# Representations of $q$-Orthogonal Polynomials

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**Abstract**

The *linearization problem* is the problem of finding the coefficients $C_k(m,n)$ in the expansion of the product $P_n(x)Q_m(x)$ of two polynomial systems in terms of a third sequence of polynomials $R_k(x)$,

$$P_n(x)Q_m(x) = \sum_{k=0}^{n+m} C_k(m,n)R_k(x).$$

Note that, in this setting, the polynomials $P_n$, $Q_m$ and $R_k$ may belong to three different polynomial families. If $Q_m(x) = 1$, we are faced with the so-called *connection problem*, which for $P_n(x) = x^n$ is known as the *inversion problem* for the family $R_k(x)$.

In this paper we use an algorithmic approach to compute the connection and linearization coefficients between orthogonal polynomials of the $q$-Hahn tableau. These polynomial systems are solution of a $q$-differential equation of the type

$$\sigma(x)D_qD_{1/q}P_n(x) + \tau(x)D_qP_n(x) + \lambda_nP_n(x) = 0,$$

where the $q$-differential operator $D_q$ is defined by

$$D_qf(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

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## 1. Structural Formulas for $q$-Orthogonal Polynomials of the $q$-Hahn Class

A family

$$y(x) = P_n(x) = k_nx^n + k_n'x^{n-1} + k_n''x^{n-2} + \cdots \qquad (n \in \mathbb{N}_{\geq 0} := \{0,1,2\ldots\},\ k_n \neq 0) \tag{1}$$

of polynomials of degree exactly $n$ is a family of classical $q$-orthogonal polynomials of the $q$-Hahn class if it is the solution of a $q$-differential equation of the type

$$\sigma(x)D_qD_{1/q}P_n(x) + \tau(x)D_qP_n(x) + \lambda_nP_n(x) = 0, \tag{2}$$

where $\sigma(x) = ax^2 + bx + c$ is a polynomial of at most second order and $\tau(x) = dx + e$ is a polynomial of first order. The $q$-differential operator $D_q$ is defined by

$$D_qf(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0,\ q \neq 1$$

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and $D_q f(0) := f'(0)$ by continuity, provided $f'(0)$ exists.

The *linearization problem* is the problem of finding the coefficients $C_k(m,n)$ in the expansion of the product $P_n(x)Q_m(x)$ of two polynomial systems in terms of a third sequence of polynomials $R_k(x)$,

$$P_n(x)Q_m(x) = \sum_{k=0}^{n+m} C_k(m,n)R_k(x). \tag{3}$$

Note that, in this setting, the polynomials $P_n$, $Q_m$ and $R_k$ may belong to three different polynomial families. When the polynomials $P_n$, $Q_m$ and $R_k$ are solutions of the same differential equation (2), this is usually called the (standard) linearization or Clebsch-Gordan-type problem for hypergeometric polynomials. On the other hand, if $Q_m(x) = 1$ in (3), we are faced with the so-called *connection problem*, which for $P_n(x) = x^n$ is known as the *inversion problem* for the family $R_k(x)$.

The polynomial systems that are solution of (2) form the $q$-Hahn tableau. These systems are contained in the so-called Askey-Wilson scheme ([4], [7], [8]). The following systems are members of the $q$-Hahn tableau: the big $q$-Jacobi polynomials denoted by $P_n(x,\alpha,\beta,\gamma;q)$, the $q$-Hahn polynomials $Q_n(x,\alpha,\beta,N;q)$, the big $q$-Laguerre polynomials $P_n(x,\alpha,\beta;q)$, the little $q$-Jacobi polynomials $p_n(x,\alpha,\beta;q)$, the alternative $q$-Charlier polynomials $K_n(x,\alpha;q)$, the little $q$-Laguerre/Wall polynomials $p_n(x,\alpha;q)$, the $q$-Meixner polynomials $M_n(x,\beta,\gamma;q)$, the $q$-Charlier polynomials $C_n(x,\alpha;q)$, the $q$-Laguerre polynomials $L_n^{(\alpha)}(x;q)$, the Stieltjes-Wigert polynomials $S_n(x;q)$, the Al-Salam-Carlitz II polynomials $V_n^{(\alpha)}(x;q)$, the discrete $q$-Hermite II polynomials $H_n(x;q) = \tilde{h}_n(-ix;q)$. The corresponding monic families will be denoted by a tilde. For these polynomial systems, the polynomials $\sigma(x)$ and $\tau(x)$ of equation (2) are given in the Table 1.

| Family | $\sigma(x)$ | $\tau(x)$ |
|---|---|---|
| Big $q$-Jacobi | $(\alpha q - x)(\gamma q - x)$ | $\frac{\gamma q - x + \alpha q(1-(\beta+\gamma)q+\beta q x)}{q-1}$ |
| $q$-Hahn | $(\alpha q - x)(q^{-N} - x)$ | $\frac{\alpha q(1+(x-1)\beta q) - x + q^{-N}(1-\alpha q)}{q-1}$ |
| Big $q$-Laguerre | $(x - \alpha q)(\beta q - x)$ | $\frac{-(\alpha+\beta)q+\alpha\beta q^2+x}{q-1}$ |
| Little $q$-Jacobi | $x(x-1)$ | $\frac{1-x+\alpha q(\beta q x-1)}{q-1}$ |
| Alternative $q$-Charlier | $x(1-x)$ | $\frac{-1+x(1+\alpha q)}{q-1}$ |
| Little $q$-Laguerre/Wall | $x(1-x)$ | $\frac{-1+\alpha q+x}{q-1}$ |
| $q$-Meixner | $\gamma(x-\beta q)$ | $\frac{\gamma(\beta q-1)+q(x-1)}{q-1}$ |
| $q$-Charlier | $\alpha x$ | $\frac{q(x-1)-\alpha}{q-1}$ |
| $q$-Laguerre | $x$ | $\frac{q^{1+\alpha}(1+x)-1}{q-1}$ |
| Stieltjes-Wigert | $x$ | $\frac{qx-1}{q-1}$ |
| Al-Salam-Carlitz II | $\alpha$ | $\frac{x-\alpha-1}{q-1}$ |
| Discrete $q$-Hermite II | $1$ | $\frac{x}{1-q}$ |

Table 1: Families in the $q$-Hahn class

Since one demands that $P_n(x)$ has exactly degree $n$, we substitute $P_n(x)$ in the $q$-differential equation (2) and by equating the coefficients of $x^n$, one gets

$$\lambda_n = -[n]_{\frac{1}{q}}[n-1]_q a - [n]_q d \tag{4}$$

where the abbreviation

$$[n]_q = \frac{1-q^n}{1-q}$$

denotes the so-called $q$-brackets. Note that $\lim_{q\to 1}[n]_q = n$.

Equating the coefficients of $x^{n-1}$ and $x^{n-2}$ gives $k'_n$, and $k''_n$, respectively, as rational multiples w.r.t. $N = q^n$ of $k_n$:

$$
\begin{aligned}
k'_n &= -\frac{(-1+N)\left(eqN - eN + bN - bq\right)q}{(q-1)\left(aq^2 - dN^2q + dN^2 - aN^2\right)}k_n \qquad (5)\\
k''_n &= -k_n\Big(q^2(-1+N)(-N+q)\left(-eq^3Nb + q^3b^2 - cq^3a + cq^2dN^2 + e^2q^2N^2\right.\\
&\quad -b^2q^2N + q^2ca - 2cqdN^2 + 2eqN^2b - 2e^2qN^2\\
&\quad \left.+cqaN^2 - qb^2N + qeNb + b^2N^2 + e^2N^2 - 2eN^2b + cdN^2 - caN^2\right)\Big)\Big/ \qquad (6)\\
&\quad \Big((1+q)(q-1)^2\left(aq^3 - dN^2q + dN^2 - aN^2\right)\left(aq^2 - dN^2q + dN^2 - aN^2\right)\Big).
\end{aligned}
$$

Here and throughout the paper[2], we will use the notations $N = q^n$ and $M = q^m$.

It can be shown (see e.g. [8], [11], [12]) that any solution of (2) satisfies a recurrence equation

$$P_{n+1}(x) = (A_nx + B_n)P_n(x) - C_nP_{n-1}(x) \qquad (n \in \mathbb{N}_{\geq 0}, P_{-1} \equiv 0) \qquad (7)$$

or equivalently

$$xP_n(x) = a_nP_{n+1} + b_nP_n(x) + c_nP_{n-1}(x) \qquad (8)$$

with

$$a_n = \frac{1}{A_n} \quad b_n = -\frac{B_n}{A_n}, \quad c_n = \frac{C_n}{A_n}. \qquad (9)$$

An important point is that the coefficients $A_n$, $B_n$ and $C_n$ appearing in this formula (7) can be computed directly in terms of the coefficients of the polynomials $\sigma(x)$ and $\tau(x)$, which completely characterize the second order $q$-differential equation (2). Their expressions are (see e.g. [8], [11], [12])

$$
\begin{aligned}
\frac{k_n}{k_{n+1}}A_n &= 1, \qquad (10)\\
\frac{k_n}{k_{n+1}}B_n &= [n+1]_q\frac{b[n]_q + eq^n}{a[2n]_q + dq^{2n}} - [n]_q\frac{b[n-1]_q + eq^{n-1}}{a[2(n-1)]_q + dq^{2(n-1)}}, \qquad (11)\\
\frac{k_{n-1}}{k_{n+1}}C_n &= \Big((N-1)(-Nd + Na + Nqd - aq^2)\\
&\quad \times(-a^2cN^4 - N^4d^2c + N^3beqd - abeNq^3 - 2adcN^2q^2\\
&\quad +2aN^2q^2be - aN^3beq + beN^3q^3d - 2beN^3q^2d\\
&\quad +2dcN^2q^3a - 2beN^2q^3a - 2dcN^4qa + beNq^4a\\
&\quad +beN^3aq^2 - b^2q^3dN^2 - a^2cq^4 + N^2b^2q^2d - N^3b^2qd\\
&\quad +aNq^3b^2 - 2ab^2N^2q^2 - ae^2N^2q^2 + aN^3b^2q + 2aN^4dc\\
&\quad +2a^2cq^2N^2 + 2d^2cN^4q - d^2cN^4q^2 + b^2N^3q^2d\\
&\quad +2e^2N^2q^3a - e^2N^2q^4a)Nq\Big)\Big/\Big((dN^2q - aq + aN^2 - dN^2)\\
&\quad \times(dN^2q - dN^2 + aN^2 - aq^2)^2(dN^2q - dN^2 + aN^2 - aq^3)\Big). \qquad (12)
\end{aligned}
$$

In fact, substitute $P_n(x)$ in the proposed equation (7) and equate the three highest coefficients. This yields $A_n$, $B_n$, and $C_n$ in terms of $a$, $b$, $c$, $d$, $e$, $q$, $q^n$, $k_{n-1}$, $k_n$, $k_{n+1}$, $k'_{n-1}$, $k'_n$, $k'_{n+1}$, $k''_{n-1}$, $k''_n$, $k''_{n+1}$ by linear algebra.

Substituting the values of $k'_{n-1}$, $k'_n$, $k'_{n+1}$, $k''_{n-1}$, $k''_n$, and $k''_{n+1}$ given by (5) and (6) yields the above formulas.

$q$-orthogonal polynomials satisfy further structure equations. One of those is given by the $q$-derivative rule (see e.g. [11])

$$\sigma(x)D_{\frac{1}{q}}P_n(x) = \alpha_nP_{n+1}(x) + \beta_nP_n(x) + \gamma_nP_{n-1}(x), \qquad (13)$$

[2]The use of $N = q^n$ should not be confused with the parameter $N$ of the $q$-Hahn polynomials

where the coefficients $\alpha_n$, $\beta_n$ and $\gamma_n$ are given by the explicit formulas

$$\begin{aligned}
\alpha_n &= a[n]_{1/q}\frac{k_n}{k_{n+1}},\\
\beta_n &= \Big((-1+N)(aN+dNq-aq)(eNaq^2-bdN^2q+Nqbq\\
&\quad -bqa+bdN^2-baN^2+bNa-eNa)q\Big)\Big/\\
&\quad \Big((-dN^2q+dN^2+aq^2-aN^2)(q-1)(dN^2q-a+aN^2-dN^2)\Big),\\
\gamma_n &= \Big((-1+N)(aN+dNq-dN-aq)(aq^2-dNq+dN-aN)\\
&\quad (2b^2N^2aq^2+q^4ca^2+b^2q^3dN^2-b^2q^2dN^2+e^2N^2aq^2\\
&\quad +cq^2N^4+cN^4a^2-2cq^3adN^2+eNbaq^3-2eN^2baq^2\\
&\quad -eq^4Nba-eq^2N^3ba+2eq^2N^3bd+2bq^3N^2ea\\
&\quad -eq^3N^3bd+eN^3baq+2cqaN^4d+e^2q^4N^2a-2e^2q^3N^2a\\
&\quad qb^2N^3d-q^3b^2Na-qb^2N^3a-b^2q^2N^3d-2cqd^2N^4\\
&\quad -2q^2N^2ca^2-2dN^4ca+cq^2d^2N^4+2q^2dcaN^2-eN^3bdq)Nq\Big)\Big/\\
&\quad \Big((-aq+dN^2q+aN^2-dN^2)(-dN^2q+dN^2+aq^3-aN^2)\\
&\quad \times(-dN^2q+dN^2+aq^2-aN^2)^2(q-1)\Big)\frac{k_n}{k_{n-1}}.
\end{aligned}$$

Now, we develop further structural identities.

