Axioms **2013**, *3*, 404-434; doi:10.3390/axioms2030404

OPEN ACCESS

axioms

ISSN 2075-1680

www.mdpi.com/journal/axioms

Article

On Solutions of Holonomic Divided-Difference Equations on Nonuniform Lattices

Mama Foupouagnigni [1,*], Wolfram Koepf [2], Maurice Kenfack-Nangho [1] and Salifou Mboutngam [3]

[1] Department of Mathematics, Higher Teachers' Training College, University of Yaounde I, PO Box 47, Yaounde, Cameroon; E-Mail: kenfnang@yahoo.fr

[2] Institute of Mathematics, University of Kassel, Heinrich-Plett Street 40, Kassel 34132, Germany; E-Mail: koepf@mathematik.uni-kassel.de

[3] Department of Mathematics, Higher Teachers' Training College, University of Maroua, PO Box 55, Maroua, Cameroon; E-Mail: mbsalif@yahoo.fr

* Author to whom correspondence should be addressed; E-Mail: foupouagnigni@gmail.com; Tel.: +237-99-71-17-23; Fax: +237-22-22-13-20.

Received: 28 May 2013; in revised form: 4 June 2013 / Accepted: 3 July 2013 / Published: 23 July 2013

Abstract: The main aim of this paper is the development of suitable bases that enable the direct series representation of orthogonal polynomial systems on nonuniform lattices (quadratic lattices of a discrete or a q-discrete variable). We present two bases of this type, the first of which allows one to write solutions of arbitrary divided-difference equations in terms of series representations, extending results given by Sprenger for the q-case. Furthermore, it enables the representation of the Stieltjes function, which has already been used to prove the equivalence between the Pearson equation for a given linear functional and the Riccati equation for the formal Stieltjes function. If the Askey-Wilson polynomials are written in terms of this basis, however, the coefficients turn out to be not q-hypergeometric. Therefore, we present a second basis, which shares several relevant properties with the first one. This basis enables one to generate the defining representation of the Askey-Wilson polynomials directly from their divided-difference equation. For this purpose, the divided-difference equation must be rewritten in terms of suitable divided-difference operators developed in previous work by the first author.

Keywords: Askey-Wilson polynomials; nonuniform lattices; difference equations; divided-difference equations; Stieltjes function

Classification: MSC Primary 33D45, Secondary 39A13

1. Introduction

Let $\mu(x)$ denote a nondecreasing real-valued, bounded function with a finite or an infinite number of points of increase in the interval, $[a,\, b]$. The latter interval may be infinite. We assume that moments of all orders exist, that is:

$$\mu_n = \int_a^b x^n \, d\mu(x)$$

exists for all $n \in \mathbb{N}$, with the integration defined in the Lebesgue-Stieltjes sense.

Definition 1. *The polynomial sequence, (P_n), is said to be orthogonal with respect to the measure, $d\mu(x)$, if:*

$$\begin{cases} \deg(P_n) = n,\; n \geq 0 \\ \\ \int_a^b P_n(x)\, P_m(x)\, d\mu(x) = d_n^2\, \delta_{n,m},\;\; n,\, m \geq 0 \end{cases} \tag{1}$$

where $d_n \neq 0,\; n \geq 0$ and $\delta_{n,m}$ is the Kronecker symbol.

To any sequence of moments, $(\mu_n)_n$ is associated the linear functional, $\mathcal{L}$, defined on $\mathbb{C}[x]$ by:

$$\langle \mathcal{L}, x^n \rangle = \mu_n,\; n \geq 0 \tag{2}$$

The functional, $\mathcal{L}$ (also called moment functional), is said to be regular if there exists a family, $(P_n)_n$, of polynomials orthogonal with respect to $\mathcal{L}$, that is:

$$\begin{cases} \deg(P_n) = n,\; n \geq 0 \\ \\ \langle \mathcal{L}, P_n P_m \rangle = d_n^2\, \delta_{n,m},\;\; n,\, m \geq 0,\;\; d_n \neq 0,\, n \geq 0 \end{cases} \tag{3}$$

It is well-known that the necessary and sufficient condition for $\mathcal{L}$ to be regular is that the principal submatrices of the infinite Hankel matrix, $(\mu_{j+k})_{j,k \in \mathbb{N}}$, are nonsingular. Furthermore, to the measure $d\mu(x)$ or to the corresponding regular functional, $\mathcal{L}$, is associated the so-called Stieltjes function $S(\mathcal{L})$, defined as:

$$S(x) = \int_a^b \frac{d\mu(t)}{x - t} = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}},\; x \notin [a,\, b] \tag{4}$$

If $S(x)$ is known, then the distribution function, μ, can be recovered from $S(x)$ by means of the Stieltjes inversion formula [1] (see, also, [2,3]):

$$\mu(t) - \mu(s) = \frac{1}{\pi} \lim_{y \to 0^+} \int_s^t \mathrm{Im}(S(x + i\, y)\, dx$$

Classical orthogonal polynomials of a continuous variable, $(P_n)_n$, can be characterized by the distributional differential Equation (usually called Pearson-type Equation) satisfied by their corresponding regular functional, $\mathcal{L}$, [4]:

$$\mathcal{D}(\phi \mathcal{L}) = \psi \mathcal{L} \tag{5}$$

where ϕ and ψ are polynomials with $\deg(\phi) \leq 2$ and $\deg(\psi) = 1$, with $\mathcal{D} = \frac{d}{dx}$ and while $\phi\mathcal{L}$ and $\mathcal{D}(\mathcal{L})$ are linear functionals defined by:

$$\langle \phi\mathcal{L}, P\rangle = \langle \mathcal{L}, \phi\, P\rangle,\ \langle \mathcal{D}(\mathcal{L}), P\rangle = -\langle \mathcal{L}, \mathcal{D}P\rangle, \forall P \in \mathbb{C}[x]$$

Equation (5) is equivalent to the following two Equations [4,5] (which also characterize classical orthogonal polynomials), namely, the second-order differential Equation satisfied by each P_n:

$$\phi(x)\, P_n''(x) + \psi(x)\, P_n'(x) + \lambda_n\, P_n(x) = 0,\ \lambda_n = -n\,[(n-1)\frac{\phi''}{2} + \psi'] \tag{6}$$

and the Riccati Equation satisfied by the Stieltjes function, $S(\mathcal{L}) := S(x)$:

$$\phi(x)S'(x) + [\phi'(x) - \psi(x)]S(x) + (\psi' - \frac{\phi''}{2})\mu_0 = 0 \tag{7}$$

Properties, similar to Equations (5)–(7) exist for classical orthogonal polynomials of a discrete variable [6–8], as well as for classical orthogonal polynomials of a q-discrete variable [7,9,10].

Semi-classical orthogonal polynomials of a continuous variable [4] are defined as those for which the corresponding linear functional, $\mathcal{L}$, satisfies a Pearson-type Equation (5), but with a more relaxed condition on the polynomials, ϕ and ψ, namely, $\deg(\psi) \geq 1$. This is equivalent to the Riccati Equation (7), but with the constant term, $(\psi' - \frac{\phi''}{2})\mu_0$, replaced by a polynomial, $D(x)$. Laguerre-Hahn orthogonal polynomials [4] are also defined by general forms of Equations (5) and (7).

Classical orthogonal polynomials on a nonuniform lattice satisfy an Equation of the type: [11,12,14]

$$\left\{\phi(x(s))\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)} + \frac{\psi(x(s))}{2}\left[\frac{\Delta}{\Delta x(s)} + \frac{\nabla}{\nabla x(s)}\right] + \lambda_n\right\} P_n(x(s)) = 0,\ n \geq 0 \tag{8}$$

where ϕ and ψ are polynomials of maximal degree, two and one, respectively, λ_n is a constant depending on the integer, n, and the leading coefficients, ϕ_2 and ψ_1, of ϕ and ψ:

$$\lambda_n = -\gamma_n\,(\phi_2\,\gamma_{n-1} + \psi_1\,\alpha_n) \tag{9}$$

and $x(s)$ is a nonuniform lattice defined by:

$$x(s) = \begin{cases} c_1\,q^s + c_2\,q^{-s} + c_3 & \text{if} \quad q \neq 1 \\ c_4\,s^2 + c_5\,s + c_6 & \text{if} \quad q = 1 \end{cases} \tag{10}$$

Here, Δ and ∇ are the forward and the backward operators:

$$\Delta f(x(s)) := f(x(s+1)) - f(x(s)),\ \nabla f(x(s)) := f(x(s)) - f(x(s-1))$$

and:

$$x_\mu(s) = x(s + \frac{\mu}{2}),\ \mu \in \mathbb{C}$$

where $\mathbb{C}$ is the set of complex numbers. The Lattices (10) satisfy:

$$x(s+k) - x(s) = \gamma_k\,\nabla\,x_{k+1}(s) \tag{11}$$

$$\frac{x(s+k) + x(s)}{2} = \alpha_k\,x_k(s) + \beta_k \tag{12}$$

for $k = 0,\, 1,\, \ldots,$ with the initial values:

$$\alpha_0 = 1,\ \alpha_1 = \alpha,\ \beta_0 = 0,\ \beta_1 = \beta, \gamma_0 = 0,\ \gamma_1 = 1$$

where:

$$\alpha = \begin{cases} 1 & \text{for} \quad q = 1 \\ \frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2} & \text{if} \quad q \neq 1 \end{cases}$$

In addition, the sequences, $(\alpha_k),\ (\beta_k),\ (\gamma_k)$, satisfy the following relations:

$$\begin{aligned} \alpha_{k+1} - 2\,\alpha\,\alpha_k + \alpha_{k-1} &= 0 \\ \beta_{k+1} - 2\,\beta_k + \beta_{k-1} &= 2\,\beta\,\alpha_k \\ \gamma_{k+1} - \gamma_{k-1} &= 2\,\alpha_k \end{aligned}$$

and are given explicitly by [11,12]:

$$\alpha_n = 1,\ \beta_n = \beta\, n^2,\ \gamma_n = n,\ \textit{for}\ \alpha = 1 \tag{13}$$

and:

$$\alpha_n = \frac{q^{\frac{n}{2}} + q^{-\frac{n}{2}}}{2},\ \beta_n = \frac{\beta(1-\alpha_n)}{1-\alpha},\ \gamma_n = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},\ \textit{for}\ \alpha = \frac{q^{\frac{1}{2}} + q^{-\frac{1}{2}}}{2} \tag{14}$$

By means of the companion operators, $\mathbb{D}_x$ and $\mathbb{S}_x$ [13,14], Equation (8) can be rewritten as:

$$\phi(x(s))\,\mathbb{D}_x^2 P_n(x(s)) + \psi(x(s))\,\mathbb{S}_x\mathbb{D}_x P_n(x(s)) + \lambda_n\, P_n(x(s)) = 0 \tag{15}$$

where:

$$\mathbb{D}_x\, f(x(s)) = \frac{f(x_{-1}(s+1)) - f(x_{-1}(s))}{x_{-1}(s+1) - x_{-1}(s)},\quad \mathbb{S}_x\, f(x(s)) = \frac{f(x_{-1}(s+1)) + f(x_{-1}(s))}{2} \tag{16}$$

The operators, $\mathbb{D}_x$ and $\mathbb{S}_x$, transform polynomials of degree n in $x(s)$ into a polynomial of degree $n-1$ and n in the same variable, respectively. In addition, they fulfill important relations—called product and quotient rules—which read, taking into account the shift (compared to the definition in [14]), in the definition of the above defined companion operators as:

Theorem 2. *[14]*

1. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the product rules I*

$$\mathbb{D}_x\left(f(x(s))g(x(s))\right) = \mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)) \tag{17}$$

$$\mathbb{S}_x\left(f(x(s))g(x(s))\right) = U_2(x_1(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)) \tag{18}$$

where $U_2(x(s)) = (\alpha^2-1)\,x^2(s) + 2\,\beta\,(\alpha+1)\,x(s) + \delta_x$, and δ_x is a constant term with respect to $x(s)$.

