB RAM.

5. Factorization of Holonomic Recurrence Operators

This section deals with the factorization of holonomic recurrence operators. An algorithm for computing a first order right factor of such operators was first given by Petkovšek in [16]. Its application is limited to the cases in which the product of the leading and trailing coefficients of the considered operators do not have too many factors, because the algorithm computes more combinations than necessary. Mark van Hoeij addressed those problems in [8] by introducing the concept of *finite singularities*. We present in the second section a different method, involving Fourier series, to compute a right factor of holonomic recurrence operators, which in some cases returns the smallest order right factor. In the first section we give some resources to achieve that goal.

5.1. Conversion of a Holonomic Recurrence Equation Into a Trigonometric Holonomic Differential Equation

Section 4 described the conversion of trigonometric holonomic differential equations into recurrence equations for the Fourier coefficients, which may be homogeneous or not. In this section we do the reverse of that conversion, focussing on homogeneous recurrence equations. To do so we look if for a given homogeneous holonomic recurrence equation RE one may find coefficients α_{pl} and β_{pl} (resp. γ_{pl} and δ_{pl}) such that RE is the conversion of a differential equation of the form (3.2) (resp. (3.3)) with some initial values. In this case the sought trigonometric holonomic differential equation will be the one satisfied by a function defined in an interval $[a, b]$ such that $F^{(j)}(a) = F^{(j)}(b)$, $(j = 0, \ldots, P-1)$.

Theorem 5.1. *For a given real number $\omega = \frac{2\pi}{b-a}$ with $a < b$, each holonomic recurrence equation can be converted into a differential equation with side conditions either of the form*

$$\begin{cases} \sum\limits_{p=0}^{P}\sum\limits_{l=0}^{L} \left(\alpha_{pl}\cos(l\omega t) + \beta_{pl}\sin(l\omega t)\right) f^{(p)}(t) = 0 \\ f^{(j)}(a) = f^{(j)}(b) \qquad (j = 0, \ldots, P-1) \end{cases} \tag{5.1}$$

for appropriate integers $P \geqq 1$ and $L \geqq 0$, where α_{pl} and β_{pl} are constants, or of the form

$$\begin{cases} \sum\limits_{p=0}^{P}\sum\limits_{l=0}^{L} \left(\gamma_{pl}e^{-il\omega t} + \delta_{pl}e^{il\omega t}\right) f^{(p)}(t) = 0 \\ f^{(j)}(a) = f^{(j)}(b) \qquad (j = 0, \ldots, P-1) \end{cases} \tag{5.2}$$

for appropriate integers $P \geqq 1, L \geqq 0$, where γ_{pl} and δ_{pl} are constants.

Proof. This can be handled by using the equations in Theorem 4.5 in the backward direction. For details, see [15], page 74. □

We summarize this search in Algorithm 4. Examples will be given in the next subsection.

Algorithm 4: Conversion of a holonomic recurrence equation into a differential equation with side conditions (ReverseTHDEtoRE).

input : A holonomic recurrence equation RE and an interval $[a, b]$.
output : A differential equation with side conditions of the form (5.1) (resp. (5.2)).

```
begin
    ω ← 2π/(b−a).
    Search for coefficients α_pl and β_pl (resp. γ_pl and δ_pl) such that an equation of the form
    (4.5) − RE = 0 (resp. (4.6) − RE = 0) is valid.
    Return a differential equation with side conditions of the form (5.1) (resp. (5.2)).
end
```

5.2. Holonomic Recurrence Operators

Let $\mathbb{K}$ be a field of characteristic zero. We denote by $\mathbb{K}^{\mathbb{N}}$ the set of all sequences $(a_n)_{n=0}^{\infty}$ whose terms belong to $\mathbb{K}$.

Definition 5.2. The function $N : \mathbb{K}^{\mathbb{N}} \longrightarrow \mathbb{K}^{\mathbb{N}}$ which acts on each a_n in the following way: $N(a_n) = a_{n+1}$, is called the *shift operator*.

Note that N is linear, and that the set of all linear operators with addition defined pointwise and with the functional composition as multiplication is a (non-commutative) ring. N satisfies the commutation relation $Nn = (n+1)N$.