**Proposition 1** (See e.g. [13])**.** *If a function $y(x)$ is solution of (2), then $Y(x) = D_q y(x)$ satisfies*

$$\sigma_1(x)D_qD_{\frac{1}{q}}Y(x)+\tau_1(x)D_qY(x)+\lambda_{n,q,1}Y(x)=0 \tag{14}$$

*with $\sigma_1(x)=\sigma(x)$, $\tau_1(x)=q(D_q\sigma(x)+\tau(qx))$ and $\lambda_{n,q,1}=q(D_q\tau(x)+\lambda_{n,q})$.*

*Proof.* We recall the following identities

$$\begin{aligned}
D_{\frac{1}{q}}D_q &= qD_qD_{\frac{1}{q}} &(15)\\
D_q(f(x)g(x)) &= f(qx)D_qg(x)+g(x)D_qf(x). &(16)
\end{aligned}$$

Applying identity (16) with $g(x)=\sigma(x)$ and $f(x)=D_qD_{\frac{1}{q}}y(x)$, we obtain

$$D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)\right)=\sigma(x)D_q\left(D_qD_{\frac{1}{q}}y(x)\right)+D_q\sigma(x)D_qD_{\frac{1}{q}}y(qx).$$

Using

$$D_{\frac{1}{q}}y(qx)=\frac{y(x)-y(qx)}{\left(\frac{1}{q}-1\right)qx}=\frac{y(qx)-y(x)}{(q-1)x}=D_qy(x),$$

we obtain

$$D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)\right)=\sigma(x)D_q\left(D_qD_{\frac{1}{q}}y(x)\right)+D_q\sigma(x)D_q\left(D_qy(x)\right).$$

Now, using identity (15) for the first term of the right hand side of the previous relation, we get

$$D_q\left(\sigma(x)D_qD_{\frac{1}{q}}y(x)\right)=\frac{1}{q}\sigma(x)D_qD_{\frac{1}{q}}\left(D_qy(x)\right)+D_q\sigma(x)D_q\left(D_qy(x)\right).$$

Identity (16) with $g(x)=D_qy(x)$ and $f(x)=\tau(x)$ yields

$$D_q(\tau(x)D_qy(x))=\tau(qx)D_q\left(D_qy(x)\right)+D_q\tau(x)D_qy(x).$$

Finally,

$$
\begin{aligned}
(2) \quad \Rightarrow \quad & D_q\Big(\sigma(x)D_qD_{\frac{1}{q}}P_n(x)+\tau(x)D_qP_n(x)+\lambda_{n,q}P_n(x)\Big)=0\\
\Rightarrow \quad & \frac{1}{q}\sigma(x)D_qD_{\frac{1}{q}}\left(D_qy(x)\right)+D_q\sigma(x)D_q\left(D_qy(x)\right)\\
& +\tau(qx)D_q\left(D_qy(x)\right)+D_q\tau(x)D_qy(x)+\lambda_{n,q}D_qy(x)=0\\
\Rightarrow \quad & \sigma(x)D_qD_{\frac{1}{q}}\left(D_qy(x)\right)+q\left(D_q\sigma(x)+\tau(qx)\right)D_q\left(D_qy(x)\right)\\
& +q\left(D_q\tau(x)+\lambda_{n,q}\right)D_qy(x)=0,
\end{aligned}
$$

which proves the assertion. □

A computation shows that

$$\sigma_1(x)=a'x^2+b'x+c', \quad \tau_1(x)=d'x+c', \quad \lambda_{n,q,1}=q(\lambda_{n,q}+d) \tag{17}$$

where

$$a'=a,\ b'=b,\ c'=c \quad d'=\left((a+d)q+a\right)q,\ \ e'=(b+e)q. \tag{18}$$

From this, we deduce that the equation

$$xD_qP_n(x)=\alpha_n^\star D_qP_{n+1}(x)+\beta_n^\star D_qP_n(x)+\gamma_n^\star D_qP_{n-1}(x) \tag{19}$$

namely a recurrence equation for the family $D_qP_n(x)$, is valid, and from (18) it follows that

$$\alpha_n^\star=a_n(a,b,c,\left((a+d)q+a\right)q,(b+e)q), \qquad \beta_n^\star=b_n(a,b,c,\left((a+d)q+a\right)q,(b+e)q),$$

and

$$\gamma_n^\star=c_n(a,b,c,\left((a+d)q+a\right)q,(b+e)q)$$

where $a_n(a,b,c,d,e)$, $b_n(a,b,c,d,e)$ and $c_n(a,b,c,d,e)$, are given by (9) and the explicit formulas for $A_n$, $B_n$ and $C_n$.

**Proposition 2** (see e.g. [2], [11], Th.6)**.** *Assume $P_n(x)$ is a solution family of (2). Then a structure formula of the type*

$$P_n(x)=\hat{a}_nD_qP_{n+1}(x)+\hat{b}_nD_qP_n(x)+\hat{c}_nD_qP_{n-1}(x) \tag{20}$$

*is valid for $P_n(x)$.*

In order to obtain those coefficients $\hat{a}_n$, $\hat{b}_n$ and $\hat{c}_n$, we use the same algorithm described in the determination of $A_n$, $B_n$ and $C_n$ where (7) is substituted now by (20). This gives the following

**Theorem 3.** *Assume $P_n(x)$ is a monic solution family of (2), then the coefficients $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$ of (20) are given by:*

$$
\begin{aligned}
\hat{a}_n &= \frac{1}{[n+1]_q}\\
\hat{b}_n &= \Big(N(q-1)(-bdNq^2+eaq^2+edN^2q^2+eN^2qa+bdN^2q-2\,edN^2q\\
&\quad -bNqa+bqa-eaq-eaN^2-bdN^2+baN^2+bNd+edN^2-bNa)\Big)\Big/\\
&\quad \Big(\left(-a+dN^2q-dN^2+aN^2\right)\left(-aq^2+dN^2q-dN^2+aN^2\right)\Big)\\
\hat{c}_n &= \Big(\left(q-1\right)(2\,dN^2q^2ac+ebNq^3a-ebNq^4a+2\,dN^4qac+ebN^3qa-ebN^3q^2a\\
&\quad +2\,bedN^3q^2-bedN^3q^3-bedN^3q-2\,ebN^2q^2a-2\,dN^2q^3ac+2\,ebN^2q^3a\\
&\quad +cq^4a^2+N^4a^2c+N^4cd^2-b^2dN^2q^2+b^2dN^2q^3+d^2N^4q^2c-2\,d^2N^4qc\\
&\quad -b^2N^3qa-2\,N^4acd-b^2dN^3q^2+b^2dN^3q-2\,a^2N^2q^2c-2\,e^2N^2q^3a+2\,b^2N^2q^2a\\
&\quad +e^2q^4N^2a+e^2N^2q^2a-b^2Nq^3a)\left(a+dq-d\right)N^2q\left(N-1\right)\Big)\Big/\\
&\quad \Big(\left(aN^2+dN^2q-dN^2-aq^3\right)\left(-aq^2+dN^2q-dN^2+aN^2\right)^2\left(aN^2+dN^2q-dN^2-aq\right)\Big).
\end{aligned}
$$

Note that, applying the operator $D_q$ to the equation (8), we obtain the following

$$D_q(xP_n(x)) = a_n D_q P_{n+1}(x) + b_n D_q P_n(x) + c_n D_q P_{n-1}(x). \tag{21}$$

Using (16) for the left hand side of (21), we get

$$xD_q P_n(x) = a_n D_q P_{n+1}(x) + b_n D_q P_n(x) + c_n D_q P_{n-1}(x) - P_n(x). \tag{22}$$

Now, we use the structure relation (19) to get the following

**Proposition 4.** *The coefficients $\alpha_n^\star$, $\beta_n^\star$ and $\gamma_n^\star$ of the relation (19) are linked to the coefficients $a_n$, $b_n$ and $c_n$ of the three-term recurrence relation (8) and the coefficients $\hat{a}_n$, $\hat{b}_n$, and $\hat{c}_n$ of (20) by the following formulas:*

$$\alpha_n^\star = \frac{a_n - \hat{a}_n}{q}, \qquad \beta_n^\star = \frac{b_n - \hat{b}_n}{q}, \qquad \gamma_n^\star = \frac{c_n - \hat{c}_n}{q}, \tag{23}$$

*and are given explicitly in the monic case by:*

$$\begin{aligned}
\alpha_n^\star &= \frac{1-N}{1-qN} \\
\beta_n^\star &= -N\Big(-eNq^2a + 2\,bdN^2q - 2\,edN^2q - 2\,bNqa - eaN^2 + eNa + 2\,baN^2 - 2\,bNa \\
&\quad + edN^2 - eaq + 2\,bqa + eaq^2 - 2\,bdN^2 - bdNq^2 + edN^2q^2 + eN^2qa + bNd\Big)\Big/ \\
&\quad \left(-a + dN^2q - dN^2 + aN^2\right)\left(-aq^2 + dN^2q - dN^2 + aN^2\right) \\
\gamma_n^\star &= -\Big(2\,dN^2q^2ac + ebNq^3a - ebNq^4a + 2\,dN^4qac + ebN^3qa \\
&\quad - ebN^3q^2a + 2\,bedN^3q^2 - bedN^3q^3 - bedN^3q - 2\,ebN^2q^2a - \\
&\quad 2\,dN^2q^3ac + 2\,ebN^2q^3a + cq^4a^2 + N^4a^2c + N^4cd^2 - b^2dN^2q^2 + b^2dN^2q^3 \\
&\quad + d^2N^4q^2c - 2\,d^2N^4qc - b^2N^3qa - 2\,N^4acd - b^2dN^3q^2 + b^2dN^3q - 2\,a^2N^2q^2c \\
&\quad - 2\,e^2N^2q^3a + 2\,b^2N^2q^2a + e^2q^4N^2a + e^2N^2q^2a - b^2Nq^3a\Big) \\
&\quad \times Nq\,(N-1)\left(-dN - aq + aN + dNq\right)\Big/ \\
&\quad \left(aN^2 + dN^2q - dN^2 - aq^3\right)\left(-aq^2 + dN^2q - dN^2 + aN^2\right)^2\left(aN^2 + dN^2q - dN^2 - aq\right).
\end{aligned}$$

**Proposition 5.** *Assume a family $P_n(x)$ is a solution of (2). Then a structure formula of the type*

$$(\sigma(x) + (q-1)x\tau(x))D_q P_n(x) = S_n P_{n+1}(x) + T_n P_n(x) + R_n P_{n-1}(x), \tag{24}$$

*is valid for $P_n(x)$ where*

$$S_n = \alpha_n + (1-q)a_n\lambda_n,\; T_n = \beta_n + (1-q)b_n\lambda_n,\; R_n = \gamma_n + (1-q)c_n\lambda_n,$$

*and are in the monic case given explicitly by*

$$
\begin{aligned}
S_n &= \frac{\big(a+dq-d\big)\big(q^n-1\big)}{q-1} \\
T_n &= \Big(-(eq^2N^2d-q^2Nbd+eaq^2-2\,eN^2qd-aqNb+N^2qbd+eaN^2q-eqa \\
&\quad +bqa-bdN^2-bNa-eN^2a+eN^2d+bdN+baN^2)\,(N-1)\,\big(aN+qNd-dN-aq\big)\Big)\Big/ \\
&\quad \Big(\big(q-1\big)\big(-a+N^2qd+aN^2-dN^2\big)\big(-aq^2+N^2qd+aN^2-dN^2\big)\Big) \\
R_n &= \Big((-2\,caN^4d-2\,N^4qd^2c-2\,e^2aN^2q^3+2\,aq^2N^2b^2+e^2aq^2N^2+N^2q^3b^2d-2\,a^2q^2N^2c \\
&\quad -aN^3qb^2+N^3qb^2d+q^2N^4d^2c-N^3q^2b^2d+e^2aN^2q^4-q^2N^2b^2d-aNq^3b^2+ca^2N^4 \\
&\quad +cd^2N^4+cq^4a^2+eaNq^3b-eN^3qbd+2\,eN^3q^2bd+2\,aN^4qdc-2\,eaq^2N^2b \\
&\quad -2\,aN^2q^3dc+2\,eaN^2q^3b+2\,aq^2N^2dc-eN^3q^3bd-eaN^3q^2b+eaN^3qb-eaNq^4b) \\
&\quad (N-1)\,\big(aN-aq^2-dN+qNd\big)\,\big(aN+qNd-dN-aq\big)\,q\Big)\Big/ \\
&\quad \Big(\big(q-1\big)\big(-aq+N^2qd+aN^2-dN^2\big)\big(-aq^3+N^2qd+aN^2-dN^2\big) \\
&\quad \big(-aq^2+N^2qd+aN^2-dN^2\big)^2\Big).
\end{aligned}
$$

*Proof.* The $q$-differential operator obeys the identity

$$D_qD_{1/q}f(x)=\frac{D_qf(x)-D_{1/q}f(x)}{(q-1)x},$$

so that the $q$-differential equation (2) can be rewritten in the form

$$(\sigma(x)+(q-1)x\tau(x))D_qP_n(x)(x)-\sigma(x)D_{1/q}P_n(x)+(q-1)\lambda_nxP_n(x)=0.$$

Next, use the three-term recurrence relation (8) to get rid of the $xP_n(x)$ term and the structure formula (13) to get rid of the $\sigma(x)D_{1/q}P_n(x)$ term. The structure formula (24) is obtained by simplification. □

## 2. *q*-Hypergeometric Representations

Of course, for all polynomials of the $q$-Hahn tableau, $q$-hypergeometric representations are well-known. Nevertheless we would like to recompute these with the methods of this paper algorithmically.