2. *The operators, $\mathbb{D}_x$ and $\mathbb{S}_x$, also satisfy the product rules II:*

$$\mathbb{D}_x\,\mathbb{S}_x = \alpha\,\mathbb{S}_x\,\mathbb{D}_x + U_1(s)\,\mathbb{D}_x^2 \tag{19}$$

$$\mathbb{S}_x^2 = U_1(s)\,\mathbb{S}_x\,\mathbb{D}_x + \alpha\,U_2(s)\,\mathbb{D}_x^2 + \mathbb{I} \tag{20}$$

where:

$$U_1(s) := U_1(x(s)) = (\alpha^2-1)\,x(s) + \beta\,(\alpha+1),\quad U_2(s) := U_2(x(s)) \tag{21}$$

For illustration, the Askey-Wilson polynomials [7] $P_n(x;a,b,c,d|q)$ are defined by:

$$P_n(x;a,b,c,d|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} \middle| q;q\right), \; x = \cos\theta \tag{22}$$

By taking $e^{i\theta} = q^s$, the lattice reads as $x(s) = \cos\theta = \frac{q^s+q^{-s}}{2}$. By using the orthogonality relation and the Pearson-type Equation satisfied by the weight of the Askey-Wilson polynomials, Foupouagnigni [14] showed that the polynomials, $P_n(x;a,b,c,d|q)$, satisfy a divided-difference equation of the type from (8) with:

$$\begin{aligned} \phi(x(s)) &= 2(dcba+1)x^2(s) - (a+b+c+d+abc+abd+acd+bcd)\,x(s) \\ &\quad + ab+ac+ad+bc+bd+cd-abcd-1 \\ \psi(x(s)) &= \frac{4(abcd-1)\,q^{\frac{1}{2}}\,x(s)}{q-1} + \frac{2(a+b+c+d-abc-abd-acd-bcd)\;q^{\frac{1}{2}}}{q-1} \end{aligned} \tag{23}$$

Costas-Santos and Marcellán [15], using the corresponding weight function, gave four equivalent characterizations for classical orthogonal polynomials on nonuniform lattices: More precisely, they proved the equivalence between the second-order divided-difference Equation (8), the orthogonality of the divided-difference derivatives, the Rodrigues-type formula, as well as the structure relation.

However, despite the important results by Costas-Santos and Marcellán, the characterization of classical orthogonal polynomials on nonuniform lattices is still incomplete. In fact, there are still missing many properties, in particular:

① The characterization in terms of the Pearson-type Equation for the corresponding regular functional (similar to Equation (5));

② The characterization in terms of the Riccati Equation for the Stieltjes function of the corresponding regular functional (similar to Equation (7));

③ The method to recover orthogonal polynomials, P_n, from the second-order divided-difference Equation (8) (similar to the one of Fuchs-Frobenius method used to solve Equation (6)); this problem has already been raised by Ismail (see [16], page 518);

④ The building of a bridge between the theory of Magnus based mainly on the Riccati Equation satisfied by the formal Stieltjes function [17,18] and the theory of orthogonal polynomials on nonuniform lattices based on the functional approach. Such a bridge would enable the construction of concrete examples of semi-classical and Laguerre-Hahn orthogonal polynomials, since it is easier and more convenient to modify the functional rather than the formal Stieltjes function.

These characterizations are very important, since they are the key for the definition and the characterizations of new classes of orthogonal polynomials on nonuniform lattices, namely, the semi-classical and the Laguerre-Hahn orthogonal polynomials on nonuniform lattices.

To provide a solution to the four problems raised above and taking into consideration the expansions of P_n in the monomial basis $(x(s)^n)_n$,

$$P_n(x(s)) = \sum_{k=0}^{n} a_k\, x(s)^k$$

and the one of the formal Stieltjes function [18]:

$$S_0\left[\mathcal{L}\right](x(z)) = \int_a^b \frac{d\mu(x(s))}{x(z)-x(s)} = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}(z)}, \text{ with } \mu_n = \int_a^b x^n(s)d\mu(x(s)),\ x(z) \notin [a,\, b] \quad (24)$$

there is a strong need to control the actions of the operators, $\mathbb{D}_x$ and $\mathbb{S}_x$, on the monomial, $x^n(s)$, and also on the function, $\frac{1}{x^n(s)}$. Unfortunately, the application of these operators to the monomial, $x^n(s)$, produces a linear combination (with complicated coefficients) of all monomials of a degree less than or equal to $n-1$ and n, respectively. Also, the application of $\mathbb{D}_x$ and $\mathbb{S}_x$ to the function, $\frac{1}{x^n(s)}$, produces a rational function of the variable, $x(s)$, with coefficients being very difficult to handle [14]. These constraints makes the monomial basis, $(x^n(s))_n$, not appropriate for the aforementioned operators and justifies our decision to search for an appropriate basis for the operators, $\mathbb{D}_x$ and $\mathbb{S}_x$.

The aim of this work is:

① To provide an appropriate basis for the companion operators, that is, a basis $(F_n(x(s)))_n$, such that each $F_n(x(s))$ is a polynomial of degree, n, in $x(s)$ fulfilling:

$$\begin{aligned}
\mathbb{D}_x F_n(x(s)) &= a_n\, F_{n-1}(x(s)),\ \ \mathbb{D}_x \frac{1}{F_n(x(s))} = \frac{b_n}{F_{n+1}(x(s))} \\
\mathbb{S}_x F_n(x(s)) &= c_n\, F_n(x(s)) + d_n\, F_{n-1}(x(s)),\ \mathbb{S}_x \frac{1}{F_n(x(s))} = \frac{e_n}{F_n(x(s))} + \frac{f_n}{F_{n+1}(x(s))}
\end{aligned}$$

where a_n, b_n, c_n, d_n, e_n and f_n are given constants.

② To provide an algorithmic method to solve Equation (15) as a series in terms of the new basis and to extend this result to solve arbitrary linear divided-difference equations with polynomial coefficients involving only products of operators, $\mathbb{D}_x$ and $\mathbb{S}_x$.

③ To use another appropriate basis for the operators, $\mathbb{D}_x^2$ and $\mathbb{S}_x\,\mathbb{D}_x$, to derive Representation (15) for the Askey-Wilson polynomials from the hypergeometric Representation (22) without making use of the weight function.

④ To solve explicitly an Equation of type (15) and to extend this result to solve arbitrary linear divided-difference equations with polynomial coefficients involving only products of operators $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$.

⑤ To provide a new representation of the formal Stieltjes function of a given linear functional on a nonuniform lattice and deduce from it various important properties connecting the functional approach and the one based on the Riccati Equation for the formal Stieltjes function.

The content of this paper is organized as follows. In Section 1, we recall necessary preliminaries, while in the second section, we provide the basis, $(F_k)_k$, compatible with the companion operators, $\mathbb{D}_x$ and $\mathbb{S}_x$. The third section deals with the algorithmic series solutions of divided-difference equations in terms of the basis, $(F_k)_k$. In Section 4, we give the second basis, $(B_k)_k$, compatible not with the companion operators, but rather with their products, $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$, and use this basis in the fifth section to find the algorithmic series solutions of some divided difference equations in terms of the basis, $(B_k)_k$.

In the last section, we apply the basis, $(F_k)_k$, to provide new representation of the formal Stieltjes series and deduce its corresponding properties. Basic exponential and basic trigonometric functions have also been expanded in terms of the basis, $(F_k)_k$.

2. A New Basis Compatible with the Companion Operators

Using the generalized power on the lattice, $x_m(z),\ m \geq 0$, defined by Suslov [12] (see, also, [11]) as:

$$[x_m(z) - x_m(s)]^{(n)} = \prod_{j=0}^{n-1}[x_m(z) - x_m(s-j)],\ n \geq 1,\ m \geq 0,\ [x_m(z) - x_m(s)]^{(0)} \equiv 1 \tag{25}$$

we define the function, $F_n(s)$, by:

$$F_n(s) = (-1)^n\, [x(\epsilon) - x_{n-1}(s)]^{(n)},\ n \geq 1,\ F_0(s) \equiv 1 \tag{26}$$

where ϵ is the unique solution of the Equation (in the variable, t):

$$x_1(t) = x(t) \Longleftrightarrow x(t + \frac{1}{2}) = x(t) \tag{27}$$

provided that the coefficients, c_j, of (10) fulfill $c_1\, c_2 \neq 0$ or $c_4 \neq 0$ for the quadratic lattice of the q-discrete and discrete variable, respectively. More precisely, ϵ fulfills:

$$q^{2\epsilon} = \frac{c_2}{c_1} q^{\frac{-1}{2}} \tag{28}$$

for the q-quadratic lattice, $x(s) = c_1\, q^s + c_2\, q^{-s} + \frac{\beta}{1-\alpha},\ q \neq 1,\ c_1\, c_2 \neq 0$, or the relation:

$$\text{and } \epsilon = -\frac{1}{4} - \frac{c_5}{2c_4} \tag{29}$$

for the quadratic lattice, $x(s) = c_4\, s^2 + c_5\, s + c_6,\ c_4 \neq 0$.

Although the function, $F_n(s)$, from Relation (26) does not appear explicitly as a function of $x(s)$, $F_n(s)$, as will be shown in the following theorem, it is a polynomial of degree n in the variable, $x(s)$. Therefore, we adopt the notation, $F_n(x(s))$, instead of $F_n(s)$, and obtain the following results:

Theorem 3.

$$\mathbb{D}_x F_n(x(s)) = \gamma_n\, F_{n-1}(x(s)) \tag{30}$$

$$\mathbb{S}_x F_n(x(s)) = \alpha_n\, F_n(x(s)) + \frac{\gamma_n}{2}\, \nabla x_{n+1}(\epsilon) F_{n-1}(x(s)) \tag{31}$$

$$\mathbb{D}_x \frac{1}{F_n(x(s))} = -\frac{\gamma_n}{F_{n+1}(x(s))} \tag{32}$$

$$\mathbb{S}_x \frac{1}{F_n(x(s))} = \frac{\alpha_n}{F_n(x(s))} + \frac{\gamma_n}{2} \frac{\nabla x_{n+2}(\epsilon)}{F_{n+1}(x(s))} \tag{33}$$

$$\begin{aligned}
F_{n+1}(x(s)) &= (x(s) - x_{n+1}(\epsilon))\, F_n(x(s)) = \prod_{j=1}^{n+1} (x(s) - x_j(\epsilon)),\ n \geq 0 && (34)\\
F_n(x_k(\epsilon)) &\neq 0,\ \ \forall n \geq 0,\ \forall k > n \geq 0 && (35)\\
\frac{F_n(x(s))}{F_1(x(s))} &= F_{n-1}(x(s)) + \sum_{j=0}^{n-2} E_j\, F_j(x(s)),\ \textit{with } E_j = \prod_{i=j+2}^{n} (x(\epsilon) - x_i(\epsilon)) && (36)\\
\mathbb{D}_x^k F_n(x(s)) &= \left[\prod_{j=0}^{k-1} \gamma_{n-j}\right] F_{n-k}(x(s)) = \frac{\gamma_n!}{\gamma_{n-k}!} F_{n-k}(x(s)),\ k \leq n && (37)\\
F_k(x(s))\, F_n(x(s)) &= \sum_{j=0}^{k} C_{n+j}(n,k)\, F_{n+j}(x(s)),\ \textit{with} && (38)\\
C_{n+j}(n,k) &= \Theta_{\epsilon + \frac{n+j}{2}} \circ \Theta_{\epsilon + \frac{n+j-1}{2}} \ldots \circ \Theta_{\epsilon + \frac{n+1}{2}} F_k\left(x(s)\right)|_{s=\epsilon + \frac{n+j+1}{2}},\ 1 \leq j \leq k \leq n\\
C_n &= F_k\left(x_{n+1}(\epsilon)\right)
\end{aligned}$$

In order to prove Theorem 3, we first prove the following lemmas:

Lemma 4. *The following relation holds:*

$$[x_m(z) - x_m(s)]^{(n)} = \prod_{j=0}^{n-1} [x_{m-n+1}(z+j) - x_{m-n+1}(s)],\ n,\ m \in \mathbb{N} \tag{39}$$

where $\mathbb{N}$ is the set of nonnegative integer numbers.