Definition 5.3. Operators of the form $L = \sum\limits_{k=0}^{r} a_k N^k$, where $a_k \in \mathbb{K}^{\mathbb{N}}$ and N is the shift operator are called *recurrence operators* on $\mathbb{K}^{\mathbb{N}}$, see [17]. If $a_r \neq 0$ and $a_0 \neq 0$, then the *order* of L is r. Equations of the form

$$L(u_n) = 0, \quad \text{i.e.} \quad a_r u_{n+r} + a_{r-1} u_{n+r-1} + \cdots + a_0 u_n = 0$$

are called *recurrence equations*. A recurrence equation is *holonomic* if $a_k \in \mathbb{K}[n]$ $(0 \leqq k \leqq r)$.

5.2.1. A New Factorization Method. The idea behind the method described in this study is to consider the holonomic recurrence equation RE corresponding to a given holonomic recurrence operator as the one satisfied by the Fourier coefficients of a trigonometric holonomic function in an interval $[a, b]$, which may be w.l.o.g. the interval $[0, 2\pi]$. To do so, we compute the differential equation with side conditions corresponding to RE either of the form (5.1) or of the form (5.2). Then we solve it to get possible choices of the corresponding trigonometric holonomic functions. Applying Algorithm 2, we may find a lower order trigonometric holonomic differential equation satisfied by f, which may lead to a recurrence equation of lower order. The corresponding holonomic recurrence operator clearly is a right factor of the input recurrence operator since the Fourier coefficients of f are solutions of both. Algorithm 5 below provides an overview of this procedure.

Algorithm 5: Search for a right factor of a holonomic recurrence operator.

input : A holonomic recurrence operator L.
output : A right factor of L or "A right factor of L cannot be found using this method, since no solution of DE found".

begin
- Convert L into a recurrence equation RE.
- Use Algorithm 4 to convert RE into a differential equation with side conditions DE.
- Solve that differential equation with side conditions.
- **if** *DE cannot be solved* **then**
  - **return** *A right factor of the given operator cannot be found using this algorithm, since DE cannot be solved*
- **else**
  - Set f a solution of DE.
  - Apply Algorithm 2 to f to search for a new holonomic differential equation DE satisfied by f of as low order as possible which leads this time to the lowest order holonomic recurrence equation RE.
  - Convert this RE into a holonomic recurrence operator.
  - **return** *the right factor of the given holonomic recurrence operator*
- **end**

end

Instead of searching for the smallest recurrence, one could also try to compute several recurrences, and then compute the greatest common right divisor.

5.2.2. Some Factorization Examples of Holonomic Recurrence Operators.

Example 5.4. Consider the following holonomic recurrence operator, as given in [17].

$$L = (n-1)N^2 - (n^2+3n-2)N + 2n(n+1)\ .$$

The corresponding holonomic recurrence equation with respect to c_n is

$$RE_1 : (n-1)c_{n+2} - (n^2+3n-2)c_{n+1} + 2n(n+1)c_n = 0\ . \tag{5.3}$$

Applying Algorithm 4 returns the differential equation with side conditions

$$DE_1 : \begin{cases} (-4e^{-it}+3e^{-2it})f(t) + i(-e^{-it}+e^{-2it}+2)f'(t) - (e^{-it}-2)f''(t) = 0 \\ f^{(j)}(0) = f^{(j)}(2\pi) \qquad (j=0,1)\ . \end{cases}$$

Solving this DE_1 we get one of the solutions

$$f(t) = e^{-it-e^{-it}} \in \mathrm{sTH}\ .$$

Applying Algorithm 2 to f with $\omega = \frac{2\pi}{2\pi-0} = 1$ leads to

$$DE_2 : (-1+e^{it})f(t) - ie^{it}f'(t) = 0\ .$$

Now converting DE_2, considering the interval $[0, 2\pi]$, into a holonomic recurrence equation we get

$$RE_2 : c_{n+1} - (n+1)c_n = 0\ .$$

The above RE_2 shows that a first order right factor of the holonomic recurrence operator L is given by $N - n - 1$, which exactly corresponds to one of the right factors found in [17].

Of course this is a trivial example since it can be easily solved using Petkovšek's or van Hoeij's algorithm generating the two right factors $N - n - 1$ and $N - 2$ corresponding to the solution basis $(2^n, \Gamma(n+1))$ of (5.3).