**Definition 6.** *The basic hypergeometric or $q$-hypergeometric function ${}_r\phi_s$ is defined by the series*

$${}_r\phi_s\left(\begin{array}{c}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array}\middle|\,q;z\right):=\sum_{k=0}^{\infty}A_k=\sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_k}{(b_1,\cdots,b_s;q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k},$$

*where*

$$(a_1,\cdots,a_r)_k:=(a_1;q)_k\cdots(a_r;q)_k,\ \ with\ \ (a_i;q)_k=\begin{cases}\prod\limits_{j=0}^{k-1}(1-a_iq^j) & if\ k=1,2,3,\cdots\\ 1 & if\ k=0\end{cases}.$$

*The summand $A_k$ of the $q$-hypergeometric series is a $q$-hypergeometric term, i. e. $\frac{A_{k+1}}{A_k}\in\mathbb{Q}(q,q^k)$ is a rational function.*
*A recurrence equation is called $q$-holonomic if it is linear and if the coefficients $c_k(q^n;q^m;q)$ are $q$-hypergeometric terms w. r. t. $q^n$ and $q^m$, i. e. the ratios*

$$\frac{c_k(q^{n+1};q^m;q)}{c_k(q^n;q^m;q)},\frac{c_k(q^n;q^{m+1};q)}{c_k(q^n;q^m;q)}\in\mathbb{Q}(q^n,q^m,q)$$

*are rational functions.*

For the polynomial systems of the $q$-Hahn tableau we get

**Theorem 7.** *Let $P_n(x)$ be a polynomial system given by the $q$-differential equation (2) with $\sigma(x) = ax^2+bx+c$, and $\tau(x) = dx+e$. Then, the power series coefficients $C_m(n)$ given by*

$$P_n(x) = \sum_{m=0}^{n} C_m(n)x^m \tag{25}$$

*satisfy the recurrence equation*

$$\begin{aligned}(a[m]_{\frac{1}{q}}[m-1]_q + d[m]_q - \lambda_n)C_m(n) + (b[m+1]_{\frac{1}{q}}[m]_q + e[m+1]_q)C_{m+1}(n)\\ +c[m+2]_{\frac{1}{q}}[m+1]_q C_{m+2}(n) = 0,\end{aligned} \tag{26}$$

*with $C_n(n) = 1, C_{n+1}(n) = 0$. In particular, if $c = 0$, then the recurrence equation*

$$(a[m]_{1/q}[m-1]_q + d[m]_q - \lambda_n)C_m(n) + (b[m+1]_{1/q}[m]_q + e[m+1]_q)C_{m+1}(n), \tag{27}$$

*is valid, and therefore $P_n(x)$ has the following $q$-hypergeometric representation up to a constant $K_n$:*

$$P_n(x) = K_n\, {}_2\phi_1\left(\begin{array}{c} q^{-n}, \frac{a-d+dq}{a}q^{n-1} \\ \frac{b-e+eq}{b} \end{array}\middle| q; -\frac{aq}{b}x\right), \qquad ab \neq 0, \tag{28}$$

$$P_n(x) = K_n\, {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ \frac{b-e+eq}{b} \end{array}\middle| q; \frac{d(1-q)q^n}{b}x\right) \qquad a = 0,\, b \neq 0, \tag{29}$$

$$P_n(x) = K_n\, {}_1\phi_0\left(\begin{array}{c} q^{-n} \\ - \end{array}\middle| q; -\frac{dq^n}{e}x\right), \qquad a = b = 0. \tag{30}$$

*Proof.* Substituting the power series (25) into the $q$-differential equation (2), and equating the coefficients yields the recurrence equation (26).

For $c = 0$ this recurrence equation degenerates to a two-term recurrence equation, and hence establishes the $q$-hypergeometric representations (28)-(30), using the initial value $C_n(n) = 1$, $C_{n+1}(n) = 0$. □

We would like to mention that the recurrence equation (26) carries complete information about the $q$-hypergeometric representations given in the theorem.

**Theorem 8.** *Let $P_n(x)$ be a polynomial system given by the $q$-differential equation (2) with $\sigma(x) = ax^2+bx+c$, and $\tau(x) = dx+e$. Then, the power series coefficients $C_m(n)$ given by*

$$P_n(x) = \sum_{m=0}^{n} C_m(n)(x;q)_m \tag{31}$$

*satisfy the recurrence equation*

$$\begin{aligned}&q^n\left(q^{m+2}-1\right)\left(q^{m+1}-1\right)\left(a+q^{m+1}b+cq^{2m+2}\right)C_{m+2}(n) - \left(q^{m+1}-1\right)q\Big(-q^{n+1}a - aq^n\\ &+q^{n+2m+1}b - q^{m+1+n}b + q^{2m+2+n}e - q^{n+2m+1}e + q^{m+2n}a + q^{m+2n+1}d - q^{m+2n}d\\ &+q^{m+1}a\Big)C_{m+1}(n) - \left(-q^m+q^n\right)\left(q^{n+m}a + q^{m+1+n}d - q^{n+m}d - aq\right)q^2C_m(n) = 0,\end{aligned} \tag{32}$$

*where $m = -2, -1, 0, \cdots, n$ and $C_m(n) = 0$ outside the set of $(n,m)$ such that $0 \le m \le n$, with $C_n(n) = 1, C_{n+1}(n) = 0$.*

*Proof.* We first remark the following relations:

$$D_q(x;q)_m = -\frac{[m]_q}{1-x}(x;q)_m \text{ or } D_q(x;q)_m = -[m]_q(qx;q)_{m-1},$$
$$D_{\frac{1}{q}}(x;q)_m = -[m]_q(x;q)_{m-1},$$
$$x(qx;q)_n = q^{-n-1}(qx;q)_n - q^{-n-1}(qx;q)_{n+1}.$$

From these relations, we obtain

$$\begin{aligned}
(x;q)_m &= (1-q^{-m})(qx;q)_{m-1} + q^{-m}(qx;q)_m,\\
x(qx;q)_{m-1} &= q^{-m}(qx;q)_{m-1} - q^{-m}(qx;q)_m,\\
x(qx;q)_{m-2} &= q^{-m+1}(qx;q)_{m-2} - q^{-m+1}(qx;q)_{m-1},\\
x^2(qx;q)_{m-2} &= q^{-2m+2}(qx;q)_{m-2} - (q^{-2m+2}+q^{-2m+1})(qx;q)_{m-1} + q^{-2m+1}(qx;q)_m,\\
(qx;q)_m &= \frac{(x;q)_m}{1-x}.
\end{aligned}$$

Next, we substitute $P_n(x)$ in the $q$-differential equation (2) and obtain (using the preceding relations and simplification)

$$\sum_{m=0}^{n} C_m(n)(x;q)_{m+1}\left(a[m]_q[m-1]_q q^{-2m+1} + d[m]_q q^{-m} + \lambda_n q^{-m}\right)$$
$$+\sum_{m=-1}^{n-1} C_{m+1}(n)(x;q)_{m+1}\Big(-a[m+1]_q[m]_q(q^{-2m}+q^{-2m-1}) - b[m+1]_q[m]_q q^{-m}$$
$$-d[m+1]_q q^{-m-1} - e[m+1]_q + \lambda_n(1-q^{-m-1})\Big) + \sum_{m=-2}^{n-2} C_{m+2}(n)(x;q)_{m+1}\times$$
$$\Big(a[m+2]_q[m+1]_q q^{-2m-2} + b[m+2]_q[m+1]_q q^{-m-1} + c[m+2]_q[m+1]_q\Big) = 0.$$

Since $(x;q)_m$ is a linearly independent family, equating the coefficient of $(x;q)_n$ yields the constant

$$\lambda_n = -a[n]_{\frac{1}{q}}[n-1]_q - d[n]_q.$$

Equating the coefficients of $(x;q)_{m+1}$, yields the desired recurrence equation satisfied by the coefficients $C_m(n)$. □

The above computations show that in the general case, we get a $q$-holonomic three-term recurrence equation for $C_m(n)$. In order to find solutions which are $q$-hypergeometric terms—hence satisfying a first-order $q$-holonomic recurrence—in some specific situations, we can use a $q$-version of Petkovšek's algorithm (see e. g. [9], [15]) which was given by Abramov, Paule and Petkovšek [1] and by Böing and Koepf [5]. This algorithm can be used utilizing the `qrecsolve` command of the `qsum` package in Maple [5].

The $q$-Petkovšek algorithm can be successfully used for several instances in this paper. However, this algorithm is rather inefficient and therefore not at all suitable for many of our complicated questions posed. Fortunately Horn [6] published a refined version which is much more efficient. Sprenger [16] presented a Maple implementation of this refined version `qHypergeomSolveRE` in his package `qFPS` which finds easily the $q$-hypergeometric term solutions of all $q$-recurrence equations of this paper. It is due to this algorithm that we can state all our results, especially in § 5.

In particular, we can solve the recurrence equations of Theorems 7 and 8 for all particular systems and therefore obtain the $q$-hypergeometric representations up to a constant $K_n$. Here, we consider the monic cases such that by equating the highest coefficients of (25) and (31), we can recover the constant $K_n$. This method yields

**Corollary 9.** *The following representations of monic orthogonal polynomials of the q-Hahn class are valid:*

$$\begin{aligned}
\tilde{P}_n(x,\alpha,\beta,\gamma;q) &= \frac{(\alpha q;q)_n(\gamma q;q)_n}{(\alpha\beta q^{n+1};q)_n}{}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},x \\ \alpha q,\gamma q\end{array}\middle| q;q\right) \text{ for the Big q-Jacobi family}\\
\tilde{Q}_n(x,\alpha,\beta,N,q) &= \frac{(\alpha q;q)_n(q^{-N};q)_n}{(\alpha\beta q^{n+1};q)_n}{}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},x \\ \alpha q,q^{-N}\end{array}\middle| q;q\right) \text{ for the q-Hahn family}\\
\tilde{P}_n(x,\alpha,\beta,q) &= (\alpha q;q)_n(\beta q;q)_n {}_3\phi_2\left(\begin{array}{c} q^{-n},0,x \\ \alpha q,\beta q\end{array}\middle| q;q\right) \text{ for the Big q-Laguerre family}\\
\tilde{p}_n(x;\alpha,\beta|q) &= \frac{(-1)^n q^{\frac{n(n-1)}{2}}(\alpha q;q)_n}{(\alpha\beta q^{n+1};q)_n}{}_2\phi_1\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1} \\ \alpha q\end{array}\middle| q;qx\right), \text{ for the Little q-Jacobi}\\
\tilde{K}_n(x,\alpha,q) &= \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(-\alpha q^n;q)_n}{}_2\phi_1\left(\begin{array}{c} q^{-n},-\alpha q^n \\ 0\end{array}\middle| q;qx\right) \text{ for Alternative q-Charlier}\\
\tilde{p}_n(x,\alpha|q) &= (-1)^n q^{\frac{n(n-1)}{2}}(\alpha q;q)_n {}_2\phi_1\left(\begin{array}{c} q^{-n},0 \\ \alpha q\end{array}\middle| q,qx\right) \text{ for Little q-Laguerre/Wall}\\
\tilde{M}_n(x,\beta,\gamma;q) &= (-\gamma)^n q^{n^2}(\beta q;q)_n {}_2\phi_1\left(\begin{array}{c} q^{-n},x \\ \beta q\end{array}\middle| q;-\frac{q^{n+1}}{\gamma}\right) \text{ for the q-Meixner family}\\
\tilde{C}_n(x,\alpha,q) &= (-\alpha)^n q^{-n^2} {}_2\phi_1\left(\begin{array}{c} q^{-n},x \\ 0\end{array}\middle| q;-\frac{q^{n+1}}{\alpha}\right) \text{ for the q-Charlier family}\\
\tilde{L}_n^{(\alpha)}(x;q) &= \frac{(-1)^n(q^{\alpha+1}:q)_n}{q^{n(n+\alpha)}}{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1}\end{array}\middle| q;-xq^{\alpha+n+1}\right) \text{ for q-Laguerre}\\
\tilde{S}_n(x;q) &= (-1)^n q^{-n^2} {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ 0\end{array}\middle| q;-xq^{n+1}\right) \text{ for Stieltjes-Wigert}\\
\tilde{V}_n^{(\alpha)}(x,q) &= (-\alpha)^n q^{-\frac{n(n-1)}{2}}) {}_2\phi_0\left(\begin{array}{c} q^{-n},x \\ -\end{array}\middle| q;\frac{q^n}{\alpha}\right) \text{ for the Al Salam-Carlitz II family}\\
\tilde{H}_n(x,q) &= q^{-\frac{n(n-1)}{2}} {}_2\phi_0\left(\begin{array}{c} q^{-n},x \\ -\end{array}\middle| q,-q^n\right) \text{ for the Discrete q-Hermite II family.}
\end{aligned}$$

*Here and throughout the paper, the discrete q-Hermite II polynomials $H_n(x;q) = \tilde{h}_n(-ix;q)$ are evaluated in $x$ instead of $ix$.*

## 3. Power and $q$-Pochhammer Representation

Whereas in the last section we considered the specific connection problems for $Q_m(x) = x^m$ and $Q_m(x) = (x;q)_m$, in this section the opposite problems, having $P_n(x) = x^n$ and $P_n(x) = (x;q)_n$, is studied.