Lemma 5. *The following relations are satisfied:*

$$\mathbb{D}_x f_n(x_{n-1}(z), x_{n-1}(s)) = -\gamma_n\, f_{n-1}\left(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s)\right) \tag{40}$$

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \alpha_n\, f_n\left(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)\right)\\
&\quad -\frac{\gamma_n}{2} \nabla x_{2n}(z) f_{n-1}\left(x_{n-2}\left(z+\frac{1}{2}\right), x_{n-2}(s)\right)
\end{aligned} \tag{41}$$

where the function, f_n, is defined as:

$$f_n(x_m(z), x_m(s)) = \prod_{j=0}^{n-1} [x_m(z) - x_m(s-j)],\ n \geq 1,\ m \geq 0,\ f_0(x_m(z), x_m(s)) \equiv 1 \tag{42}$$

Proof of Lemma 4

Proof: Relation (39) is obtained by iteration of the following relation:

$$[x_m(s) - x_m(z)]^{(n)} = [x_{m-n+1}(s+n-1) - x_{m-n+1}(z)]\, [x_{m-1}(s) - x_{m-1}(z)]^{(n-1)}$$

given by Suslov ([12], Equation 2.8, page 235). Therefore:

$$f_n(x_m(z), x_m(s)) = g_n(x_m(z), x_m(s)),\ n,\ m \geq 0$$

where the function, g_n, is defined as:

$$g_n(x_m(z), x_m(s)) = \prod_{j=0}^{n-1} [x_{m-n+1}(z+j) - x_{m-n+1}(s)],\ n \geq 1,\ m \geq 0,\ g_0((x_m(z), x_m(s))) \equiv 1 \qquad (43)$$

Proof of Lemma 5

Proof: Relation (40) is proven by combination of the definition of $\mathbb{D}_x f(x(s))$, given by (16), and the following relation, given by Suslov ([12], Equation (2.22), page 239):

$$\frac{\Delta_s [x_m(s) - x_m(z)]^{(n)}}{\Delta_s x_{m-n+1}(s)} = -\gamma(n)\, [x_m(s) - x_m(z)]^{(n-1)}$$

but with $m = n-1$ and s replaced by $s - \frac{1}{2}$. Here, Δ_s means that the forward operator acts on the variable, s.

The proof of Relation (41) will be done in the following steps:

In the first step, we apply operator $\mathbb{S}_x$ on both sides of Relation (42) and adjust the index, j, to obtain:

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \frac{1}{2}\prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j+\frac{1}{2})) + \frac{1}{2}\prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&= \frac{1}{2}[x_{n-1}(z) - x_{n-1}(s+\frac{1}{2})]\prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&+ \frac{1}{2}[x_{n-1}(z) - x_{n-1}(s-n+\frac{1}{2})]\prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&= \frac{1}{2}[2\,x_{n-1}(z) - x_{n-1}(s-n+\frac{1}{2}) - x_{n-1}(s+\frac{1}{2})]\prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2}))
\end{aligned}$$

In the second step, we use the relation:

$$x_{n-1}(s-n+\frac{1}{2}) + x_{n-1}(s+\frac{1}{2}) = x_n(s-n) + x_n(s) = 2\alpha_n\, x_{2n}(s-n) + 2\beta_n = 2\alpha_n\, x(s) + 2\beta_n$$

obtained from (12) and the relation, $x_\mu(z) = x_\mu(z + \frac{\mu}{2})$, to get:

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\, f_{n-1}(x_{n-1}(z), x_{n-1}(s-\frac{1}{2}))
\end{aligned}$$

In the third step, we use Relation (39) to transform the previous Equation into:

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\, f_{n-1}(x_{n-1}(z), x_{n-1}(s-\frac{1}{2})) \\
&= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\, g_{n-1}(x_{n-1}(z), x_{n-1}(s-\frac{1}{2})) \\
&= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n] \prod_{j=0}^{n-2} (x_1(z+j) - x_1(s-\frac{1}{2})) \\
&= \left[\alpha_n [x(z+\frac{1}{2}+n-1) - x(s)] + x_{n-1}(z) - \alpha_n\, x(z+\frac{1}{2}+n-1) - \beta_n\right] \\
&\times \prod_{j=0}^{n-2} (x(z+\frac{1}{2}+j) - x(s)) \\
&= \alpha_n \prod_{j=0}^{n-1} (x(z+\frac{1}{2}+j) - x(s)) \\
&+ \left[x_{n-1}(z) - \alpha_n\, x(z-\frac{1}{2}+n) - \beta_n\right] \prod_{j=0}^{n-2} (x(z+\frac{1}{2}+j) - x(s)) \\
&= \alpha_n\, g_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) \\
&+ \left[x_{n-1}(z) - \alpha_n\, x(z-\frac{1}{2}+n) - \beta_n\right] g_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s))
\end{aligned}$$

Therefore, using, again, Relation (39), we get:

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \alpha_n\, f_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) \\
&+ \left[x_{n-1}(z) - \alpha_n\, x_n(z-\frac{1}{2}+\frac{n}{2}) - \beta_n\right] f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s))
\end{aligned} \tag{44}$$

In the fourth step, we use the following relation obtained from (12):

$$x(z-\frac{1}{2}+\frac{n}{2}) + x(z-\frac{1}{2}+\frac{n}{2}+n) = 2\alpha_n\, x_n(z-\frac{1}{2}+\frac{n}{2}) + 2\beta_n$$

to Transform (44) as:

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \alpha_n\, f_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) \\
&+ \left[x_{n-1}(z) - \frac{x_n(z-\frac{1}{2}) + x_n(z-\frac{1}{2}+n)}{2}\right] f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s)) \\
&= \alpha_n\, f_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) \\
&+ \left[\frac{x_n(z-\frac{1}{2}) - x_n(z-\frac{1}{2}+n)}{2}\right] f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s)) \\
&= \alpha_n\, f_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) \\
&+ \left[\frac{x_{n-1}(z) - x_{n-1}(z+n)}{2}\right] f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s))
\end{aligned}$$

In the last step, we use the following relation obtained from (11):

$$\frac{x_{n-1}(z) - x_{n-1}(z+n)}{2} = -\gamma_n\, \nabla x_{2n}(z)$$

to obtain the Equation:

$$\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) = \alpha_n\, f_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) - \frac{\gamma_n}{2}\, \nabla x_{2n}(z) f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s))$$

and the proof is complete. □

Proof of Theorem 3

Proof: By replacing: z by $z - \frac{n-1}{2}$ in Equations (40) and (41), we obtain:

$$\begin{aligned} \mathbb{D}_x f_n(x(z), x_{n-1}(s)) &= -\gamma_n\, f_{n-1}(x(z), x_{n-2}(s)) \\ \mathbb{S}_x f_n(x(z), x_{n-1}(s)) &= \alpha_n\, f_n(x_1(z), x_{n-1}(s)) - \frac{\gamma_n}{2}\, \nabla x_{n+1}(z) f_{n-1}(x(z), x_{n-2}(s)) \end{aligned}$$

Therefore, for $\mathbb{S}_x f_n(x(z), x_{n-1}(s))$ to be a linear combination of $f_n(x(z), x_{n-1}(s))$ and $f_{n-1}(x(z), x_{n-2}(s))$, it is necessary for the parameter, z, to be a solution of the following Equation in the variable, t:

$$x_1(t) = x(t) \Longleftrightarrow x(t+\frac{1}{2}) = x(t)$$

This solution is unique, denoted by ϵ (see Equation (27)), provided that the coefficients, c_j, of (10) fulfill $c_1\, c_2 \neq 0$ or $c_4 \neq 0$ for the quadratic lattice of the q-discrete and discrete variable, respectively. The resulting basis reads:

$$F_n(x(s)) := (-1)^n\, f_n(x(\epsilon), x_{n-1}(s))$$

This basis fulfills Relations (30) and (31).

The proof of Relations (32) and (33) is similar to those of Relations (30) and (31).

For the proof of Relation (34), we use Relations (39), (42) and (43) for fixed nonnegative integer, n, and the fact that $x_1(\epsilon) = x(\epsilon)$ with $x_\mu(z) = x(z + \frac{\mu}{2})$ to obtain:

$$\begin{aligned} F_{n+1}(x(s)) &= (-1)^{n+1}\, f_{n+1}(x(\epsilon), x_n(s)) \\ &= (-1)^{n+1}\, f_{n+1}(x_1(\epsilon), x_n(s)) \\ &= (-1)^{n+1}\, f_{n+1}(x_n(\epsilon + \frac{1-n}{2}), x_n(s)) \\ &= (-1)^{n+1}\, g_{n+1}(x_n(\epsilon + \frac{1-n}{2}), x_n(s)) \\ &= (-1)^{n+1} \prod_{j=0}^{n} (x(\epsilon + \frac{1-n}{2} + j) - x(s)) \\ &= ((x(s) - x_{n+1}(\epsilon) - x(s)))\, (-1)^n \prod_{j=0}^{n-1} (x(\epsilon + \frac{1-n}{2} + j) - x(s)) \\ &= ((x(s) - x_{n+1}(\epsilon) - x(s)))\, (-1)^n\, g_n(x_{n-1}(\epsilon + \frac{1-n}{2}), x_{n-1}(s)) \\ &= ((x(s) - x_{n+1}(\epsilon) - x(s)))\, (-1)^n\, f_n(x_{n-1}(\epsilon + \frac{1-n}{2}), x_{n-1}(s)) \\ &= ((x(s) - x_{n+1}(\epsilon) - x(s)))\, (-1)^n\, f_n(x(\epsilon), x_{n-1}(s)) \\ &= (x(s) - x_{n+1}(\epsilon))\, F_n(x(s)) \end{aligned}$$

Relation (35) is satisfied, since, for integers, n, j and k, such that $k \geq 0$ and $1 \leq j \leq n$, we get, by direct computation using (11), that:

$$x_{n+k+1}(\epsilon) - x_j(\epsilon) \neq 0$$

Therefore, $F_n(x_k(\epsilon)) \neq 0,\ k > n$.

Relation (36) is proven by induction on n. Relation (37) is obtained by Iterating (30). We split the proof of Relation (38) into three steps:

In the first step, for fixed $n,\ k \geq 1$, we expand $F_k\, F_n$ in the basis F_l:

$$F_k(x(s))\, F_n(x(s)) = \sum_{l=0}^{n+k} C_l(n,k)\, F_l(x(s)) \tag{45}$$

and use the following relation, due to (34):

$$F_k(x_j(\epsilon)) = 0,\ 1 \leq j \leq k \tag{46}$$

to get $C_0(n,k) = F_k(x_1(\epsilon))\, F_n(x_1(\epsilon)) = 0$. Considering (45) for $x(s) = x_2(\epsilon)$ and $C_0(n,k) = 0$, we get—using, again, (46):

$$C_1(n,k)F_1(x_2(\epsilon)) = F_k(x_2(\epsilon))\, F_n(x_2(\epsilon)) = 0,\ n \geq 2$$

Therefore, $C_1(n,k) = 0$, thanks to (35). Progressively, we obtain in a similar way for a fixed integer, j, using (35), (45) and (46), that:

$$C_0(n,k) = C_1(n,k) = \cdots = C_j(n,k) = 0,\ n \geq j+1$$

In the second step, we rewrite, accordingly, Relation (45):

$$F_k(x(s))\, F_n(x(s)) = \sum_{j=0}^{k} C_{n+j}(n,k)\, F_{n+j}(x(s))$$

and obtain, using (34):

$$F_k(x(s)) = \sum_{j=0}^{k} C_{n+j}(n,k)\, \frac{F_{n+j}(x(s))}{F_n(x(s))} = \sum_{j=0}^{k} C_{n+j}(n,k)\, g_{n+j,n+1}(x(s)) \tag{47}$$

where:

$$g_{n,j}(x(s)) = \prod_{l=j}^{n} (x(s) - x_l(\epsilon)),\ 1 \leq j \leq n,\ g_{n,n+1}(x(s)) \equiv 1,\ g_{n,n+l}(x(s)) \equiv 0,\ l > 1 \tag{48}$$

Use of Equation (47) for $x(s) = x_{n+1}(\epsilon)$ gives:

$$C_n(n,k) = F_k(x_{n+1}(\epsilon))$$

taking into account Relation (46) and the fact that $g_{n,n+1} \equiv 1$. In the third step, we apply the operator, Θ_a (defined in (59)) to (47), and use the relation:

$$\Theta_{\epsilon+\frac{j}{2}}\, g_{n,j}(x(s)) = g_{n,j+1}(x(s)),\ 1 \leq j \leq n$$

derived by direct computation, to obtain the relation:

$$\Theta_{\epsilon+\frac{n+1}{2}}\, F_k(x(s)) = \sum_{j=1}^{k} C_{n+j}(n,k)\, g_{n+j,n+2}(x(s)) \tag{49}$$

from which we deduce using, again, (46) that:

$$C_{n+1}(n,k) = \Theta_{\epsilon+\frac{n+1}{2}}\, F_k(x_{n+2}(\epsilon))$$

The remaining coefficients, $C_{n+l}(n,k),\ l \geq 2$, are obtained in the same way by successive application of $\Theta_{\epsilon+\frac{n+l}{2}},\ 2 \leq l \leq k$ on (49) and use of the Equations, $g_{n,j}(x_j(\epsilon)) = 0,\ 1 \leq j \leq n$. $\square$

As corollary of Theorem 3 (Equation (34)), we would like to give, explicitly, a representation of the basis, F_n.