Example 5.5. Consider the following holonomic recurrence operator L of order 6

$$L = N^6 - 5N^4 + (14-n)N^3 + (-n^2-n+2)N^2 + (n^2+11n-14)N + (24-12n)\ .$$

Its lowest order right factor, which is of order 3, was computed by Horn in [9]. The following shows the computation of the same factor with the method of this study. The conversion of L into a recurrence equation for c_n is

$$\begin{aligned} RE_1 : c_{n+6} - 5c_{n+4} + (14-n)c_{n+3} + (-n^2-n+2)c_{n+2} \\ +(n^2+11n-14)c_{n+1} + (24-12n)c_n = 0\ . \end{aligned}$$

The differential equation with side conditions corresponding to RE_1 is

$$DE_1 : \begin{cases} -(-24+24e^{-it}-17e^{-3it}+5e^{-4it}-e^{-6it})f(t) \\ -i(3e^{-2it}-12+9e^{-it}-e^{-3it})f'(t) + (-e^{-it}+e^{-2it})f''(t) = 0 \quad 0 \\ f^{(j)}(0) = f^{(j)}(2\pi) \qquad (j=0,1)\ . \end{cases}$$

One of the solutions of this DE_1 is

$$f(t) = e^{12e^{it}-\frac{1}{2}e^{-2it}}\ .$$

Applying Algorithm 2 to $f(t)$ shows that $f(t)$ satisfies the trigonometric holonomic differential equation

$$DE_2 : (1+12e^{3it})f(t) + ie^{2it}f'(t) = 0$$

which leads to the third order holonomic recurrence equation

$$RE_2 : c_{n+3} + (-n-1)c_{n+1} + 12c_n = 0\ .$$

Converting the above RE_2 in terms of operators leads to

$$N^3 + (-n-1)N + 12$$

which corresponds exactly to the one found in [9].

Remark 5.6. The method presented in this section is much more time efficient than most previous methods. For the particular case described in Example 5.5 the computation time of the right factor is less than a second and needs much less memory capacity in comparison to the computation method described in [9], which needs more than 21 hours and utilizes 32 GB memory capacity to get the same result. The timing in [9] is high because the algorithm first needs to compute a left factor of a recurrence operator of order 20.

Additionally an analogous time efficient method via power series can be used to get the previous right factors. This algorithm is summarized as follows: Convert the holonomic recurrence operator `RE` given to a holonomic differential operator for its generating function (see e. g. [10]) and factorize this differential operator using van Hoeij's factorization algorithm [7]. Take the right factor of lowest order and convert it back to a recurrence operator for the coefficients. The result is a right factor of `RE`.

This method does not require to solve a differential equation, which in some instances turns out to be complicated. However, unfortunately this method does not always find a right factor even if one exists.

For the following example, a right factor cannot be found using the method involving power series which shows that this method is quite rigid whereas the new approach is more flexible.

Example 5.7. Consider the following holonomic recurrence operator

$$L = (-5-n)N^4 + (-20-4n)N^3 - 4N^2 + (4n+4)N + (n+1) \ .$$

The corresponding holonomic recurrence equation is

$$RE_1 : (n+1)c_n + (4n+4)c_{n+1} - 4c_{n+2} + (-20-4n)c_{n+3} + (-5-n)c_{n+4}$$

which converted into a differential equation with side conditions returns

$$DE_1 : \begin{cases} (1-4e^{-2it}-8e^{-3it}-e^{-4it})f(t) - i(1+4e^{-it}-4e^{-3it}-e^{-4it})f'(t) = 0 \ . \\ f(0) = f(2\pi) \ . \end{cases}$$

One of the solutions of the above DE is

$$f(t) = \frac{e^{it} - e^{-it}}{e^{2it} + 4e^{it} + 1}$$

whose Fourier coefficients in the interval $[0, 2\pi]$ satisfy the second order holonomic recurrence equation

$$RE_2 : (n+1)(n+3)c_{n+2} + 4(n+1)(n+3)c_{n+1} + (n+1)(n+3)c_n$$

from which we deduce the second order right factor of L

$$L_2 : (n+1)(n+3)N^2 + 4(n+1)(n+3)N + (n+3)(n+1) \ .$$

Conclusion and Acknowledgments

In this work, we presented an algorithm for the symbolic computation of the Fourier coefficients of trigonometric holonomic functions which finds either a closed form or at least a holonomic recurrence equation. The latter gives always an efficient way to compute a finite number of Fourier coefficients. Furthermore, an algorithm for the factorization of recurrence operators was given.

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