In many applications, one wants to develop a given polynomial in terms of a given orthogonal polynomial system. In this case, handy formulas for the powers $x^n$ or/and $(x;q)_n$ are very welcome.

### *3.1. Structure Formulas of the Bases $x^n$ and $(x;q)_n$.*

We need the following three relations in order to solve the inversion problem.

$$\begin{aligned}
xP_n(x) &= a_nP_{n+1}(x)+b_nP_n(x)+c_nP_{n-1}(x) &(33)\\
\sigma(x)D_{\frac{1}{q}}P_n(x) &= \alpha_nP_{n+1}(x)+\beta_nP_n(x)+\gamma_nP_{n-1}(x) &(34)\\
xD_qP_n(x) &= \alpha_n^{\star}D_qP_{n+1}(x)+\beta_n^{\star}D_qP_n(x)+\gamma_n^{\star}D_qP_{n-1}(x). &(35)
\end{aligned}$$

*3.1.1. The case $x^n$*

**Proposition 10.** *The polynomial $x^n$ satisfies (33), (34) and (35) with*

*(a)* $a_n = 1$, $b_n = c_n = 0$;

*(b)* $\alpha_n = \bar{a}[n]_{\frac{1}{q}}$, $\beta_n = \bar{b}[n]_{\frac{1}{q}}$ *and* $\gamma_n = \bar{c}[n]_{\frac{1}{q}}$;

*(c)* $\alpha_n^\star = \frac{q^n-1}{q^{n+1}-1}$; $\beta_n^\star = \gamma_n^\star = 0$.

*3.1.2. The case $(x;q)_n$*

**Proposition 11.** *The polynomial $(x;q)_n$ satisfies (33), (34) and (35) with*

*(a)* $a_n = -q^{-n}$, $\qquad b_n = q^{-n}$, $\qquad c_n = 0$;

*(b)* $\alpha_n = -\bar{a}[n]_q a_n a_{n-1}$, $\qquad \beta_n = -[n]_q\left(\bar{a}a_{n-1}(b_n + b_{n-1}) + \bar{b}a_{n-1}\right)$ *and* $\gamma_n = -[n]_q\left(\bar{a}b_{n-1}^2 + \bar{b}b_{n-1} + \bar{c}\right)$;

*(c)* $\alpha_n^\star = \frac{q^{-n}-1}{q^{n+1}-1}$; $\beta_n^\star = q^{-n}$ *and* $\gamma_n^\star = 0$.

*Proof.* (a) Since $(x;q)_{n+1} = (x;q)_n(1-q^n x)$, we have

$$x(x;q)_n = q^{-n}(x;q)_n - q^{-n}(x;q)_{n+1},$$

from which we deduce the coefficients $a_n$, $b_n$ and $c_n$.

(b) First we remark that $D_{\frac{1}{q}}(x;q)_n = -[n]_q(x;q)_{n-1}$. Next we multiply both members of this equality by $\bar{\sigma}(x)$ to get

$$\bar{\sigma}(x)D_{\frac{1}{q}}(x;q)_n = -[n]_q\left(\bar{a}x^2(x;q)_{n-1} + \bar{b}x(x;q)_{n-1} + \bar{c}(x;q)_{n-1}\right).$$

We use the three-term recurrence relation (33) to get

$$x(x;q)_{n-1} = a_{n-1}(x;q)_n + b_{n-1}(x;q)_{n-1}.$$

We repeat the process to obtain

$$x^2(x;q)_{n-1} = a_n a_{n-1}(x;q)_{n+1} + a_{n-1}(b_n + b_{n-1})(x;q)_n + b_{n-1}^2(x;q)_{n-1}.$$

The coefficients $\alpha_n$, $\beta_n$ and $\gamma_n$ are obtained by simplification.

(c) Using the fact that $D_q(x;q)_n = -[n]_q(qx;q)_{n-1}$, we obtain

$$\begin{aligned} xD_q(x;q)_n &= -[n]_q x(qx;q)_{n-1} \\ &= -[n]_q\left\{q^{-n}(qx,q)_{n-1} - q^{-n}(qx;q)_n\right\} \\ &= -[n]_q q^{-n}(qx;q)_{n-1} - \frac{[n]_q}{[n+1]_q}[n+1]_q(qx;q)_n \\ &= q^{-n}D_q(x;q)_n - \frac{[n]_q}{[n+1]_q}q^{-n}D_q(x;q)_{n+1}, \end{aligned}$$

and the result follows. □

**Remark 12.** *The following structure formula*

$$P_n(x) = \hat{a}_n D_q P_{n+1}(x) + \hat{b}_n D_q P_n(x) + \hat{c}_n D_q P_{n-1}(x) \tag{36}$$

*can be also important. The polynomial basis $\mathcal{V}_n(x) = x^n$ satisfies (36) with*

$$\hat{a}_n = \frac{1}{[n+1]_q}, \qquad \hat{b}_n = 0, \qquad \hat{c}_n = 0;$$

*and the polynomial basis $\mathcal{V}_n(x) = (x;q)_n$ satisfies (36) with*

$$\hat{a}_n = -\frac{q^{-n}}{[n+1]_q}, \qquad \hat{b}_n = \frac{q^{-n}-1}{[n]_q}, \qquad \hat{c}_n = 0.$$

*3.2. Power and q-Pochhammer Representation*

**Theorem 13.** *Let $Q_m(x)$ be monic polynomial system given by the $q$-differential equation (2) with $\overline{\sigma}(x) = \bar{a}x^2 + \bar{b}x + \bar{c}$, and $\overline{\tau}(x) = \bar{d}x + \bar{e}$. Then, the coefficients $C_m(n)$ of the power representation*

$$x^n = \sum_{m=0}^{n} C_m(n) Q_m(x) \tag{37}$$

*satisfy the recurrence equation*

$$\begin{aligned}
&Mq(q-1)(M-1)(Mq-1)\left(-\bar{a}q^2 + M^2q\bar{d} - M^2\bar{d} + \bar{a}M^2\right)\\
&\times\Big(-M^3q^2\bar{d}\bar{e}\bar{b} - \bar{a}M^3q\bar{e}\bar{b} - \bar{a}qM\bar{e}\bar{b} + 2\,\bar{c}\bar{a}qM^4\bar{d} + 2\,M^3q\bar{d}\bar{e}\bar{b} + 2\,\bar{a}M^2q\bar{e}\bar{b} - 2\,\bar{c}\bar{a}M^2q\bar{d} + \bar{c}\bar{a}^2 + 2\,\bar{a}M^2\bar{b}^2\\
&-M^2\bar{d}\bar{b}^2 + \bar{c}\bar{d}^2M^4 + M^3\bar{d}\bar{b}^2 + \bar{c}\bar{a}^2M^4 - \bar{a}M^3\bar{b}^2 - 2\,\bar{c}\bar{a}^2M^2 - \bar{a}M\bar{b}^2 + \bar{a}M^2\bar{e}^2 - M^3\bar{d}\bar{e}\bar{b} + \bar{a}M^3\bar{e}\bar{b} + M^2q\bar{d}\bar{b}^2\\
&+\bar{a}M^2q^2\bar{e}^2 + \bar{c}q^2M^4\bar{d}^2 - M^3q\bar{d}\bar{b}^2 + 2\,\bar{c}\bar{a}M^2\bar{d} - 2\,\bar{a}M^2\bar{e}\bar{b} - 2\,\bar{c}\bar{a}M^4\bar{d} - 2\,\bar{c}qM^4\bar{d}^2 - 2\,\bar{a}M^2q\bar{e}^2 + \bar{a}M\bar{e}\bar{b}\Big)\\
&\times\left(-\bar{a} + M\bar{a}N + M\bar{d}Nq - \bar{d}MN\right)C_{m+1}(n) + (q-1)\left(-\bar{a} + M^2q\bar{d} - M^2\bar{d} + \bar{a}M^2\right)(M-1)Mq\Big(\bar{b}\bar{a}M^2 + M^2q\bar{d}\bar{b}\\
&-\bar{b}\bar{d}M^2 - \bar{a}Mqb - \bar{b}M\bar{a} + \bar{b}q\bar{a} - \bar{e}\bar{a}q^2M + \bar{e}M\bar{a} + qNM^2\bar{a}\bar{e} - NM^2\bar{a}\bar{e} + q^2NM^2\bar{d}\bar{e} - 2\,qNM^2\bar{d}\bar{e} + NM^2\bar{d}\bar{e}\\
&-\bar{a}qN\bar{e} + q^2N\bar{a}\bar{e} + NM^2\bar{a}\bar{b} + qNM^2\bar{d}\bar{b} - NM^2\bar{d}\bar{b} - NMq\bar{a}\bar{b} - NM\bar{a}\bar{b} - q^2NM\bar{d}\bar{b} + NM\bar{d}\bar{b} + \bar{a}qN\bar{b}\Big)\\
&\times\left(-\bar{a}q + M^2q\bar{d} - M^2\bar{d} + \bar{a}M^2\right)\left(-\bar{a} + q^2M^2\bar{d} - M^2q\bar{d} + \bar{a}qM^2\right)C_m(n) - (q-1)\left(-\bar{a}q + M^2q\bar{d} - M^2\bar{d} + \bar{a}M^2\right)\\
&\left(q^2M^2\bar{d} - M^2q\bar{d} + \bar{a}qM^2 - \bar{a}\right)\left(M^2q\bar{d} - M^2\bar{d} + \bar{a}M^2 - \bar{a}q^2\right)\left(M^2q\bar{d} - M^2\bar{d} + \bar{a}\bar{a}M^2 - \bar{a}\bar{a}\right)^2\\
&\left(M - qN\right)C_{m-1}(n).
\end{aligned} \tag{38}$$

*Proof.* For $P_n(x) = x^n$, we substitute $xP_n(x)$ and $xQ_m(x)$ in $xP_n(x) = \sum_{m=0}^{n} C_m(n)xQ_m(x)$ using the three-term recurrence equations

$$\begin{aligned}
xP_n(x) &= a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x),\\
xQ_m(x) &= \bar{a}_mQ_{m+1}(x) + \bar{b}_mQ_m(x) + \bar{c}_mQ_{m-1}(x),
\end{aligned}$$

and after an index shift, we obtain

$$a_nC_m(n+1) + b_nC_m(n) + c_nC_m(n-1) = \bar{a}_{m-1}C_{m-1}(n) + \bar{b}_mC_m(n) + \bar{c}_{m+1}C_{m+1}(n). \tag{39}$$

Using both recurrence equations for the $q$-derivatives yields by the same process

$$\alpha_n^\star C_m(n+1) + \beta_n^\star C_m(n) + \gamma_n^\star C_m(n-1) = \bar{\alpha}_{m-1}^\star C_{m-1}(n) + \bar{\beta}_m^\star C_m(n) + \bar{\gamma}_{m+1}^\star C_{m+1}(n). \tag{40}$$

In a similar way, we also get

$$\alpha_nC_m(n+1) + \beta_nC_m(n) + \gamma_nC_m(n-1) = \bar{\alpha}_{m-1}C_{m-1}(n) + \bar{\beta}_mC_m(n) + \bar{\gamma}_{m+1}C_{m+1}(n), \tag{41}$$

To obtain a pure recurrence equation with respect to $m$, from (39),(40) and (41), we eliminate the variables $C_m(n+1)$ and $C_m(n-1)$ by linear algebra and substitute the coefficients of Proposition 10 to get the result. □