Corollary 6. ① *The basis, F_n, is explicitly defined on the lattices, $x(s) = c_1q^{-s} + c_2q^s + c_3$ (with $c_1c_2 \neq 0$), by:*

$$F_n(x(s)) = \left(-c_1q^{-(\frac{n}{4}+\epsilon+\frac{1}{4})}\right)^n \left(\frac{c_2}{c_1}q^{\epsilon+\frac{1}{2}}q^s; q^{1/2}\right)_n \left(q^{\epsilon+\frac{1}{2}}q^{-s}; q^{1/2}\right)_n \tag{50}$$

$$= \left(-c_1q^{-\epsilon}\right)^n \left(q^{\epsilon+\frac{1-n}{2}}q^{-s}, \frac{c_2}{c_1}q^{\epsilon+\frac{1-n}{2}}q^s; q\right)_n \tag{51}$$

where ϵ is defined by (28) and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$.

② *In the particular case of the Askey-Wilson lattice, $x(s) = \frac{q^s+q^{-s}}{2}$ ($c_1 = c_2 = \frac{1}{2}$, $c_3 = 0$, $\epsilon = -\frac{1}{4}$), the previous Equations read:*

$$F_n(x(s)) = \left(-\frac{q^{-\frac{n}{4}}}{2}\right)^n \left(q^{\frac{1}{4}}q^s; q^{\frac{1}{2}}\right)_n \left(q^{\frac{1}{4}}q^{-s}; q^{\frac{1}{2}}\right)_n \tag{52}$$

$$= \left(-\frac{q^{\frac{1}{4}}}{2}\right)^n \left(q^{\frac{1}{4}-\frac{n}{2}}q^s, q^{\frac{1}{4}-\frac{n}{2}}q^{-s}; q\right)_n \tag{53}$$

③ *The basis, F_n, is explicitly defined on the lattices, $x(s) = c_1s^2 + c_2s + c_3$ (with $c_1c_2 \neq 0$), by:*

$$F_n(x(s)) = \left(-\frac{c_1}{4}\right)^n \left(-2s + \frac{1}{2} - \frac{c_2}{c_1}\right)_n \left(2s + \frac{1}{2} + \frac{c_2}{c_1}\right)_n \tag{54}$$

$$= (-c_1)^n \left(s + \frac{1}{4} - \frac{n}{2} - \frac{c_2}{2c_1}\right)_n \left(-s + \frac{1}{4} - \frac{n}{2} + \frac{c_2}{2c_1}\right)_n \tag{55}$$

where the Pochhammer symbol, $(a)_n$, is defined as $(a)_n = \prod_{k=0}^{n-1}(a+k)$, $n \geq 1$, $(a)_0 = 1$.

④ *In the particular case of the Racah lattice, $x(s) = s(s+\gamma+\delta+1)$ ($c_1 = 1$, $c_2 = \gamma+\delta+1$, $c_3 = 0$, $\epsilon = -\frac{1}{4} - \frac{\gamma+\delta+1}{2}$), the previous Equations read:*

$$F_n(x(s)) = \left(\frac{-1}{4}\right)^n \left(-2s - \gamma - \delta - \frac{1}{2}\right)_n \left(2s + \gamma + \delta + \frac{3}{2}\right)_n$$

$$= (-1)^n \left(s - \frac{1}{4} - \frac{n}{2} - \frac{\gamma+\delta}{2}\right)_n \left(-s + \frac{3}{4} - \frac{n}{2} + \frac{\gamma+\delta}{2}\right)_n$$

Remark 7. *It should be mentioned that for the specific case of the Askey-Wilson lattice our basis, F_n, coincides (up to a multiplicative factor) to the basis, ϕ_n, used by Ismail [19] (Equation (1.4), page 261):*

$$\phi_n(\cos\theta) = \left(q^{\frac{1}{4}}q^s; q^{\frac{1}{2}}\right)_n \left(q^{\frac{1}{4}}q^{-s}; q^{\frac{1}{2}}\right)_n = \left(-\frac{q^{\frac{-n}{4}}}{2}\right)^{-n} F_n(x),\ x = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{q^s+q^{-s}}{2}$$

to provide the Taylor expansion of a polynomial in terms of the basis, ϕ_n.

3. Algorithmic Series Solutions of Divided-Difference Equations

In this section, we provide an algorithmic method to solve a divided-difference equation in terms of the bases, F_n and $B_n(a,s)$, with the latter, which is defined in Proposition 14, being more appropriate for the Askey-Wilson lattice.

3.1. Algorithmic Series Solutions of Divided-Difference Equations in Terms of F_n

The basis, F_n, is relevant for the companion operators and provides a method to obtain series solutions of divided-difference equations.

Theorem 8.
If:

$$y(x(s)) = \sum_{k=0}^{\infty} d_k\, F_k(x(s)) \tag{56}$$

is a polynomial solution of the Equation:

$$\phi(x(s))\,\mathbb{D}_x^2 y(x(s)) + \psi(x(s))\,\mathbb{S}_x\mathbb{D}_x y(x(s)) + \lambda\, y(x(s)) = 0 \tag{57}$$

where λ is a constant, ϕ and ψ are polynomials a degree of at most two and one, respectively, and given by:

$$\phi(x(s)) = \phi_2\, F_2(x(s)) + \phi_1\, F_1(x(s)) + \phi_0, \psi(x(s)) = \psi_1\, F_1(x(s)) + \psi_0$$

then, the coefficients, $(d_n)_n$, satisfy a second-order recurrence Equation:

$$A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k = 0,\ k \geq 0 \tag{58}$$

with:

$$\begin{aligned} A_k &= \left[\phi\left(x_{k+1}(\epsilon)\right) + \frac{\nabla x_{k+2}(\epsilon)}{2}\psi\left(x_{k+1}(\epsilon)\right)\right]\gamma_{k+1}\,\gamma_{k+2} \\ B_k &= \left[\gamma_k\,\Theta_{\epsilon+\frac{k}{2}}\phi\left(x_{k+1}(\epsilon)\right) + \alpha_k\,\psi\left(x_{k+1}(\epsilon)\right) + \frac{\gamma_k \nabla x_{k+1}(\epsilon)}{2}\psi_1\right]\gamma_{k+1} \\ C_k &= \gamma_k\,\gamma_{k-1}\,\phi_2 + \gamma_k\,\alpha_{k-1}\psi_1 + \lambda \end{aligned}$$

where:

$$\Theta_a f(x(s)) = \frac{f(x(s)) - f(x(a))}{x(s) - x(a)} \tag{59}$$

Proof: In the first step, we apply the companion operators to (56) and, taking into Account (30) and (31), we get:

$$\mathbb{D}_x^2 y(x(s)) = \sum_{k=2}^{\infty} d_k\,\gamma_k\,\gamma_{k-1}\,F_{k-2}(x(s)) \tag{60}$$

$$\mathbb{S}_x\mathbb{D}_x y(x(s)) = \sum_{k=1}^{\infty} d_k\,\gamma_k\,\alpha_{k-1}\,F_{k-1}(x(s)) + \sum_{k=2}^{\infty}\frac{d_k\,\gamma_k\,\gamma_{k-1}\,\nabla x_k(\epsilon)}{2}F_{k-2}(x(s)) \tag{61}$$

In the next step, we use (60) and (61) in (57) and the following relations obtained by Iterating (34):

$$\begin{aligned} F_1(x(s))\,F_n(x(s)) &= F_{n+1}(x(s)) + F_1(x_{n+1}(\epsilon))\,F_n(x(s)) \\ F_2(x(s))\,F_n(x(s)) &= F_{n+2}(x(s)) + \Theta_{\epsilon+\frac{n+1}{2}}F_2\left(x_{n+2}(\epsilon)\right)\,F_{n+1}(x(s)) \\ &\quad + F_2(x_{n+1}(\epsilon))\,F_n(x(s)) \end{aligned}$$

to get an Equation of type:

$$\sum_{n=0}^{\infty} A_{k-2}\,d_k\,F_{k-2}(x(s)) + B_{k-1}\,d_k\,F_{k-1}(x(s)) + C_k\,d_k\,F_k(x(s)) = 0,\ \text{with } A_{-j} = B_{-j} = 0,\ j \geq 1$$

The proof is completed by transforming the previous Equation into:

$$\sum_{k=0}^{\infty}(A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k)\, F_k(x(s)) = 0$$

and using the fact that $(F_k)_k$ is a basis of $\mathbb{C}[x(s)]$. □

As an application of Theorem 8, in the following proposition, we provide two linearly independent solutions of Equation (8) (or Equation (15)), where ϕ and ψ are those coefficients of the Askey-Wilson polynomials given by (23) with $b = aq^{\frac{1}{2}}$ and $d = cq^{\frac{1}{2}}$.

Proposition 9. *For $b = aq^{\frac{1}{2}}$ and $d = cq^{\frac{1}{2}}$, two linearly-independent solutions of the divided-difference Equation (8) for the Askey-Wilson polynomials are represented in terms of the basis, F_n, as:*

$$\begin{aligned} P_n(x(s); a, c \mid q) &= \sum_{k=0}^{n} \frac{(q^{-\frac{n}{2}}; q^{\frac{1}{2}})_k (-acq^{\frac{n}{2}}; q^{\frac{1}{2}})_k (-2)^k q^{\frac{k(k+2)}{4}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_k (-q^{\frac{1}{2}}; q^{\frac{1}{2}})_k (aq^{\frac{1}{4}}; q^{\frac{1}{2}})_k (cq^{\frac{1}{4}}; q^{\frac{1}{2}})_k} F_k(x(s)) \\ &= {}_4\phi_3\left(\begin{matrix} q^{-\frac{n}{2}}, -ac\,q^{\frac{n}{2}}, q^{\frac{1}{4}}e^{i\theta}, q^{\frac{1}{4}}e^{-i\theta} \\ -q^{\frac{1}{2}}, a\,q^{\frac{1}{4}}, c\,q^{\frac{1}{4}} \end{matrix} \,\middle|\, q^{\frac{1}{2}}; q^{\frac{1}{2}}\right) \end{aligned} \tag{62}$$

$$\begin{aligned} Q_n(x(s); a, c \mid q) &= \sum_{k=0}^{+\infty} \frac{(-q^{-\frac{n}{2}}; q^{\frac{1}{2}})_k (acq^{\frac{n}{2}}; q^{\frac{1}{2}})_k (-2)^k q^{\frac{k(k+2)}{4}}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_k (-q^{\frac{1}{2}}; q^{\frac{1}{2}})_k (aq^{\frac{1}{4}}; q^{\frac{1}{2}})_k (cq^{\frac{1}{4}}; q^{\frac{1}{2}})_k} F_k(x(s)) \\ &= {}_4\phi_3\left(\begin{matrix} -q^{-\frac{n}{2}}, ac\,q^{\frac{n}{2}}, q^{\frac{1}{4}}e^{i\theta}, q^{\frac{1}{4}}e^{-i\theta} \\ -q^{\frac{1}{2}}, a\,q^{\frac{1}{4}}, c\,q^{\frac{1}{4}} \end{matrix} \,\middle|\, q^{\frac{1}{2}}; q^{\frac{1}{2}}\right) \end{aligned} \tag{63}$$

where $e^{i\theta} = q^s$, $P_n(x(s); a, c \mid q)$ is a polynomial and $Q_n(x(s); a, c \mid q)$ is not.

Proof: Following the method described in the previous theorem, we obtain for ϕ and ψ, given by (23), with $b = aq^{\frac{1}{2}}$ and $d = cq^{\frac{1}{2}}$, the second-order difference Equation for the expansion coefficient, $d_{n,k}$ (see (58)):

$$\begin{aligned} &4\,p^{4k+10}\,(p^4-1)^3\,(p^{4k}-p^{4n})(a^2c^2p^{4n+4k}-1)\,d_n(k) \\ +\;& 2\,p^{4n+2k+5}\,(p^4-1)^3(p^2+1)\,(p^{4k+4}-1)\,(cp^{2k+1}-1)\,(ap^{2k+1}-1)\,(acp^{4k+2}-1)\,d_n(k+1) \\ +\;& p^{4n}\,(p^4-1)^3(p^{4k+4}-1)\,(p^{4k+8}-1)\,(cp^{2k+1}-1)\,(cp^{2k+3}-1)\,(ap^{2k+1}-1)(ap^{2k+3}-1)\,d_n(k+2) = 0 \end{aligned} \tag{64}$$

with the change of variable $p = q^{\frac{1}{4}}$.