**Theorem 14.** *Let $Q_m(x)$ be monic polynomial system given by the $q$-differential equation (2) with $\sigma(x) = \bar{a}x^2 + \bar{b}x + \bar{c}$, and $\tau(x) = \bar{d}x + \bar{e}$. Then, the coefficients $C_m(n)$ of the power representation*

$$(x;q)_n = \sum_{m=0}^{n} C_m(n) Q_m(x) \tag{42}$$

*satisfy the recurrence equation*

$$Mq\left(q-1\right)\left(M-1\right)\left(-1+Mq\right)\left(-\bar{a}q^{2}+M^{2}q\bar{d}-M^{2}\bar{d}+\bar{a}M^{2}\right)\Big(-M^{3}q^{2}\bar{d}\bar{e}\bar{b}-\bar{a}M^{3}q\bar{e}\bar{b}-\bar{a}qM\bar{e}\bar{b}$$
$$+2\,\bar{c}\bar{a}qM^{4}\bar{d}+2\,M^{3}q\bar{d}\bar{e}\bar{b}+2\,\bar{a}M^{2}q\bar{e}\bar{b}-2\,\bar{c}\bar{a}M^{2}q\bar{d}+\bar{c}\bar{a}^{2}+2\,\bar{a}M^{2}\bar{b}^{2}-M^{2}\bar{d}\bar{b}^{2}+\bar{c}\bar{d}^{2}M^{4}+M^{3}\bar{d}\bar{b}^{2}+\bar{c}\bar{a}^{2}M^{4}$$
$$-\bar{a}M^{3}\bar{b}^{2}-2\,\bar{c}\bar{a}^{2}M^{2}-\bar{a}M\bar{b}^{2}+\bar{a}M^{2}\bar{e}^{2}-M^{3}\bar{d}\bar{e}\bar{b}+\bar{a}M^{3}\bar{e}\bar{b}+M^{2}q\bar{d}\bar{b}^{2}+\bar{a}M^{2}q^{2}\bar{e}^{2}+\bar{c}q^{2}M^{4}\bar{d}^{2}-M^{3}q\bar{d}\bar{b}^{2}$$
$$+2\,\bar{c}\bar{a}M^{2}\bar{d}-2\,\bar{a}M^{2}\bar{e}\bar{b}-2\,\bar{c}\bar{a}M^{4}\bar{d}-2\,\bar{c}qM^{4}\bar{d}^{2}-2\,\bar{a}M^{2}q\bar{e}^{2}+\bar{a}M\bar{e}\bar{b}\Big)\left(-\bar{a}+M\bar{a}N+M\bar{d}Nq-\bar{d}MN\right)c_{m+1}\left(n\right)$$
$$+q\left(q-1\right)\left(M-1\right)\left(-\bar{a}+M^{2}q\bar{d}-M^{2}\bar{d}+\bar{a}M^{2}\right)\left(-\bar{a}+q^{2}M^{2}\bar{d}-M^{2}q\bar{d}+\bar{a}qM^{2}\right)\left(-\bar{a}q+M^{2}q\bar{d}-M^{2}\bar{d}+\bar{a}M^{2}\right)$$
$$\times\Big(\bar{a}M^{2}\bar{e}+\bar{a}Mq\bar{b}-\bar{b}\bar{a}M^{2}+\bar{b}q^{n}\bar{a}Mq-\bar{b}q^{n}\bar{a}qM^{2}+\bar{e}q^{n}M\bar{a}q^{2}-\bar{b}q^{n}q^{2}M^{2}\bar{d}+\bar{b}q^{n}M^{3}q\bar{d}-\bar{e}q^{n}\bar{a}Mq$$
$$+\bar{e}q^{n}M^{3}q^{2}\bar{d}+\bar{e}q^{n}M^{3}q\bar{a}-2\,\bar{e}q^{n}M^{3}q\bar{d}+\bar{a}^{2}q^{2}-M^{2}q^{2}\bar{a}^{2}+M^{4}q^{2}\bar{d}^{2}+\bar{a}M^{2}\bar{d}-2\,M^{4}q\bar{d}^{2}-2\,M^{4}\bar{d}\bar{a}-M^{3}\bar{b}\bar{d}$$
$$+M^{3}\bar{b}\bar{a}-\bar{b}q^{n}\bar{a}M^{2}+\bar{b}q^{n}M^{2}\bar{d}-\bar{a}M^{2}q\bar{b}-\bar{a}M^{2}q\bar{d}+\bar{a}q^{2}M^{2}\bar{d}-\bar{a}q^{3}M^{2}\bar{d}+M^{3}\bar{b}q\bar{d}-M^{2}\bar{e}\bar{a}q^{2}-\bar{e}q^{n}\bar{a}M^{3}$$
$$+\bar{e}q^{n}M^{3}\bar{d}+\bar{b}q^{n}\bar{a}M^{3}-\bar{b}q^{n}M^{3}\bar{d}+2\,M^{4}q\bar{d}\bar{a}-\bar{a}^{2}M^{2}+M^{4}\bar{d}^{2}+\bar{a}^{2}M^{4}\Big)c_{m}\left(n\right)$$
$$-\left(q-1\right)\left(-\bar{a}q+M^{2}q\bar{d}-M^{2}\bar{d}+\bar{a}M^{2}\right)\left(-\bar{a}+q^{2}M^{2}\bar{d}-M^{2}q\bar{d}+\bar{a}qM^{2}\right)\left(-\bar{a}q^{2}+M^{2}q\bar{d}-M^{2}\bar{d}+\bar{a}M^{2}\right)$$
$$\left(-\bar{a}+M^{2}q\bar{d}-M^{2}\bar{d}+\bar{a}M^{2}\right)^{2}\left(M-qN\right)c_{m-1}\left(n\right). \tag{43}$$

*Proof.* Here we use the structure formulas of Proposition 11 and proceed as in Theorem 13. □

If we use one of the variants of $q$-Petkovšek's algorithm we can again solve the recurrences of Theorems 13 and 14 leading to

**Corollary 15.** *(Compare [2], Table 3) The inversion coefficients of each monic polynomial systems of the $q$-Hahn class are given by Table 2.*

## 4. Connection Coefficients Using the Inversion Formulas

If

$$P_n(x)=\sum_{j=0}^{n}A_j(n)\mathcal{V}_j(x)\text{ and }\mathcal{V}_j(x)=\sum_{m=0}^{j}B_m(j)Q_m(x)$$

where $\mathcal{V}_j(x)=x^j$ or $\mathcal{V}_j(x)=(x;q)_j$ then

$$P_n(x)=\sum_{j=0}^{n}A_j(n)\Big(\sum_{m=0}^{j}B_m(j)Q_m(x)\Big),$$

and by rearranging the order of summation gives

$$P_n(x)=\sum_{m=0}^{n}C_m(n)Q_m(x),\text{ with }C_m(n)=\sum_{j=m}^{n}A_j(n)B_m(j)=\sum_{j=0}^{n-m}A_{j+m}(n)B_m(j+m).$$

We can also use the relations

$$x^n=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(-1)^mq^{\frac{m(m+1)}{2}}}{q^{mn}}(x;q)_m,\text{ and }(x;q)_n=\sum_{m=0}^{n}(-1)^mq^{\frac{m(m-1)}{2}}\begin{bmatrix} n\\ m\end{bmatrix}_q x^m$$

to obtain a connection between two families which are not represented in the same basis. In fact, we suppose that

$$P_n(x)=\sum_{j=0}^{n}A_j(n)\mathcal{V}_j(x),\ Q_m(x)=\sum_{k=0}^{m}d_k(m)\theta_k(x), \tag{44}$$

| Family | Basis | $C_m(n)$ |
|---|---|---|
| Big $q$-Jacobi | $\{(x;q)_n\}_n$ | $(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(\alpha q^{m+1},\gamma q^{m+1};q)_{n-m}}{(\alpha\beta q^{2(m+1)};q)_{n-m}}$ |
| $q$-Hahn | $\{(x;q)_n\}_n$ | $\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\frac{m(m-1)}{2}} (\alpha q^{m+1},q^{m-N};q)_{n-m}}{(\alpha\beta q^{2(m+1)};q)_m(\alpha q;q)_{n-m}(\alpha\beta q^{n+2};q)_m}$ |
| Big $q$-Laguerre | $\{(x;q)_n\}_n$ | $(-1)^m q^{\frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (\alpha q^{m+1},\beta q^{m+1};q)_{n-m}$ |
| $q$-Meixner | $\{(x;q)_n\}_n$ | $(-1)^n \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(3m+1)}{2}-n(m+1)} \gamma^{n-m}(\beta q^{m+1};q)_{n-m}$ |
| $q$-Charlier | $\{(x;q)_n\}_n$ | $(-1)^n \alpha^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(3m+1)}{2}-n(m+1)}$ |
| Al-Salam-Carlitz II | $\{(x;q)_n\}_n$ | $(-1)^n \alpha^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)+\frac{n(n-1)}{2}}$ |
| Discrete $q$-Hermite II | $\{(x;q)_n\}_n$ | $(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)+\frac{n(n-1)}{2}}$ |
| Little $q$-Jacobi | $\{x^n\}_n$ | $\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha q^{m+1};q)_{n-m}}{(\alpha\beta q^{2(m+1)};q)_{n-m}}$ |
| Alternative $q$-Charlier | $\{x^n\}_n$ | $\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{1}{(-\alpha q^{2m+1};q)_{n-m}}$ |
| Little $q$-Laguerre Wall | $\{x^n\}_n$ | $\begin{bmatrix} n \\ m \end{bmatrix}_q (\alpha q^{m+1};q)_{n-m}$ |
| $q$-Laguerre | $\{x^n\}_n$ | $\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{(m-n)(2\alpha+3m+n+1)}{2}} (q^{\alpha+m+1};q)_{n-m}$ |
| Stieltjes-Wigert | $\{x^n\}_n$ | $\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{(m-n)(3m+n+1)}{2}}$ |

Table 2: Inversion coefficients for each monic family inside the $q$-Hahn tableau

where the expanding bases $\mathcal{V}_j(x)$ and $\theta_j(x)$ can be any of the polynomials $x^j$, $(x;q)_j$,

$$\mathcal{V}_j(x) = \sum_{k=0}^{j} B_k(j)\theta_k(x), \text{and } \theta_k(x) = \sum_{m=0}^{k} D_m(k)Q_m(x). \tag{45}$$

From these three expansions, we obtain the representation as multiple sum

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x) \text{ with } C_m(n) = \sum_{j=0}^{n-m} \sum_{k=0}^{j} A_{j+m}(n) B_{k+m}(j+m) D_m(k+m).$$

## 5. Connection Coefficients for $\sigma(x) = \bar{\sigma}(x)$

Koepf and Schmersau [10] solved the problem to determine the connection coefficients between different polynomial systems (continuous case and discrete case) by an algorithmic approach. In this section, following the same algorithmic method, we consider the same problem for the $q$-case.

Here, we assume that $P_n(x) = k_n x^n + \cdots$ denotes a family of polynomials of degree exactly $n$ and $Q_m(x) = \bar{k}_m x^m + \cdots$ denotes a family of polynomials of degree exactly $m$. We want to determine the connection coefficients $C_m(n)$, $(n \in \mathbb{N}, m = 0, \cdots, n)$, between the systems $P_n(x)$ and $Q_m(x)$,

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x). \tag{46}$$

We assume that $C_m(n) = 0$ outside the above $n \times m$ region. We will denote all coefficients connected with $Q_m(x)$ by dashes.

We substitute $xP_n(x)$ and $xQ_m(x)$ in $xP_n(x) = \sum_{m=0}^{n} C_m(n) x Q_m(x)$ using the three-term recurrence equations

$$\begin{aligned} xP_n(x) &= a_n P_{n+1}(x) + b_n P_n(x) + c_n P_{n-1}(x), \\ xQ_m(x) &= \bar{a}_m Q_{m+1}(x) + \bar{b}_m Q_m(x) + \bar{c}_m Q_{m-1}(x), \end{aligned}$$

and after an index shift, we obtain

$$a_n C_m(n+1) + b_n C_m(n) + c_n C_m(n-1) = \bar{a}_{m-1} C_{m-1}(n) + \bar{b}_m C_m(n) + \bar{c}_{m+1} C_{m+1}(n). \tag{47}$$

Using both recurrence equations for the $q$-differentials yields by the same process

$$\alpha_n^\star C_m(n+1) + \beta_n^\star C_m(n) + \gamma_n^\star C_m(n-1) = \bar{\alpha}_{m-1}^\star C_{m-1}(n) + \bar{\beta}_m^\star C_m(n) + \bar{\gamma}_{m+1}^\star C_{m+1}(n). \tag{48}$$

In a similar way, we also get

$$\hat{a}_n C_m(n+1) + \hat{b}_n C_m(n) + \hat{c}_n C_m(n-1) = \hat{\bar{a}}_{m-1} C_{m-1}(n) + \hat{\bar{b}}_m C_m(n) + \hat{\bar{c}}_{m+1} C_{m+1}(n). \tag{49}$$

In the specific case $\sigma(x) = \bar{\sigma}(x)$ one can use the same procedure to obtain

$$\alpha_n C_m(n+1) + \beta_n C_m(n) + \gamma_n C_m(n-1) = \bar{\alpha}_{m-1} C_{m-1}(n) + \bar{\beta}_m C_m(n) + \bar{\gamma}_{m+1} C_{m+1}(n). \tag{50}$$

To obtain a pure recurrence equation with respect to $m$, from (47), (48) and (50), we eliminate the variables $C_m(n+1)$ and $C_m(n-1)$, by linear algebra. Using this approach, we get the following recurrence equation.