To solve this recurrence Equation, we use the refined version of q-Petkovšek's algorithm published by Horn [20,21] and implemented in Maple by Sprenger [23,24]) (in his package `qFPS.mpl`) by the command `qHypergeomsolveRE`, which finds the q-hypergeometric term solutions of q-holonomic recurrence Equations. We obtain two solutions expressed in terms of p-Pochhammer and p^2-Pochhammer terms. Details of this computation can be found in a Maple file made available on [22]. Finally, we transform these solutions using the relation (see [25], Equation 1.2.40, page 6):

$$(a, p)_k\,(-a, p)_k = (a^2, p^2)_k \tag{65}$$

and the relation obtained by direct computation:

$$(1+p^{2k})(1+p^k)\,(-1, p)_k\,(p, p)_k\,(-1, p^2)_k = 4\,(p^4, p^4)_k$$

to obtain the coefficients of F_k in the Expansions (62) and (63), given above, after replacing p by $q^{\frac{1}{4}}$.

The q-hypergeometric representations of P_n and Q_n are deduced by direct computation using the Expansions (62) and (63) and Proposition 6. These solutions are linearly-independent, since the first is a polynomial, while the second is an infinite series expansion in terms of F_k. □

Remark 10.

① *For fixed positive integer, n, the polynomial solution given by (62) is proportional to the Askey-Wilson polynomial given by (22) with $b = aq^{\frac{1}{2}}$ and $d = cq^{\frac{1}{2}}$.*

② *The non-polynomial solution of Equation (64) given by (63), which is convergent, proves clearly that our method described in the previous theorem can be applied to recover convergent series solutions of divided difference Equations in terms of the basis, F_n.*

The previous theorem can be extended to solve divided-difference Equations of arbitrary order with polynomial coefficients, using Equations (37) and (38).

Theorem 11. *The coefficients c_n of a polynomial solution:*

$$y(x(s)) = \sum_{n=0}^{\infty} c_n\, F_n(x(s)) \tag{66}$$

of any divided-difference Equation of the form:

$$\sum_{i,j=0}^{N} P_{i,j}(x(s))\, \mathbb{S}_x^i\, \mathbb{D}_x^j\, y(x(s)) = Q(x(s)) \tag{67}$$

where $k \in \mathbb{N}$ and $P_{i,j}(x(s))$ and $Q(x(s))$ are polynomials of arbitrary (but fixed) degree in the variable, $x(s)$, are solution of a linear difference Equation.

Proof: Equation (67) can be transformed into an Equation of the type:

$$\sum_{i=0}^{1} \sum_{j=0}^{M} \tilde{P}_{i,j}(x(s))\, \mathbb{S}_x^i\, \mathbb{D}_x^j\, y(x(s)) = Q(x(s)) \tag{68}$$

where $M \in \mathbb{N}$ and $\tilde{P}_{i,j}(x(s))$ are polynomials of arbitrary (but fixed) degree in the variable, $x(s)$, using Relations (19) and (20). The proof of the theorem is completed in the same way as in Theorem 8, Substituting (66) in (68) and making use of Equations (37) and (38). □

Remark 12. *The previous result generalizes the one given by Atakishiyev and Suslov [26] in which they provide a method to construct particular solutions to hypergeometric-type difference Equations on a nonuniform lattice.*

Remark 13. *Theorems 8 and 11 provide a method of expanding solutions of Askey-Wilson operator Equations in terms of the basis, ϕ_n (see Remark 7), or F_n, providing, therefore, the solution of the problem raised by Ismail ([16], page 518).*

The coefficients of the expansion of the polynomial solution given in (62) is not q-hypergeometric, but rather $q^{\frac{1}{2}}$-hypergeometric. Therefore, there is the need of an additional transformation in order to prove that P_n are effectively the known Askey-Wilson polynomials. In order to provide an explicit, direct and simple representation of polynomial solutions of divided-difference Equations, such as (15), we provide, in the next subsection, a second basis, which is not compatible with the operators, $\mathbb{D}_x$ and $\mathbb{S}_x$, but rather, with $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$ and are, therefore, very useful when searching for series solutions of divided-difference Equations with polynomials coefficients, involving linear combination of the products of $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$. This basis allows, for example, to recover from the divided-difference Equation (15), without the need of any further transformation, the defining representation of the Askey-Wilson polynomials given by (22).

3.2. Algorithmic Series Solutions of Divided-Difference Equations in Terms of $B_n(a,s)$

In the first step, we define the basis $B_n(a,s)$ and study its properties.

Expressing the Askey-Wilson polynomials (22) in terms of q-Pochhammer symbols:

$$P_n(x;a,b,c,d|q) = \sum_{k=0}^{n} \frac{(q^{-n},q)_k\ (a\,b\,c\,d\,q^{n-1},q)_k\ (a\,q^{s},q)_k\ (a\,q^{-s},q)_k}{(a\,b,q)_k\ (a\,c,q)_k\ (a\,d,q)_k} \frac{q^k}{(q,q)_k} \tag{69}$$

and the fact that these polynomials fulfill (15) suggests the study of the action of the companion operators on the function:

$$B(a,x(s),n) = (a\,q^{s},q)_n\ \left(a\,q^{-s},q\right)_n,\ n\geq 1,\ B(a,x(s),0)\equiv 1 \tag{70}$$

which happens to be a polynomial of degree n in $x(s)=\frac{q^s+q^{-s}}{2}$. By considering a more general situation, we get:

Proposition 14. *The general q-quadratic lattice:*

$$x(s) = u\,q^{s} + v\,q^{-s}$$

and the corresponding polynomial basis:

$$\hat{B}_n(a,u,v,x(s)) = (2a\,u\,q^{s},q)_n\ \left(2a\,v\,q^{-s},q\right)_n,\ n\geq 1,\ \hat{B}_0(a,u,v,x(s))\equiv 1$$

which we relabel as:

$$B_n(a,s)\equiv \hat{B}_n(a,u,v,x(s))$$

fulfilling the relations:

$$\begin{aligned}
\mathbb{D}_x B_n(a,s) &= \eta(a,n) B_{n-1}(a\sqrt{q},s) && (71)\\
\mathbb{S}_x B_n(a,s) &= \beta_1(a,n)\, B_{n-1}(a\sqrt{q},s) + \beta_2(n)\, B_n(a\sqrt{q},s) && (72)\\
B_1(a,s)\mathbb{D}_x^2\, B_n(a,s) &= \eta(a,n)\,\eta(a\,\sqrt{q},n-1)\, B_{n-1}(a,s) && (73)\\
B_1(a,s)\mathbb{S}_x\mathbb{D}_x\, B_n(a,s) &= \eta(a,n)\ [\beta_1(a\sqrt{q},n-1)\, B_{n-1}(a,s) + \beta_2(n-1)\, B_n(a,s)] && (74)\\
x(s)\, B_n(a,s) &= \mu_1(a,n)\, B_n(a,s) + \mu_2(n)\ \ B_{n+1}(a,s) && (75)\\
B_1(a,s)\, B_n(a,s) &= \nu_1(a,n)\, B_n(a,s) + \nu_2(n)\, B_{n+1}(a,s) && (76)\\
B_1(a,s)B_n(a\,q,s) &= B_{n+1}(a,s) && (77)
\end{aligned}$$

where:

$$\begin{aligned}
\mu_1(a,n) &= \frac{1+4a^2uv\,q^{2n}}{2a\,q^n}, \quad \mu_2(a,n)=\frac{-1}{2a\,q^n}\\
\nu_1(a,n) &= (1-q^{-n})\,(1-4a^2uv\,q^n), \quad \nu_2(n)=q^{-n}, \quad \eta(a,n)=\frac{2a(1-q^n)}{q-1}\\
\beta_1(a,n) &= \frac{1}{2}(1-4a^2uv\;q^{2n-1})(1-q^{-n}), \quad \beta_2(n)=\frac{1}{2}+\frac{1}{2q^n}
\end{aligned}$$

Proof: The proof is obtained by direct computation. □

Remark 15. *It should, however, be noted that for $u=v=\frac{1}{2}$, Relation (71) appears as exercise in [27], page 34. It also appears in [16], Equation (20.3.11), page 518.*

The following proposition provides the connection coefficients between the bases, F_n and $B_n(a,x)$:

Proposition 16. *The bases, $(F_k)_k$ and $(B_k(a,s))_k$, are connected in the following ways:*

$$F_n(x(s))=\sum_{j=0}^{n} r_{n,j}\,B_j(a,s), \;\; B_n(a,s)=\sum_{j=0}^{n} s_{n,j}\,F_j(x(s)) \tag{78}$$

where:

$$r_{n,k} = \frac{\gamma_n!}{\gamma_{n-k}!}\frac{F_{n-k}(\epsilon_{0,k})}{\prod_{l=0}^{k-1}\eta\left(aq^{\frac{l}{2}},k-l\right)}, \;\; 0\le k\le n, \;\; n\ge 1 \tag{79}$$

$$s_{n,k} = \frac{1}{\gamma_k!}\;B_{n-k}\left(aq^{\frac{k}{2}},\epsilon+\frac{1}{2}\right)\prod_{l=0}^{k-1}\eta\left(aq^{\frac{l}{2}},n-l\right), \;\; 0\le k\le n, \;\; n\ge 1 \tag{80}$$

and:

$$\epsilon_{j,k}=\frac{1+4a^2uvq^{2j+k}}{2aq^{j+\frac{k}{2}}} \tag{81}$$

Proof: The proof is obtained first by applying the operator, $\mathbb{D}_x^k$, to both members of (78) for fixed non-negative integers, $n\ge 1$ and $k\le n$, and using (37) and (71), then by observing that:

$$\hat{B}_n(a,u,v,\epsilon_{j,0})=0, \;\; \forall n\ge 1, \;\; \forall j\le n, \;\; \text{and; } F_n(x_1(\epsilon))=0, \;\; \forall n\ge 1$$

□

From Proposition 14, it appears clearly that because of the appearance of $a\sqrt{q}$ in Relations (71) and (72), the action of $\mathbb{D}_x$ and $\mathbb{S}_x$ on $B_n(a,s)$ cannot be written as finite (number of terms not depending on n) linear combination of the elements of the basis $(B_k(a,s))_k$. However, this problem is solved by using the operators, $B_1(a,s)\mathbb{D}_x^2$ and $B_1(a,s)\mathbb{S}_x\mathbb{D}_x$, instead, to obtain Relations (73) and (74). Equation (15) can therefore be solved using the known coefficients, ϕ and ψ, of Askey-Wilson.

3.3. Algorithmic Series Solutions of Divided-Difference Equations in Terms of $B_n(a,s)$

Theorem 17.
If:

$$y(x(s)) = \sum_{k=0}^{\infty} d_k\, B_k(a,s) \tag{82}$$

is a polynomial solution of the Equation:

$$\phi(x(s))\,\mathbb{D}_x^2 y(x(s)) + \psi(x(s))\,\mathbb{S}_x\mathbb{D}_x y(x(s)) + \lambda\, y(x(s)) = 0 \tag{83}$$

where λ is a constant, ϕ and ψ are polynomials of a degree of at most two and one, respectively:

$$\phi(x(s)) = \phi_2\, x^2(s) + \phi_1\, x(s) + \phi_0, \quad \psi(x(s)) = \psi_1\, x(s) + \psi_0 \tag{84}$$

then, the coefficients, $(d_k)_n$, satisfy a second-order difference Equation:

$$A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k = 0,\ k \geq 0 \tag{85}$$

with:

$$\begin{aligned}
A_k &= \eta(a,k+2)\,[\eta(a\sqrt{q},k+1)\,\phi\,(\mu_1(a,k+1)) + \beta_1(a\sqrt{q},k+1)\psi\,(\mu_1(a,k+1))] \\
B_k &= \eta(a,k+1)\ \{\eta(a\sqrt{q},k)\ (\phi_2\ \mu_1(a,k)\mu_2(a,k) + \phi_2\ \mu_1(a,k+1)\mu_2(a,k) + \phi_1\ \mu_2(a,k)\ ) \\
&\quad +\beta_1(a\sqrt{q},k)\ (\psi_1\ \mu_2(a,k) + \beta_2(k)\,(\psi_1\ \mu_1(a,k+1) + \psi_0)\ )\} + \lambda\ \nu_1(a,k+1) \\
C_k &= \phi_2\ \mu_2(a,k-1)\ \mu_2(a,k)\,\eta(a,k)\,\eta(a\sqrt{q},k-1) + \psi_1\ \eta(a,k)\,\beta_2(k-1)\mu_2(a,k) + \lambda\ \nu_2(k)
\end{aligned}$$

Proof: The proof is similar to the one of Theorem 8 using the properties of the basis $B_n(a,s)$ given in Proposition 14. □

The previous theorem can be extended to divided-difference Equations of arbitrary order with polynomial coefficients, involving linear combinations of powers of the operators, $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$. Such operators can be rewritten, using Relations (19) and (20), as linear combination of $\mathbb{D}_x^{2j}$ and $\mathbb{S}_x\mathbb{D}_x^{2j+1}$, $j \geq 0$. For this extension, we will need the following results, obtained by iteration of Relations (71)–(77).

Proposition 18. *The basis, $(B_n(a,s))_n$, satisfies the following relations:*

$$\begin{aligned}
\mathbb{D}_x^{2k} B_n(a,s) &= \pi_{n,k}\, B_{n-2k}(aq^k,s),\ 0 \leq 2k \leq n \\
L_k(s)B_n(aq^k,s) &= B_{n+k}(a,s) \\
L_k(s)\mathbb{D}_x^{2k} B_n(a,s) &= \pi_{n,k}\, B_{n-k}(a,s) \\
L_{k+1}(s)\mathbb{S}_x\mathbb{D}_x^{2k+1} B_n(a,s) &= I_{n,k}\, B_{n-k-1}(a,s) + J_{n,k}\, B_{n-k}(a,s)
\end{aligned}$$

where:

$$\begin{aligned}
L_k(s) &= \prod_{j=0}^{k-1} B_1(aq^j,s),\ \pi_{n,k} = \prod_{j=0}^{2k-1} \eta(aq^{\frac{j}{2}},n-j) \\
I_{n,k} &= \pi_{n,k}\,\eta(aq^k,n-2k)\,\beta_1(aq^{k+\frac{1}{2}},n-2k-1) \\
J_{n,k} &= \pi_{n,k}\,\eta(aq^k,n-2k)\,\beta_2(aq^{k+\frac{1}{2}},n-2k-1)
\end{aligned}$$

We now state the following theorem, which can be proven in the same way as Theorem 17, but using, instead, the Equations of the previous proposition.

Theorem 19. *If*

$$y(x(s)) = \sum_{k=0}^{\infty} d_n\, B_n(a,s)$$

is a polynomial solution of the divided-difference Equation:

$$\left[\sum_{j=0}^{M} P_j(x(s))\mathbb{D}_x^{2j} + \sum_{j=0}^{N} Q_j(x(s))\mathbb{S}_x\mathbb{D}_x^{2j+1}\right] y(x(s)) = T(x(s)) \tag{86}$$

where P_j, Q_j and T are polynomials in the variable, $x(s)$, then the coefficients, $(d_k)_n$, satisfy a linear difference Equation of maximal order, $\max(2M, 2N+1)$.

As the corollary of Theorem 17, we have recovered the representation of the Askey-Wilson polynomials by solving Equation (8).

In fact, we have replaced the coefficients, ϕ and ψ, of the Askey-Wilson polynomials (see (23)) in Equation (85) to obtain the following difference Equation of the expansion coefficient, d_k:

$$\begin{aligned}
&4\,q^n q\left(q^{k+1}-1\right)\left(a^2q^{k+1}-1\right)\left(acq^{k+1}-1\right)\left(adq^{k+1}-1\right)\left(qq^{k+1}-1\right)\left(abq^{k+1}-1\right)d_{k+2}\\
&+4\left(q^{k+1}-1\right)\Big\{-\left(q^{k+1}\right)^3a^3q^nbcd-\left(q^{k+1}\right)^3a^3qq^nbcd+\left(q^{k+1}\right)^2q^nabcdq+\left(q^{k+1}\right)^2a^2qq^nbc\\
&+\left(q^{k+1}\right)^2a^3q\left(q^n\right)^2bcd+\left(q^{k+1}\right)^2q^2a^2+\left(q^{k+1}\right)^2a^2qq^ncd+\left(q^{k+1}\right)^2a^2qq^nbd-q^{k+1}q^2\\
&-q^{k+1}q\left(q^n\right)^2abcd-q^{k+1}a^2q^2q^n-q^{k+1}aq^2q^nc-q^{k+1}aq^2q^nd-q^{k+1}aq^2q^nb+q^3q^n+q^2q^n\Big\}\,d_{k+1}\\
&-4\left(q^nq-q^{k+1}\right)q\left(q^{k+1}abcdq^n-q^2\right)d_k=0
\end{aligned} \tag{87}$$

In [28,29]—see, also, [30]—algorithms were presented to find all solutions of an arbitrary q-holonomic difference Equation in terms of linear combinations of q-hypergeometric terms. This algorithm was tuned and made much more efficient in [20,21], and a Maple implementation was made available in [23]. For the purpose of solving the second order q-difference Equation (87) in terms of hypergeometric terms, we have used the command, `qrecsolve`, from the q-sum package [29] to obtain the expansion coefficients, d_k, of the expansion of the Askey-Wilson polynomial (69), with $B_k(a,s) = (a\,q^s, q)_k\,(a\,q^{-s}, q)_k$. Details of this computation can be found in a Maple file made available on [22].

4. Applications and Illustrations

In this section, we provide three applications for the basis, F_k: The first gives a formal expansion of a given function in the basis, F_n, the second provides a new representation of the formal Stieltjes series in terms of the basis, F_k, while the third gives a representation of the basic exponential and trigonometric functions in terms of the basis, F_k, after solving explicitly a second-order divided-difference Equation in terms of F_n.

4.1. Properties of the New Representation of the Formal Stieltjes Function

In this subsection, we treat series representations of functions on our lattices.

Theorem 20. *Let N be a positive integer. Then, any polynomial, $f_N(x(s))$, of degree, N, in the variable, $x(s)$, can be expanded in the basis, $F_k(x(s))$, as:*

$$f_N(x(s)) = \sum_{k=0}^{N} d_k\, F_k(x(s)) \tag{88}$$

where:

$$d_k = \frac{\mathbb{D}_x^k f(x(\epsilon))}{\gamma_k!}, \quad \gamma_k! = \prod_{j=1}^{k} \gamma_j, \; k \geq 1, \quad \gamma_0! = 1$$

Proof: We write: $f_N(x(s)) = \sum\limits_{k=0}^{N} d_k\, F_k(x(s))$ and obtain for $0 \leq k \leq N$,

$$\mathbb{D}_x^k f_N(x(s))\big|_{s=\epsilon} = \gamma_k!\, d_k$$

since $x_1(\epsilon) = x(\epsilon)$ and $F_k(x_1(\epsilon)) = 0, \; \forall k \geq 1$. □

Remark 21. *The Expansion (88) constitutes a generalization to all polynomials of a quadratic or q-quadratic variable of the Taylor expansion given by Ismail [19] (Theorem 2.1, page 262), but for the basis, F_n, restricted only to the Askey-Wilson case given above in Proposition 6.*

The following theorem shows that the previous expansion, restricted to polynomials, can be extended to some non-polynomial function, but with an additional expression.

Theorem 22.
The following expansion holds for $|q| < 1$, $z \neq s$ and $z \neq \epsilon + \frac{j}{2}$, $j \geq 0$:

$$\frac{1}{x(z) - x(s)} = \sum_{k=0}^{\infty} \frac{F_k(x(s))}{F_{k+1}(x(z))} + \frac{1}{(x(z) - x(s))} \lim_{N\to\infty} \frac{F_N(x(s))}{F_N(x(z))} \tag{89}$$

Proof: We will organize the proof in the following steps:

In the first step, we state the fundamental relation for any $N \in \mathbb{N}^*$:

$$[x(z) - x(s)] \sum_{k=0}^{N-1} \frac{F_k(x(s))}{F_{k+1}(x(z))} = 1 - \frac{F_N(x(s))}{F_N(x(z))}, \; z \neq \epsilon + \frac{j}{2}, \, 0 \leq j \leq N \tag{90}$$

which can be obtained by direct computation using the relation:

$$[x(z) - x(s)] \frac{F_k(x(s))}{F_{k+1}(x(z))} = \frac{F_k(x(s))}{F_k(x(z))} - \frac{F_{k+1}(x(s))}{F_{k+1}(x(z))}$$

In the second step, we clear the convergence issue of the series, $\sum_{k=0}^{\infty} \frac{F_k(x(s))}{F_{k+1}(x(z))}$, by remarking that for a fixed s and z, if $a_k = \frac{F_k(x(s))}{F_{k+1}(x(z))}$, then:

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{x(s) - x_{k+1}(\epsilon)}{x(z) - x_{k+2}(\epsilon)} = \lim_{k\to\infty} \frac{x_{k+1}(\epsilon)}{x_{k+2}(\epsilon)} = q^{\frac{1}{2}}$$

since $\lim_{k\to\infty} x_k(\epsilon) = \infty$, due to the fact that $|q| < 1$. Therefore, $\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \sqrt{|q|} < 1$ and the series converges.

In the third step, we deduce, for fixed s and z, the convergence when N goes to infinity of the expression $\frac{F_N(x(s))}{F_N(x(z))}$, by observing that:

$$\frac{F_N(x(s))}{F_N(x(z))} = \prod_{k=1}^{N} \frac{1 - \frac{x(s)}{x_k(\epsilon)}}{1 - \frac{x(z)}{x_k(\epsilon)}}$$

and that $\prod_{k=1}^{N} \left(1 - \frac{x(s)}{x_k(\epsilon)}\right)$ and $\prod_{k=1}^{N} \left(1 - \frac{x(z)}{x_k(\epsilon)}\right)$ converge when N goes to infinity, since $\lim_{k\to\infty} x_k(\epsilon) = \infty$. The proof is completed by taking the limit when N goes to infinity in Equation (90).

By taking the integral of both sides of Relation (90) (divided by $x(z) - x(s)$) and taking, formally, the limit when N goes to infinity, we obtain:

$$S_0(\mathcal{L})(x(z)) = S_1(\mathcal{L})(x(z)) + \lim_{N\to\infty} \int_a^b \left[\frac{1}{(x(z) - x(s))} \frac{F_N(x(s))}{F_N(x(z))}\right] d\mu(x(s)) \tag{91}$$

with $z \neq s,\ z \neq \epsilon + \frac{j}{2},\ j \geq 0,\ x(z) \notin [a,\, b]$;

$$S_1(\mathcal{L})(x(z)) = \sum_{n=0}^{\infty} \frac{\hat{\mu}_n}{F_{n+1}(x(z))} \tag{92}$$

where $S_0(\mathcal{L})(x(z))$ is the usual Stieltjes function defined by (24) and:

$$\hat{\mu}_n = \int_a^b F_n(x(s))\, d\mu(x(s)) := \langle \mathcal{L}, F_n(x(s)) \rangle$$

Remark 23. *The function, $S_1(\mathcal{L})(x(z))$, defined above will be called the new representation of the formal Stieltjes function. The fact that we call $S_1(\mathcal{L})(x(z))$ a new representation of the formal Stieltjes function is justified, for classical orthogonal polynomials on a nonuniform lattice, by the fact that, on the one hand, the Riccati Equation satisfied by the formal Stieltjes function, $S_0(\mathcal{L})(x(z))$, is equivalent to the second-order divided-difference equation (defining completely the polynomials) satisfied by classical orthogonal polynomials on a nonuniform lattice [17]. On the other hand, the Riccati Equation satisfied by the so-called new representation of the formal Stieltjes function, $S_1(\mathcal{L})(x(z))$, is equivalent to the same second-order divided-difference equation satisfied by classical orthogonal polynomials on a nonuniform lattice [31].*

Using $S(\mathcal{L})(x(z)) := S_1(\mathcal{L})(x(z))$, we prove the following theorem by defining, formally, the actions of the operators, $\mathbb{D}_x$ and $\mathbb{S}_x$, on $S(\mathcal{L})$ as follows:

$$\mathbb{O}_x\, S(\mathcal{L})(x(z)) = \sum_{n=0}^{\infty} \hat{\mu}_n\, \mathbb{O}_x\, \frac{1}{F_{n+1}(x(z))},\; z \neq \epsilon + \frac{j}{2},\; \mathbb{O}_x \in \{\mathbb{D}_x,\, \mathbb{S}_x\}$$