**Theorem 16.** *Let $P_n(x)$ be a monic polynomial system given by the $q$-differential equation (2) with $\sigma(x) = ax^2 + bx + c$, and $\tau(x) = dx + e$ and $Q_m(x)$ be a monic polynomial system given by (2) with $\bar{\sigma}(x) = \sigma(x)$, and $\tau(x) = \bar{d}x + \bar{e}$. Then the relation (46) is valid, $C_m(n)$ satisfying the second order recurrence equation with respect to $m$*

$$
\begin{aligned}
&(Mq-1)\big(cM^4a^2+cM^4\bar{d}^2-2\,cM^2a^2-b^2M^3a+b^2M^3\bar{d}+2\,b^2aM^2-b^2\bar{d}\,M^2-b^2Ma+\bar{e}^2aM^2+ca^2\\
&+2\,cM^4a\bar{d}\,q-2\,cM^2a\bar{d}\,q-\bar{e}\,bM^3qa-\bar{e}\,bM^3q^2\bar{d}+2\,\bar{e}\,bM^3\bar{d}\,q+2\,\bar{e}\,baqM^2-\bar{e}\,bMaq-2\,cM^4a\bar{d}\\
&+cM^4q^2\bar{d}^2-2\,cM^4q\bar{d}^2+2\,cM^2a\bar{d}-b^2M^3\bar{d}\,q+b^2\bar{d}\,M^2q-2\,\bar{e}^2aqM^2+\bar{e}^2M^2aq^2+\bar{e}\,bM^3a-\bar{e}\,bM^3\bar{d}\\
&-2\,\bar{e}\,baM^2+\bar{e}\,bMa\big)\big(-aq^3+dN^2q+aN^2-dN^2\big)\big(-a+dN^2q+aN^2-dN^2\big)\big(-aq+dN^2q+aN^2-dN^2\big)\big(-aq^2\\
&+dN^2q+aN^2-dN^2\big)^2(M-qN)\big(MaN+M\bar{d}\,Nq-M\bar{d}\,N-a\big)C_m(1+n)+(Mq-1)\big(cM^4a^2+cM^4\bar{d}^2-2\,cM^2a^2\\
&-b^2M^3a+b^2M^3\bar{d}+2\,b^2aM^2-b^2\bar{d}\,M^2-b^2Ma+\bar{e}^2aM^2+ca^2+2\,cM^4a\bar{d}\,q-2\,cM^2a\bar{d}\,q\\
&-\bar{e}\,bM^3qa-\bar{e}\,bM^3q^2\bar{d}+2\,\bar{e}\,bM^3\bar{d}\,q+2\,\bar{e}\,baqM^2-\bar{e}\,bMaq-2\,cM^4a\bar{d}+cM^4q^2\bar{d}^2-2\,cM^4q\bar{d}^2\\
&+2\,cM^2a\bar{d}-b^2M^3\bar{d}\,q+b^2\bar{d}\,M^2q-2\,\bar{e}^2aqM^2+\bar{e}^2M^2aq^2+\bar{e}\,bM^3a-\bar{e}\,bM^3\bar{d}-2\,\bar{e}\,baM^2+\bar{e}\,bMa\big)N(qN-1)\\
&\times\big(-aq+dN^2q+aN^2-dN^2\big)\big(-aq^2+dN^2q+aN^2-dN^2\big)\big(-aq^3+dN^2q+aN^2-dN^2\big)\big(-M\bar{e}\,N^4d^2q^3\\
&-MbN^4d^2q^2+bN^2adq-2\,eN^3adq^2+eN^3adq+\bar{d}\,MeN^2aq^2-\bar{d}\,MeN^2aq-2\,MbN^4adq+MbN^3adq^2\\
&+MbN^3adq+MbN^2adq^3-2\,M\bar{e}\,N^4adq^2+4\,M\bar{e}\,N^4adq+M\bar{e}\,N^2adq^4+M^2eN^3adq^2-2\,M^2eN^3adq\\
&+M^2bN^3adq+b\bar{d}\,M^2N^3aq+b\bar{d}\,M^2N^3dq^2-2\,b\bar{d}\,M^2N^3dq+b\bar{d}\,MN^3aq^2-b\bar{d}\,MN^3aq+b\bar{d}\,MN^3dq^3\\
&+b\bar{d}\,MN^3dq-2\,b\bar{d}\,MN^3dq^2-b\bar{d}\,MN^2aq^3+b\bar{d}\,MN^2aq+\bar{d}\,M^2eN^3aq^2-2\,\bar{d}\,M^2eN^3aq\\
&+\bar{d}\,M^2eN^3dq^3-3\,\bar{d}\,M^2eN^3dq^2+3\,\bar{d}\,M^2eN^3dq-MeN^2adq^3+MeN^2adq^2+MeN^2adq+MbN^3d^2+M\bar{e}\,N^4a^2\\
&+M\bar{e}\,N^4d^2-M\bar{e}\,N^2a^2-M^2eN^3a^2+M^2bN^3a^2-bN^2qa^2-bN^2a^2q^2+eN^3a^2q^2-eN^3qa^2\\
&-MbN^4a^2-MbN^4d^2+MbN^3a^2-\bar{e}\,Ma^2q^2N^2+dbM^2aN^2-dbM^2N^2\bar{d}+bNM^2qa^2\\
&+bNa^2q^2M+bNMa^2q-bN^2M^2qa^2+bN^2M^2a\bar{d}-2\,bN^2Ma^2q+eM^2Na^2q^2-eNM^2qa^2+\bar{e}\,N^2Ma^2q\\
&-\bar{d}\,MeN^2q^4a+eN^3adq^3+\bar{d}\,MeN^2aq^3+bN^3qa^2-bN^2adq^3+bN^3adq^2-bN^3adq+MeN^2a^2\\
&-M\bar{e}\,N^4qa^2-2\,MbN^3d^2q+M\bar{e}\,N^2ad+M^2eN^3qa^2+M^2eN^3ad-bN^2M^2aq^2\bar{d}-bNMaq^2\bar{d}\\
&-bNM^2a\bar{d}\,q+bNMaq^3\bar{d}+bNM^2aq^2\bar{d}+eM^2Naq^3\bar{d}-2\,eNM^2aq^2\bar{d}+eNM^2a\bar{d}\,q\\
&-dbM^2N^2aq^2-dbM^2N^2q^3\bar{d}+dbM^2N^2q^2\bar{d}+dbM^2N^2\bar{d}\,q+dbMN^2aq-2\,dbMN^2aq^2+dbMNaq^2\\
&-dbMaNq-2\,d\bar{e}\,MN^2aq^3+2\,d\bar{e}\,MN^2aq^2-2\,d\bar{e}\,MN^2aq-M^2bN^3ad-b\bar{d}\,M^2N^3a+b\bar{d}\,M^2N^3d\\
&+\bar{d}\,M^2eN^3a-\bar{d}\,M^2eN^3d-MeN^2q^3a^2-MeN^2a^2q^2+MeN^2qa^2-MeN^2ad-3\,M\bar{e}\,N^4d^2q+M\bar{e}\,N^2q^3a^2\\
&+3\,M\bar{e}\,N^4d^2q^2+MbN^3d^2q^2-2\,M\bar{e}\,N^4ad-ba^2q^2M+bNa^2q^2-bN^2M^2a^2+eNq^3a^2-eNa^2q^2+\bar{e}\,a^2q^2M\\
&-\bar{e}\,Mq^3a^2-2\,MbN^3ad+MbN^3qa^2+2\,MbN^4d^2q+2\,MbN^4ad\big)C_m(n)+(Mq-1)\big(cM^4a^2+cM^4\bar{d}^2\\
&-2\,cM^2a^2-b^2M^3a+b^2M^3\bar{d}+2\,b^2aM^2-b^2\bar{d}\,M^2-b^2Ma+\bar{e}^2aM^2+ca^2+2\,cM^4a\bar{d}\,q-2\,cM^2a\bar{d}\,q\\
&-\bar{e}\,bM^3qa-\bar{e}\,bM^3q^2\bar{d}+2\,\bar{e}\,bM^3\bar{d}\,q+2\,\bar{e}\,baqM^2-\bar{e}\,bMaq-2\,cM^4a\bar{d}+cM^4q^2\bar{d}^2-2\,cM^4q\bar{d}^2+2\,cM^2a\bar{d}\\
&-b^2M^3\bar{d}\,q+b^2\bar{d}\,M^2q-2\,\bar{e}^2aqM^2+\bar{e}^2M^2aq^2+\bar{e}\,bM^3a-\bar{e}\,bM^3\bar{d}-2\,\bar{e}\,baM^2+\bar{e}\,bMa\big)N^2\\
&\times(N-1)(qN-1)\big(-a+dN^2q+aN^2-dN^2\big)\big(MaN-aq^2+dMqN-dMN\big)\big(Maq+Mq^2\bar{d}-M\bar{d}\,q-aN\\
&-dqN+dN\big)\big(-bdeq^3N^3-beaq^2N^3+beaq^3N-bdeN^3q+beaN^3q-beaq^4N+2\,bdeq^2N^3\\
&+2\,acdqN^4+2\,beaN^2q^3-2\,acdN^2q^3+2\,acdq^2N^2-2\,beaq^2N^2+cq^4a^2-2\,caN^4d-2\,ca^2q^2N^2\\
&+2\,b^2aq^2N^2-2\,e^2aN^2q^3-2\,cd^2qN^4-b^2aq^3N+db^2N^3q-db^2q^2N^2+cd^2q^2N^4-db^2q^2N^3\\
&+e^2aq^4N^2-b^2aN^3q+e^2aq^2N^2+db^2N^2q^3+ca^2N^4+cd^2N^4\big)C_m(n-1)=0.
\end{aligned}
$$

To solve this recurrence equation, we use Horn's variant of $q$-Petkovšek's algorithm [6] using Sprenger's Maple implementation [16]. Otherwise we could not solve these recurrence equations and obtain their closed form solutions. With the aid of this algorithm, we obtain the following connection coefficients.

**Corollary 17.** *The following connection relations between the classical monic orthogonal polynomials are valid:*

$$
\tilde{P}_n(x,\alpha,\beta,\gamma;q)=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(\alpha\beta q^{n+1})^{n-m}(\alpha q^{m+1};\gamma q^{m+1};\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};\alpha\beta q^{n+m+1};q)_{n-m}}\tilde{P}_m(x,\alpha,\beta_1,\gamma;q);
$$

$$
\tilde{Q}_n(x,\alpha,\beta,N;q)=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q\frac{(\alpha\beta q^{n+1})^{n-m}(\alpha q^{m+1},q^{m-N},\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)},\alpha\beta q^{n+m+1};q)_{n-m}}\tilde{Q}_m(x,\alpha,\beta_1,N;q);
$$

$$\tilde{p}_n(x,\alpha,\beta;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha q^n)^{n-m}(\beta q^{m+1}, \frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha_1\beta q^{2(m+1)}, \alpha\beta q^{n+m+1};q)_{n-m}} \tilde{p}_m(x,\alpha_1,\beta;q);$$

$$\tilde{p}_n(x,\alpha,\beta;q) = \sum_{m=0}^{n} (-\alpha\beta q^{\frac{3n+m+1}{2}})^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha q^{m+1}, \frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)}, \alpha\beta q^{n+m+1};q)_{n-m}} \tilde{p}_m(x,\alpha,\beta_1;q);$$

$$\tilde{K}_n(x,\alpha;q) = \sum_{m=0}^{n} (\alpha q^{\frac{3n+m-1}{2}})^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(-\alpha_1 q^{2m+1}, -\alpha q^{n+m};q)_{n-m}} \tilde{K}_m(x,\alpha_1;q);$$

$$\tilde{p}_n(x,\alpha;q) = \sum_{m=0}^{n} (\alpha q^n)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q (\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m} \tilde{p}_m(x,\alpha_1;q).$$

*where* $\tilde{P}_n(x,\alpha,\beta,\gamma;q)$, $\tilde{Q}_n(x,\alpha,\beta,N;q)$, $\tilde{p}_n(x,\alpha,\beta;q)$, $\tilde{K}_n(x,\alpha;q)$, $\tilde{p}_n(x,\alpha;q)$ *stand for the monic Big* $q$*-Jacobi, the* $q$*-Hahn, the Little* $q$*-Jacobi, the Alternative* $q$*-Charlier and the Little* $q$*-Laguerre/Wall polynomials respectively.*