Theorem 24. *The following results hold:*

$$\begin{aligned} S\left(\mathbb{D}_x\mathcal{L})(s)\right) &= \mathbb{D}_x S(\mathcal{L})(s) & (93)\\ S\left(\mathbb{S}_x\mathcal{L})(s)\right) &= \alpha\, \mathbb{S}_x S\left(\mathcal{L}\right)(s) + U_1 \mathbb{D}_x S\left(\mathcal{L}\right)(s) & (94)\end{aligned}$$

where the actions of $\mathbb{D}_x$ *and* $\mathbb{S}_x$ *on* $\mathcal{L}$ *are defined as:*

$$\langle \mathbb{D}_x\mathcal{L}, P\rangle = -\langle \mathcal{L}, \mathbb{D}_x P\rangle,\; \langle \mathbb{S}_x\mathcal{L}, P\rangle = \langle \mathcal{L}, \mathbb{S}_x P\rangle,\; \forall P \in \mathbb{C}[x(s)]$$

and the product of a polynomial, f*, by a linear functional,* $\mathcal{L}$*,* $f\mathcal{L}$*, is defined by:*

$$\langle f\mathcal{L}, P\rangle = \langle \mathcal{L}, fP\rangle,\; \forall P \in \mathbb{C}[x(s)]$$

Proof: Relation (93) is obtained by direct computation using Equations (30) and (32). The proof of (94) uses the following results:

$$\begin{aligned} S(U_1(x(s))\mathbb{D}_x\mathcal{L}) &= U_1(x(s))S(\mathbb{D}_x\mathcal{L}) & (95)\\ \mathbb{S}_x F_n + \mathbb{D}_x(U_1 F_n) &= \alpha(\alpha_{n+1}F_n + \frac{\gamma_n}{2}\nabla x_{n+2}(\epsilon)F_{n-1}) & (96)\end{aligned}$$

Relation (95) is obtained using the well-known result by Maroni [32]:

$$S\left[f\mathcal{L}\right](x) = f(x)\, S\left[\mathcal{L}\right](x) + (\mathcal{L}\theta_0 f)(x),\; f \in \mathbb{C}[x] \tag{97}$$

where:

$$\theta_0 f(x) = \frac{f(x) - f(0)}{x}$$

and:

$$\mathcal{L}g(x(s)) = \sum_{k=0}^{n} g_k \sum_{j=0}^{k} \langle \mathcal{L}, x^j(s)\rangle x^{k-j}(s),\;\; with\; g(x(s)) = \sum_{k=0}^{n} g_k\, x^k(s),\; n \geq 0$$

Relation (96) is derived by direct computation using (10), (21), (28), (30) and (31).

To prove Relation (94), we combine (31), (33), (95), (96), and the fact that $\gamma_0 = 0$ to get:

$$\begin{aligned}
S(\mathbb{S}_x\mathcal{L})(x(s)) - U_1(x(s))\mathbb{D}_x(S(\mathcal{L}))(x(s)) &= S(\mathbb{S}_x\mathcal{L} - U_1(x(s))\mathbb{D}_x\mathcal{L}) \\
&= \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, \mathbb{S}_x F_n + \mathbb{D}_x(U_1 F_n)\rangle}{F_{n+1}(x(s))} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, (\alpha_{n+1}F_n + \frac{\gamma_n}{2}\nabla x_{n+2}(\epsilon)F_{n-1}\rangle}{F_{n+1}(x(s))} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, \alpha_{n+1}F_n\rangle}{F_{n+1}(x(s))} + \alpha \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, \gamma_n \nabla x_{n+2}(\epsilon)F_{n-1}\rangle}{2F_{n+1}(x(s))} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, \alpha_{n+1}F_n\rangle}{F_{n+1}(x(s))} + \alpha \sum_{n=1}^{\infty} \frac{\langle \mathcal{L}, \gamma_n \nabla x_{n+2}(\epsilon)F_{n-1}\rangle}{2F_{n+1}(x(s))} \\
&= \alpha \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, \alpha_{n+1}F_n\rangle}{F_{n+1}(x(s))} + \alpha \sum_{n=0}^{\infty} \frac{\langle \mathcal{L}, \gamma_{n+1} \nabla x_{n+3}(\epsilon)F_{n}\rangle}{2F_{n+2}(x(s))} \\
&= \alpha \sum_{n=0}^{\infty} \langle \mathcal{L}, F_n\rangle \left(\frac{\alpha_{n+1}}{F_{n+1}(x(s))} + \frac{\gamma_{n+1}\nabla x_{n+3}(\epsilon)}{2F_{n+2}(x(s))} \right) \\
&= \alpha \sum_{n=0}^{\infty} \langle \mathcal{L}, F_n\rangle \mathbb{S}_x \frac{1}{F_{n+1}(x(s))} \\
&= \alpha \mathbb{S}_x(S(\mathcal{L}))(x(s))
\end{aligned}$$

$\square$

4.2. Series Expansion of the Basic Exponential Function

In this sub-section, we represent the basic exponential function in terms of the basis, $(F_k)_k$. The basic exponential function, $\mathcal{E}_q(x(s); w)$, can be defined using the representation by Ismail and Stanton (see [27] page 21 and references therein):

$$\mathcal{E}_q(x; w) = \frac{\left(-w; q^{\frac{1}{2}}\right)_\infty}{(qw^2; q^2)_\infty} {}_2\varphi_1\left(\left. \begin{matrix} q^{\frac{1}{4}}e^{i\theta}, q^{\frac{1}{4}}e^{-i\theta} \\ -q^{\frac{1}{2}} \end{matrix} \right| q^{\frac{1}{2}}; -w \right), \; x = \cos\theta$$

where $(a, q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$.

By putting $e^{i\theta} = q^s$, and therefore:

$$x = \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{q^s + q^{-s}}{2} = x(s)$$

$\mathcal{E}_q(x(s); w)$ satisfies the following first-order divided-difference Equation ([27], page 18)

$$\mathbb{D}_x y(x(s)) = \frac{2wq^{\frac{1}{4}}}{1-q} y(x(s))$$

By inserting the series expansion of $y(x(s))$ in terms of $(F_k)_k$:

$$y(x(s)) = \sum_{n=0}^{\infty} a_n F_n(x(s))$$

in the previous first-order divided-difference equation and following the method developed in Theorem 8, we get the following recurrence Equation for a_n:

$$a_{n+1} = \frac{2wq^{\frac{1}{4}}}{(1-q)\gamma_{n+1}} a_n$$

from which we deduce that:

$$a_n = \left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^n \frac{a_0}{\gamma_n!}, \quad \gamma_0! \equiv 1$$

Therefore we have, taking into account Equation (53), the following representation of the basic exponential function:

$$\begin{aligned} \mathcal{E}_q(x(s);w) &= a_0 \sum_{n=0}^{\infty} \left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^n \frac{F_n(x(s))}{\gamma_n!} \\ &= a_0 \sum_{n=0}^{\infty} \left(\frac{wq^{\frac{1}{2}}}{q-1}\right)^n \frac{1}{\gamma_n!} \left(q^{\frac{1-2n}{4}} q^s; q\right)_n \left(q^{\frac{1-2n}{4}} q^{-s}; q\right)_n \end{aligned}$$

where a_0 is a suitable constant, which from the fact that $F_n(x_1(\epsilon)) = 0,\ n \geq 1$, is given by:

$$a_0 = \mathcal{E}_q(x_1(\epsilon), w) = \frac{(-w; q^{\frac{1}{2}})_\infty}{(qw^2; q^2)_\infty}$$

with the last expression taken from [27] (Equation 2.3.10, page 18). Here, it should be mentioned that the previous expansion of the basic exponential function in terms of the basis, F_n, has already been given by Ismail (see [19], Equation (2.4) page 363), using another approach.

4.3. Series Expansion of the Basic Trigonometric Functions

In this subsection, we represent the basic trigonometric functions in terms of the basis, $(F_k)_k$. The basic trigonometric cosine and sine functions are defined, respectively, by (see [27], page 23):

$$\begin{aligned} C_q(x;w) &= \frac{(-w^2;q^2)_\infty}{(-qw^2;q^2)_\infty} {}_2\varphi_1\left(\begin{matrix} -qe^{2i\theta}, -qe^{-2i\theta} \\ q \end{matrix} \middle| q^2; -w^2\right) \\ S_q(x;w) &= \frac{(-w^2;q^2)_\infty}{(-qw^2;q^2)_\infty} \frac{2wq^{\frac{1}{4}}}{1-q} \cos\theta\, {}_2\varphi_1\left(\begin{matrix} -qe^{2i\theta}, -qe^{-2i\theta} \\ q \end{matrix} \middle| q^2; -w^2\right), \quad x = \cos\theta,\ |w| < 1 \end{aligned}$$

By putting $e^{i\theta} = q^s$, the functions, $C_q(x(s);w)$ and $S_q(x(s);w)$, satisfy the following second-order divided-difference Equation ([27], page 26):

$$\mathbb{D}_x^2 y(x(s)) = -\left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^2 y(x(s)) \tag{98}$$

We have the following:

Proposition 25. *The two linearly-independent solutions of (98) are $A_q(x(s),w)$ and $B_q(x(s),w)$, defined by:*

$$\begin{aligned}
A_q(x(s),w) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{F_{2n}(x(s))}{\gamma_{2n}!} = {}_4\phi_3\left(\left.\begin{matrix} q^{s+\frac14},q^{-s+\frac14},q^{s+\frac34},q^{-s+\frac34} \\ -q^{\frac12},q^{\frac12},-q\end{matrix}\,\right|\, q;-w^2\right),\ q^s=e^{i\theta}\\
B_q(x(s),w) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{F_{2n+1}(x(s))}{\gamma_{2n+1}!} = (x(s)-x(\epsilon))\,{}_4\phi_3\left(\left.\begin{matrix} q^{s+\frac34},q^{-s+\frac34},q^{s+\frac54},q^{-s+\frac54} \\ -q^{\frac32},q^{\frac32},-q\end{matrix}\,\right|\, q;-w^2\right)
\end{aligned}$$

Proof: By inserting the series expansion of $y(x(s))$ in terms of $(F_k)_k$:

$$y(x(s)) = \sum_{n=0}^{\infty} b_n F_n(x(s))$$

in (98) and following the method developed in Theorem 8, we get the following recurrence Equation:

$$b_{n+2} = -\left(\frac{2wq^{\frac14}}{1-q}\right)^2 \frac{1}{\gamma_{n+2}\gamma_{n+1}} b_n$$

from which we deduce that:

$$b_{2n} = (-1)^n\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{b_0}{\gamma_{2n}!}\ \text{ and }\ b_{2n+1} = (-1)^n\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{b_1}{\gamma_{2n+1}!}$$

Therefore, the two linearly-independent Solutions (98) are, taking into account Relation (65), the explicit representation of F_n given in (53) for the Askey-Wilson lattice, and the relation (see [25], Equation 1.2.39, page 6):

$$(a;q)_{2n} = (a;q^2)_n(aq;q^2)_n$$

as well as the following relation obtained by direct computation:

$$\gamma_n! = \prod_{k=1}^{n}\gamma_k = q^{\frac{-n(n-1)}{4}}\frac{(q;q)_n}{(1-q)^n}$$

$$\begin{aligned}
A_q(x(s),w) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{F_{2n}(x(s))}{\gamma_{2n}!}\\
&= \sum_{n=0}^{\infty}\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{(-1)^n}{\gamma_{2n}!}\left(\frac{-q^{-\frac{n}{2}}}{2}\right)^{2n}\left(q^{s+\frac14},q^{\frac12}\right)_{2n}\left(q^{-s+\frac14},q^{\frac12}\right)_{2n}\\
&= {}_4\phi_3\left(\left.\begin{matrix} q^{s+\frac14},q^{-s+\frac14},q^{s+\frac34},q^{-s+\frac34} \\ -q^{\frac12},q^{\frac12},-q\end{matrix}\,\right|\, q;-w^2\right)
\end{aligned}$$

and:

$$\begin{aligned}
B_q(x(s),w) &= \sum_{n=0}^{\infty}(-1)^n\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{F_{2n+1}(x(s))}{\gamma_{2n+1}!}\\
&= \sum_{n=0}^{\infty}\left(\frac{2wq^{\frac14}}{1-q}\right)^{2n}\frac{(-1)^n}{\gamma_{2n+1}!}\left(\frac{-q^{-\frac{2n+1}{4}}}{2}\right)^{2n+1}\left(q^{s+\frac14},q^{\frac12}\right)_{2n+1}\left(q^{-s+\frac14},q^{\frac12}\right)_{2n+1}\\
&= (x(s)-x(\epsilon))\,{}_4\phi_3\left(\left.\begin{matrix} q^{s+\frac34},q^{-s+\frac34},q^{s+\frac54},q^{-s+\frac54} \\ -q^{\frac32},q^{\frac32},-q\end{matrix}\,\right|\, q;-w^2\right)
\end{aligned}$$

□

Since the functions, C_q and S_q, are both solutions of (98), which is equivalent to a second-order linear homogeneous difference Equation of the form:

$$a_2(s)Y(s+1) + a_1(s)Y(s) + a_0(s)Y(s-1) = 0, \text{ with } Y(s) = y(x(s))$$

where $a_j(s)$ are functions of s, they can be expressed as linear combination of the solutions, A_q and B_q:

$$C_q(x(s), w) = u_0(s)A_q(x(s), w) + u_1(s)B_q(x(s), w), \ S_q(x(s), w) = v_0(s)A_q(x(s), w) + v_1(s)B_q(x(s), w) \tag{99}$$

where u_i and v_i are quasi-constants in s, that is, they satisfy $u_i(s+1) = u_i(s)$ and $v_i(s+1) = v_i(s)$.