The connection coefficient between $\tilde{p}_n(x,\alpha,\beta;q)$ and $\tilde{p}_m(x,\alpha_1,\beta_1;q)$ is obtained by combining the two cases: $(\alpha,\beta) \to (\alpha_1,\beta) \to (\alpha_1,\beta_1)$. In fact, we have

$$\tilde{p}_n(x,\alpha,\beta;q) = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q \frac{(\alpha q^n)^{n-j}(\beta q^{j+1}, \frac{\alpha_1 q^{j+1}}{\alpha q^n};q)_{n-j}}{(\alpha_1\beta q^{2(j+1)}, \alpha\beta q^{n+j+1};q)_{n-j}} \tilde{p}_j(x,\alpha_1,\beta;q)$$

and

$$\tilde{p}_j(x,\alpha_1,\beta;q) = \sum_{m=0}^{j} \frac{\begin{bmatrix} j \\ m \end{bmatrix}_q (-\alpha_1\beta q^{\frac{3j+m+1}{2}})^{j-m} \left(\alpha_1 q^{m+1}, \frac{\beta_1 q^{m+1}}{\beta q^j};q\right)_{j-m}}{(\alpha_1\beta_1 q^{2(m+1)}, \alpha_1\beta q^{j+m+1};q)_{j-m}} \tilde{p}_m(x,\alpha_1,\beta_1;q),$$

from which we get

$$\tilde{p}_n(x,\alpha,\beta;q) = \sum_{m=0}^{n} C_m(n)\tilde{p}_m(x,\alpha_1,\beta_1;q),$$

where

$$\begin{aligned} C_m(n) &= \sum_{j=0}^{n-m} \begin{bmatrix} n \\ j+m \end{bmatrix}_q \begin{bmatrix} j+m \\ m \end{bmatrix}_q \frac{(\alpha q^n)^{n-j-m}(\beta q^{j+m+1}, \frac{\alpha_1 q^{j+m+1}}{\alpha q^n};q)_{n-j-m}}{(\alpha_1\beta q^{2(j+m+1)}, \alpha\beta q^{n+j+m+1};q)_{n-j-m}} \times \\ &\quad \frac{(-\alpha_1\beta q^{\frac{3j+4m+1}{2}})^j (\alpha_1 q^{m+1}, \frac{\beta_1 q^{m+1}}{\beta q^{j+m}};q)_j}{(\alpha_1\beta_1 q^{2(m+1)}, \alpha_1\beta q^{j+2m+1};q)_j}. \end{aligned}$$

Calling the coefficient of this sum $a_j$, a decision algorithm to detect the rationality of $a_{j+1}/a_j$ ([9], Algorithm 2.1 (`simpcomb`)) and the use of equation (1.8.22) of [8], these coefficients can be written in the form

$$\begin{aligned} C_m(n) &= \sum_{j=0}^{n-m} \frac{(-1)^j q^{\binom{j}{2}} (q^{m-n}, \frac{\beta}{\beta_1}, \alpha_1 q^{m+1}, \alpha_1\beta q^{2m+1}, \alpha\beta q^{m+n+1};q)_j (\alpha_1\beta q^{2m+3};q^2)_j}{(q, \beta q^{m+1}, \frac{\alpha_1 q^{m+1}}{\alpha q^n}, \alpha_1\beta_1 q^{2m+2}, \alpha_1\beta q^{m+n+2};q)_j (\alpha_1\beta q^{2m+1};q^2)_j} \\ &\quad \times (\frac{\alpha_1\beta_1 q^{m+2}}{\alpha})^j \frac{(\alpha q^n)^{n-m}(\beta q^{m+1}, \frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha_1\beta q^{2(m+1)}, \alpha\beta q^{n+m+1};q)_{n-m}} \begin{bmatrix} n \\ m \end{bmatrix}_q \\ &= {}_7\phi_7\left( \left. \begin{matrix} q^{m-n}, \frac{\beta}{\beta_1}, \alpha_1 q^{m+1}, \alpha_1\beta q^{2m+1}, \alpha\beta q^{m+n+1}, \sqrt{\alpha_1\beta q} q^{m+1}, -\sqrt{\alpha_1\beta q} q^{m+1} \\ \beta q^{m+1}, \frac{\alpha_1 q^{m+1}}{\alpha q^n}, \alpha_1\beta_1 q^{2m+2}, \alpha_1\beta q^{m+n+2}, \sqrt{\alpha_1\beta q} q^m, -\sqrt{\alpha_1\beta q} q^m, 0 \end{matrix} \right| q, \frac{\alpha_1\beta_1 q^{m+2}}{\alpha} \right) \\ &\quad \times \frac{(\alpha q^n)^{n-m}(\beta q^{m+1}, \frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha_1\beta q^{2(m+1)}, \alpha\beta q^{n+m+1};q)_{n-m}} \begin{bmatrix} n \\ m \end{bmatrix}_q. \end{aligned}$$

Applying Zeilberger's algorithm [9] implemented in the `qsum` package[3], we obtain

$$C_m(n) = \frac{\left(q^{-n}, \alpha\beta q^{n+1}; q\right)_m \left(\alpha_1\beta_1 q^3, \alpha_1\beta_1 q^4; q\right)_{2m} \left(-\frac{1}{\alpha\alpha_1\beta q^3}\right)^m q^{-3\binom{m}{2}}}{\left(q, q^2, \alpha_1 q^2, \beta_1 q^2, \alpha_1\beta_1 q^{n+3}, \frac{\alpha_1\beta_1 q^2}{\alpha\beta q^n}; q\right)_m} \times \frac{\left(\alpha q, \alpha\beta q, \frac{\alpha\beta}{\alpha_1\beta_1}, \beta q, q; q\right)_n \left(-\alpha_1\alpha\beta_1 q^4\right)^n q^{3\binom{n}{2}}}{\left(\alpha\beta q, \alpha\beta q^2, q\right)_{2n} (\alpha_1\beta_1 q^3; q)_n}.$$

We already have

$$\tilde{p}_n(x,\alpha) = \sum_{m=0}^{n} C_m(n,\alpha,\alpha_1)\tilde{p}_m(x,\alpha_1),$$

so that if we multiply this equality by the leading coefficient $k_n(\alpha)$ of $p_n$, we obtain

$$p_n(x,\alpha) = k_n^{\alpha}\tilde{p}_n(x) = \sum_{m=0}^{n} \frac{k_n^{\alpha}}{k_m^{\alpha_1}} C_m(n,\alpha,\alpha_1) p_m(x,\alpha_1).$$

This means that the connection coefficients $D_m(n,\alpha,\alpha_1)$ of the non-monic families are related to the monic ones by the relation

$$D_m(n,\alpha,\alpha_1) = \frac{k_n^{\alpha}}{k_m^{\alpha_1}} C_m(n,\alpha,\alpha_1).$$

From this and the relation $\frac{(a;q)_n}{(a;q)_m} = (aq^m;q)_{n-m}$, we obtain

**Corollary 18.** *The following connection relations between the orthogonal polynomial systems of the $q$-Hahn class are valid:*

$$P_n(x,\alpha,\beta,\gamma;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha\beta q^{n+1})^{n-m} (\alpha\beta q^{n+1};q)_m (\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};q)_{n-m} (\alpha\beta_1 q^{m+1};q)_m} P_m(x,\alpha,\beta_1,\gamma;q);$$

$$Q_n(x,\alpha,\beta,N;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha\beta q^{(n+1)})^{n-m} (\alpha\beta q^{n+1};q)_m (\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};q)_{n-m} (\alpha\beta_1 q^{m+1};q)_m} Q_m(x,\alpha,\beta_1,N;q);$$

$$p_n(x,\alpha,\beta;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (-\alpha q^{\frac{n-m+1}{2}})^{n-m} \frac{(\alpha_1 q, \alpha\beta q^{n+1};q)_m (\beta q^{m+1}, \frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha_1\beta q^{2(m+1)};q)_{n-m} (\alpha_1\beta q^{m+1};q)_m (\alpha q;q)_n} p_m(x,\alpha_1,\beta;q);$$

$$p_n(x,\alpha,\beta;q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (\alpha\beta q^{n+1})^{n-m} \frac{(\alpha\beta q^{n+1};q)_m (\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{(\alpha\beta_1 q^{2(m+1)};q)_{n-m} (\alpha\beta_1 q^{m+1};q)_m} p_m(x,\alpha,\beta_1;q);$$

$$K_n(x,\alpha;q) = \sum_{m=0}^{n} (-\alpha q^n)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-\alpha q^n;q)_m (\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(-\alpha_1 q^{2m+1};q)_{n-m} (-\alpha_1 q^m;q)_m} K_m(x,\alpha_1;q);$$

$$p_n(x,\alpha;q) = \sum_{m=0}^{n} (-\alpha q^{\frac{n-m+1}{2}})^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(\alpha_1 q;q)_m (\frac{\alpha_1 q^{m+1}}{\alpha q^n};q)_{n-m}}{(\alpha q;q)_n} p_m(x,\alpha_1;q),$$

[3] `qsumrecursion(summand,q,j,C(m),rec2qhyper=true)`

## 6. Parameter Derivatives

For some applications, it is important to know the rate of change in the direction of the parameters of the $q$-orthogonal systems, given in terms of the system itself. By a limiting process, these parameter derivatives representations can be obtained from the results of Corollaries 17 and 18.

**Corollary 19.** *The following representations for the parameter derivatives of the classical q-orthogonal polynomials are valid:*

$$\begin{aligned}\frac{\partial}{\partial\beta}P_n(x,\alpha,\beta,\gamma;q) &= \sum_{m=0}^{n-1}\Bigg(\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}P_n(x,\alpha,\beta,\gamma;q)\\ &\quad+\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(\alpha\beta q^{n+1})^{n-m}(\alpha\beta q^{n+1};q)_m(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m}P_m(x,\alpha,\beta,\gamma;q)\Bigg),\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\beta}Q_n(x,\alpha,\beta,N;q) &= \sum_{m=0}^{n-1}\Bigg(\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}Q_n(x,\alpha,\beta,N;q)\\ &\quad+\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(\alpha\beta q^{n+1})^{n-m}(\alpha\beta q^{n+1};q)_m(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m}Q_m(x,\alpha,\beta,N;q)\Bigg),\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}p_n(x,\alpha,\beta;q) &= \sum_{m=0}^{n-1}\Bigg(\Big(\frac{q^{m+1}}{1-\alpha q^{m+1}}-\frac{\beta q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}\Big)p_n(x,\alpha,\beta;q)+\begin{bmatrix} n\\ m\end{bmatrix}_q (-\alpha q^{\frac{n-m+1}{2}})^{n-m}\\ &\quad\times\frac{(\alpha q,\alpha\beta q^{n+1};q)_m(\beta q^{m+1};q)_{n-m}(q^{n-m+1};q)_{n-m-1}}{\alpha(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m(\alpha q;q)_n}p_m(x,\alpha,\beta;q)\Bigg);\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\beta}p_n(x,\alpha,\beta;q) &= \sum_{m=0}^{n-1}\Bigg(\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}p_n(x,\alpha,\beta;q)\\ &\quad+\begin{bmatrix} n\\ m\end{bmatrix}_q (\alpha\beta q^{n+1})^{n-m}\frac{(\alpha\beta q^{n+1};q)_m(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};q)_{n-m}(\alpha\beta q^{m+1};q)_m}p_m(x,\alpha,\beta;q)\Bigg);\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}K_n(x,\alpha;q) &= \sum_{m=0}^{n-1}\Bigg(\frac{q^{n+m}}{1+\alpha q^{n+m}}K_n(x,\alpha;q)\\ &\quad+(-\alpha q^n)^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(-\alpha q^n;q)_m(q^{m-n+1};q)_{n-m-1}}{\alpha(-\alpha q^{2m+1};q)_{n-m}(-\alpha q^m;q)_m}K_m(x,\alpha;q)\Bigg);\end{aligned}$$

$$\begin{aligned}\frac{\partial}{\partial\alpha}p_n(x,\alpha;q) &= \sum_{m=0}^{n-1}\Bigg(\frac{q^{m+1}}{1-\alpha q^{m+1}}p_n(x,\alpha;q)\\ &\quad+(-\alpha q^{\frac{n-m+1}{2}})^{n-m}\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(\alpha q;q)_m(q^{m-n+1};q)_{n-m-1}}{\alpha(\alpha q;q)_n}p_m(x,\alpha;q)\Bigg).\end{aligned}$$