5. Conclusion and Perspectives

In this paper, we developed suitable bases (replacing the power basis, x^n $(n \in \mathbb{N}_{\geq 0})$), which enable the direct series representation of orthogonal polynomial systems on nonuniform lattices (quadratic lattices of a discrete or a q-discrete variable). We presented two bases of this type, the first of which allows one to write solutions of arbitrary divided-difference equations in terms of series representations extending results given in [23] for the q-case and in [26] for the quadratic case. Furthermore, we used this basis to give a new representation of the Stieltjes function, which we have already used (see [31]), to prove the equivalence between the Pearson Equation for the functional approach and the Riccati Equation for the formal Stieltjes function, leading, therefore, to the theory of classical orthogonal polynomials on a nonuniform lattice, using the functional approach.

When the Askey-Wilson polynomials are written in terms of this basis, we proved that the coefficients are not q-hypergeometric. Therefore, we presented a second basis, which shares several relevant properties with the first one. This basis enables one to generate the defining representation of the Askey-Wilson polynomials directly from their divided-difference equation and, also, to solve more general divided-difference equations of an arbitrary order involving the linear combination of $\mathbb{D}_x^{2j}$ and $\mathbb{S}_x\mathbb{D}_x^{2j+1}$, $(j \geq 0)$.

As perspective, we mention that this paper has led and shall lead to the following:

① The characterization of the classical and semi-classical orthogonal polynomials on a nonuniform lattice using the functional approach. Here, new important characterization should be pointed out: The equivalence between the Riccati divided-difference equation for the Stieltjes function:

$$\phi(x(s))\mathbb{D}_x S(x(s)) = \psi(x(s))\,\mathbb{S}_x S(x(s)) + D(x(s))$$

and the Pearson-type Equation for the functional:

$$\mathbb{D}_x(\phi\mathcal{L}) = \mathbb{S}_x(\psi\mathcal{L})$$

This pioneering work is mainly based on the use of the properties of (93) and (94), which have been established using the new Representation (92) of the Stieltjes function in terms of F_n. The characterization theorem of classical orthogonal polynomials on nonuniform lattices based on the present paper has already been established and published [31].

② The definition and the characterization of the Laguerre-Hahn orthogonal polynomials on nonuniform lattices in terms of the functional Equation for the corresponding linear functional. This provides the link between the functional approach and the one developed by Magnus [17,18] using the Riccati Equation for the formal Stieltjes series. Furthermore, the characterization in terms of the functional Equation allows the study of the modifications of the classical and semi-classical orthogonal polynomials on a nonuniform lattice, such as the multiplication of a classical functional by a polynomial (leading to a semi-classical functional) and the associated classical orthogonal polynomials on a nonuniform lattice (leading to Laguerre-Hahn orthogonal polynomials).

③ The algorithmic determination of the connection and linearization coefficients of the Askey-Wilson orthogonal polynomials using, mainly, the formalism developed in this paper, the operators, $\mathbb{D}_x$ and $\mathbb{S}_x$, as well as the bases, $B_n(a, x)$. These results have already been established and published [33].

④ The extension of the Hahn problem [34] to nonuniform lattices: That is, to prove that any family of orthogonal polynomials on a nonuniform lattice, satisfying a second-order linear homogeneous divided-difference equation of the form:

$$A(x(s))\mathbb{D}_x^2 Y(x(s)) + B(x(s))\mathbb{S}_x\mathbb{D}_x Y(x(s)) + C(x(s))Y(x(s)) = 0$$

where A, B and C are polynomials, is semi-classical. This work, mainly based on the basis, F_n, is almost completed [35].

⑤ It might also be used to solve specific divided-difference equations, such as the q-wave and the q-heat Equations [27]; and provide new identities in the domain of special functions.

Acknowledgment

The authors would like to acknowledge the various financial support from the Alexander von Humboldt Foundation (Bonn, Germany) and, particularly, the Research Group Linkage between the University of Kassel and the University of Yaounde I (2009–2012), which enabled their stay in Germany (June–September 2009) and their participation to the 10th Conference on Orthogonal Polynomials, Special Functions and Applications (Leuven, Belgium, 20–25 July, 2009), where part of this work was presented. Their sincere thanks go to Wolfram Koepf for his usual kind hospitality.

The authors would like to thank two anonymous reviewers whose comments and suggestions were very helpful for the improvement of the manuscript.

Conflict of Interest

The authors declare no conflict of interest.

References

1. Stieltjes, T.J. Recherches sur les fractions continues. *Ann. Fac. Sci. Toulouse Math.* **1894**, *2*, 398–566.
2. Ahiezer, N.I. Über eine eigenschaft der "elliptischen" polynome. *Commun. Soc. Math. Kharkoff* **1934**, *4*, 3–8.
3. Wall, H.S. A continued fraction related to some partition formulas of Euler. *Am. Math. Mon.* **1941**, *48*, 102–108.
4. Maroni, P. Une Théorie Algébrique des Polynômes Orthogonaux: Applications Aux Polynômes Orthogonaux Semi-Classiques. In *Orthogonal Polynomials and Applications*; Brezinski, C., Gori, L., Ronveux, A., Eds.; J. C. Baltzer Aktiengesellschaft: Basel, Switzerland, 1991; Volume 9, pp. 98–130.
5. Marcellán, F.; Branquinho, A.; Petronilho, J. Classical orthogonal polynomials: A functional approach. *Acta Appl. Math.* **1994**, *34*, 283–303.
6. García, A.G.; Marcellán, F.; Salto, L. A distributional study of discrete classical orthogonal polynomials. *J. Comput. Appl. Math.* **1995**, *5*, 147–162.
7. Koekoek, R.; Lesky, P.A.; Swarttouw, R.F. *Hypergeometric Orthogonal Polynomials and Their q-Analogues*; Springer: Berlin, Germany, 2010.
8. Nikiforov, A.F.; Suslov, S.K.; Uvarov, V.B. *Classical Orthogonal Polynomials of a Discrete Variable*, Springer Series in Computational Physics; Springer-Verlag: Berlin, Germany, 1991; pp. 18–54.
9. Medem, J.C.; Marcellán, F. q-Classical polynomials: A very classical approach. *Electron. Trans. Numer. Anal.* **1999**, *9*, 112–127.
10. Medem, J.C.; Álvarez-Nodarse, R.; Marcellán, F. On the q-Polynomials: A distributional study. *J. Comput. Appl. Math.* **2001**, *135*, 157–196.
11. Atakishiyev, N.M.; Rahman, M.; Suslov, S.K. On classical orthogonal polynomials. *Constr. Approx.* **1995**, *11*, 181–226.
12. Suslov, S.K. The theory of difference analogues of special functions of hypergeometric type. *Russ. Math. Surv.* **1989**, *44*, 227–278.
13. Foupouagnigni, M. On Difference and Differential Equations for Modifications of Classical Orthogonal Polynomials. Habilitation Thesis, University of Kassel, Kassel, Germany, 2006.
14. Foupouagnigni, M. On difference equations for orthogonal polynomials on nonuniform lattices. *J. Differ. Equ. Appl.* **2008**, *14*, 127–174.
15. Costas-Santos, R.S.; Marcellán, F. q-Classical orthogonal polynomials: A general difference calculus approach. *Acta Appl. Math.* **2010**, *111*, 107–128.
16. Ismail, M.E.H. *Classical and Quantum Orthogonal Polynomials in One Variable*; Cambridge University Press: Cambridge, UK, 2005.
17. Magnus, A.P. Associated Askey-Wilson Polynomials as Laguerre-Hahn Orthogonal Polynomials. In *Orthogonal Polynomials and Their Applications (Segovia, 1986)*; Alfaro, M., Dehesa, J.S., Marcellan, F.J., Rubio de Francia, J.L., Vinuesa, J., Eds.; Springer-Verlag: Berlin, Germany, 1988; pp. 261–278.

18. Magnus, A.P. Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points. *J. Comput. Appl. Math.* **1995**, *65*, 253–265.
19. Ismail, M.E.H.; Stanton, D. Applications of q-taylor theorems. *J. Comput. Appl. Math.* **2003**, *153*, 259–272.
20. Horn, P.; Koepf, W.; Sprenger, T. m-Fold hypergeometric solutions of linear recurrence equations revisited. *Math. Comput. Sci.* **2012**, *6*, 61–77.
21. Horn, P. Faktorisierung in Schief-Polynomringen. Ph.D. Thesis, University of Kassel, Kassel, Germany, 2008.
22. Koepf, W. Software downloads. Avialable online: www.mathematik.uni-kassel.de/~koepf/Publikationen (accessed on 1 July 2013).
23. Sprenger, T. Algorithmen für q-Holonome Funktionen und q-Hypergeometrische Reihen. Ph.D. Thesis, Universität Kassel, Kassel, Germnay, 2009. Available online: urn:nbn:de:hebis:34-2009072129130 (accessed on 1 July 2013).
24. Koepf, W.; Sprenger, T. Algorithmic determination of q-power series for q-holonomic functions. *J. Symb. Comput.* **2012**, *47*, 519–535.
25. Gasper, G.; Rahman, M. *Basic Hypergeometric Series, Encyclopedia of Mathematics and Its Applications 35*; Cambridge University Press: Cambridge, UK, 1990.
26. Atakishiyev, N.M.; Suslov, S.K. Difference Hypergeometric Functions. In *Progress in Approximation Theory: An International Perspective*; Gonchar, A.A., Saff, E.B., Eds.; Springer-Verlag: New York, NY, USA, 1992; Volume 19, pp. 1–35.
27. Suslov, S.K. *An Introduction to Basic Fourier Series, Kluwer Series Developments in Mathematics*; Kluwer Academic Publishers: London, UK, 2003.
28. Abramov, S.A.; Paule, P.; Petkovšek, M. q-Hypergeometric solutions of q-difference equations. *Discret. Math.* **1998**, *180*, 3–22.
29. Böing, H.; Koepf, W. Algorithms for q-hypergeometric summation in computer algebra. *J. Symb. Comput.* **1999**, *28*, 777–799.
30. Koepf, W. *Hypergeometric Summation*; Vieweg: Wiesbaden, Germany, 1998.
31. Foupouagnigni, M.; Kenfack-Nangho, M.; Mboutngam, S. Characterization theorem of classical orthogonal polynomials on nonuniform lattices: The functional approach. *Integral Transform. Spec. Funct.* **2011**, *22*, 739–758.
32. Maroni, P. Le Calcul des Formes Linéaires et les Polynômes Orthogonaux Semi-Classiques. In *Orthogonal Polynomials and Their Applications (Segovia, 1986)*; Alfaro, M., Dehesa, J.S., Marcellan, F.J., Rubio de Francia, J.L., Vinuesa, J., Eds.; Springer-Verlag: Berlin, Germany, 1986; pp. 279–290.
33. Foupouagnigni, M.; Koepf, W.; Tcheutia, D.D. Connection and linearization coefficients of Askey-Wilson polynomials. *J. Symb. Comput.* **2013**, *53*, 96–118.
34. Hahn, W. Uber differentialgleichungen fur orthogonalpolynome. *Monatsh. Math.* **1983**, *95*, 269–274.

35. Foupouagnigni, M.; Mboutngam, S. Hahn problem for orthogonal polynomials on q-discrete, discrete, quadratic or q-quadratic variables. *Appl. Numer. Math.* **2013**, submitted.