*For the monic cases, we have*

$$\frac{\partial}{\partial\beta}\tilde{P}_n(x,\alpha,\beta,\gamma;q)=\sum_{m=0}^{n-1}\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(\alpha\beta q^{n+1})^{n-m}(\alpha q^{m+1};\gamma q^{m+1};q)_{n-m}(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)};\alpha\beta q^{n+m+1};q)_{n-m}}\tilde{P}_m(x,\alpha,\beta,\gamma;q);$$

$$\frac{\partial}{\partial\beta}\tilde{Q}_n(x,\alpha,\beta,N;q)=\sum_{m=0}^{n-1}\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(\alpha\beta q^{n+1})^{n-m}(\alpha q^{m+1},q^{m-N};q)_{n-m}(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)},\alpha\beta q^{n+m+1};q)_{n-m}}\tilde{Q}_m(x,\alpha,\beta,N;q);$$

$$\frac{\partial}{\partial\alpha}\tilde{p}_n(x,\alpha,\beta;q)=\sum_{m=0}^{n-1}\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(\alpha q^{n})^{n-m}(\beta q^{m+1};q)_{n-m}(q^{m-n+1};q)_{n-m-1}}{\alpha(\alpha\beta q^{2(m+1)},\alpha\beta q^{n+m+1};q)_{n-m}}\tilde{p}_m(x,\alpha,\beta;q);$$

$$\frac{\partial}{\partial\beta}\tilde{p}_n(x,\alpha,\beta;q)=\sum_{m=0}^{n-1}\begin{bmatrix} n\\ m\end{bmatrix}_q \frac{(-\alpha\beta q^{\frac{3n+m+1}{2}})^{n-m}(\alpha q^{m+1};q)_{n-m}(q^{m-n+1};q)_{n-m-1}}{\beta(\alpha\beta q^{2(m+1)},\alpha\beta q^{n+m+1};q)_{n-m}}\tilde{p}_m(x,\alpha,\beta;q);$$

$$\frac{\partial}{\partial\alpha}\tilde{K}_n(x,\alpha;q)=\sum_{m=0}^{n-1}\begin{bmatrix} n\\ m\end{bmatrix}_q (\alpha q^{\frac{3n+m-1}{2}})^{n-m}\frac{(q^{m-n+1};q)_{n-m-1}}{\alpha(-\alpha q^{2m+1},-\alpha q^{n+m};q)_{n-m}}\tilde{K}_m(x,\alpha;q);$$

$$\frac{\partial}{\partial\alpha}\tilde{p}_n(x,\alpha;q)=\sum_{m=0}^{n}\begin{bmatrix} n\\ m\end{bmatrix}_q (\alpha q^{n})^{n-m}(q^{m-n+1};q)_{n-m-1}\tilde{p}_m(x,\alpha;q).$$

*Proof.* If

$$P_n^{\beta}(x)=\sum_{m=0}^{n}C_m(n,\beta,\beta_1)P_m^{\beta_1}(x),$$

then

$$\frac{\partial}{\partial\beta}P_n^{\beta}(x)=\lim_{\beta_1\to\beta}\frac{C_n(n,\beta,\beta_1)-1}{\beta-\beta_1}P_n^{\beta}(x)+\sum_{m=0}^{n-1}\lim_{\beta_1\to\beta}\frac{C_m(n,\beta,\beta_1)}{\beta-\beta_1}P_m^{\beta}(x).$$

We know that

$$\frac{d}{dx}\prod_{m=1}^{n}f_m(x)=\sum_{m=1}^{n}\frac{f_m'(x)}{f_m(x)}\prod_{m=1}^{n}f_m(x),$$

from which we get

$$\frac{\partial}{\partial\beta_1}(\alpha\beta_1 q^{n+1};q)_n=\sum_{m=0}^{n-1}\frac{-\alpha q^{n+m+1}}{1-\alpha\beta_1 q^{n+m+1}}(\alpha\beta_1 q^{n+1};q)_n.$$

Using de l'Hospital's rule, we obtain

$$\lim_{\beta_1\to\beta}\frac{(\alpha\beta q^{n+1};q)_n-(\alpha\beta_1 q^{n+1};q)_n}{(\alpha\beta_1 q^{n+1};q)_n(\beta-\beta_1)}=\sum_{m=0}^{n-1}\frac{-\alpha q^{n+m+1}}{1-\alpha\beta q^{n+m+1}}.$$

We also have

$$\lim_{\beta_1\to\beta}\frac{(\frac{\beta_1 q^{m+1}}{\beta q^n};q)_{n-m}}{\beta-\beta_1}=\frac{1}{\beta}(q^{m-n+1};q)_{n-m-1}.$$

From all the preceeding, the result follows. □

## 7. Linearization Coefficients

We can always suppose

$$P_n(x)=\sum_{i=0}^{n}A_i(n)x^i,\ Q_m(x)=\sum_{j=0}^{m}B_j(m)x^j,\text{ and }x^k=\sum_{l=0}^{k}D_l(k)R_l(x)$$

where $P_n(x), Q_m(x), R_l(x)$ are three families of classical $q$-orthogonal polynomials.

We then obtain the Cauchy product

$$P_n(x)Q_m(x)=\sum_{k=0}^{n+m}C_k(m,n)x^k,\text{ where }C_k(m,n)=\sum_{r=0}^{k}A_r(n)B_{k-r}(m).$$

Using the inversion formula, this can be rewritten as

$$P_n(x)Q_m(x)=\sum_{l=0}^{n+m} E_l(m,n)R_l(x)$$

with

$$E_l(m,n)=\sum_{k=0}^{n+m-l} C_{k+l}(m,n)D_l(k+l)=\sum_{k=0}^{n+m-l}\sum_{r=0}^{k+l} A_r(n)B_{k+l-r}(m)D_l(k+l).$$

We considered here the standard linearization problem of Clebsch-Gordan-type where $P_n(x)$, $Q_m(x)$, $R_l(x)$ belong to the *same* $q$-orthogonal family. With the data of Table 3, we apply the $q$-version of Algorithm 2.1 from [9][4] to obtain the $q$-hypergeometric representation of $\sum_{r=0}^{k+l} A_r(n)B_{k+l-r}(m)$, then we multiply by $D_l(k+l)$ and sum w. r. t. $k$.

| Monic family | $A_r(n)$ | $D_l(k+l)$ |
|---|---|---|
| Little $q$-Jacobi | $(-1)^n q^{\binom{n}{2}+r}\frac{(q^{-n};q)_r(\alpha q^{r+1};q)_{n-r}}{(q;q)_r(\alpha\beta q^{n+r+1};q)_{n-r}}$ | $\frac{\begin{bmatrix}k+l\\l\end{bmatrix}_q(\alpha q^{l+1};q)_k}{(\alpha\beta q^{2(l+1)};q)_k}$ |
| Alternative $q$-Charlier | $(-1)^n q^{\binom{n}{2}+r}\frac{(q^{-n};q)_r}{(q;q)_r(-\alpha q^{n+r};q)_{n-r}}$ | $\begin{bmatrix}k+l\\l\end{bmatrix}_q\frac{1}{(-\alpha q^{2l+1};q)_k}$ |
| Little $q$-Laguerre/Wall | $(-1)^n q^{\binom{n}{2}+r}\frac{(q^{-n};q)_r(\alpha q^{r+1};q)_{n-r}}{(q;q)_r}$ | $\begin{bmatrix}k+l\\l\end{bmatrix}_q(\alpha q^{l+1};q)_k$ |
| $q$-Laguerre | $\frac{(-1)^n q^{\binom{r}{2}+r(\alpha+n+1)}}{q^{n(n+\alpha)}}\frac{(q^{-n};q)_r(q^{\alpha+r+1};q)_{n-r}}{(q;q)_r}$ | $\begin{bmatrix}k+l\\l\end{bmatrix}_q\frac{(q^{\alpha+l+1};q)_k}{q^{\frac{k(2\alpha+k+4l+1)}{2}}}$ |
| Stieltjes-Wigert | $(-1)^n q^{\binom{r}{2}-n^2+r(n+1)}\frac{(q^{-n};q)_r}{(q;q)_r}$ | $\begin{bmatrix}k+l\\l\end{bmatrix}_q q^{-\frac{k(4l+k+1)}{2}}$ |

Table 3: Coefficients needed for computation

Using this algorithm, we obtain the following linearization coefficients.

**Theorem 20.** *The following linearization relations between the orthogonal polynomials systems of the q-Hahn class are valid:*
*(1)* ***Little q-Jacobi***

$$\tilde{p}_n(x,\alpha,\beta;q)\tilde{p}_m(x,\alpha_1,\beta_1;q)=\sum_{l=o}^{n+m} E_l(m,n)\tilde{p}_l(x,\alpha_2,\beta_2;q)$$

*with*

$$\begin{aligned}E_l(m,n) \quad = \quad & \sum_{k=0}^{n+m-l}\frac{(\alpha q;q)_n(q^{-m};q)_{k+l}(\alpha_1 q^{k+l+1};q)_{m-k-l}(\alpha_2 q^{l+1};q)_k\begin{bmatrix}k+l\\l\end{bmatrix}_q}{(\alpha\beta q^{n+1};q)_n(q;q)_{k+l}(\alpha_1\beta_1 q^{m+k+l+1};q)_{m-k-l}(\alpha_2\beta_2 q^{2l+2};q)_k}\\ & \times(-1)^{n+m}q^{\binom{n}{2}+\binom{m}{2}+k+l}{}_4\phi_3\left(\begin{matrix}q^{-k-l},q^{-n},\alpha\beta q^{n+1},\frac{q^{-k-l}}{\alpha_1}\\ \alpha q,q^{m-k-l+1},\frac{q^{-m-k-l}}{\alpha_1\beta_1}\end{matrix}\middle|\, q;\frac{q}{\beta_1}\right).\end{aligned}$$

[4]`sum2qhyper(term,q,r)`

(2) ***Alternative q-Charlier***

$$\tilde{K}_n(x,\alpha;q)\tilde{K}_m(x,\alpha_1;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{K}_l(x,\alpha_2;q)$$

*with*

$$E_l(m,n) = \sum_{k=0}^{n+m-l} \frac{(-1)^{n+m} q^{\binom{n}{2}+\binom{m}{2}+k+l}(q^{-m};q)_{k+l} \begin{bmatrix} k+l \\ l \end{bmatrix}_q}{(-\alpha q^n;q)_n (q;q)_{k+l} (-\alpha_1 q^{m+k+l};q)_{m-k-l} (-\alpha_2 q^{2l+1};q)_k}$$

$$ {}_3\phi_3\left( \begin{matrix} q^{-k-l}, q^{-n}, -\alpha q^n \\ -\frac{q^{1-m-k-l}}{\alpha_1}, q^{m-k-l+1}, 0 \end{matrix} \middle| q; -\frac{q^2}{\alpha_1 q^{k+l}} \right).$$

(3) ***Little q-Laguerre/Wall***

$$\tilde{p}_n(x,\alpha;q)\tilde{p}_m(x,\alpha_1;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{p}_l(x,\alpha_2;q), \ with$$

$$\begin{aligned} E_l(m,n) &= \sum_{k=0}^{n+m-l} \frac{q^{\binom{n}{2}+\binom{m}{2}+k+l}}{(q;q)_{k+l}} \begin{bmatrix} k+l \\ l \end{bmatrix}_q (\alpha q;q)_n (q^{-m};q)_{k+l} (\alpha_1 q^{k+l+1};q)_{m-k-l} \\ &\quad \times (-1)^{n+m} (\alpha_2 q^{l+1};q)_k {}_4\phi_2\left( \begin{matrix} q^{-k-l}, q^{-n}, \frac{q^{-k-l}}{\alpha_1}, 0 \\ \alpha q, q^{m-k-l+1} \end{matrix} \middle| q; \alpha_1 q^{m+k+l+1} \right). \end{aligned}$$

(4) ***q-Laguerre***

$$\tilde{L}_n^{\alpha}(x;q)\tilde{L}_m^{\alpha_1}(x;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{L}_l^{\alpha_2}(x;q), \ with$$

$$\begin{aligned} E_l(m,n) &= \sum_{k=0}^{n+m-l} \frac{(-1)^{n+m}}{(q;q)_{k+l}} (q^{\alpha+1};q)_n (q^{\alpha_1+k+l+1};q)_{m-k-l} (q^{\alpha_2+l+1};q)_k \\ &\quad \times (q^{-m};q)_{k+l} q^N \begin{bmatrix} k+l \\ l \end{bmatrix}_q {}_3\phi_3\left( \begin{matrix} q^{-\alpha_1-k-l}, q^{-n}, q^{-k-l} \\ q^{\alpha+1}, q^{m-k-l+1}, 0 \end{matrix} \middle| q; q^{\alpha+n+2} \right) \end{aligned}$$

*where* $N = -n(n+\alpha) + \binom{k+l}{2} + (k+l)(m+\alpha_1+1) - m(m+\alpha_1) - \frac{k}{2}(2\alpha_2+k+4l+1)$. (5) ***Stieltjes-Wigert***

$$\tilde{S}_n(x;q)\tilde{S}_m(x;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{S}_l(x;q) \ with$$

$$E_l(m,n) = \sum_{k=0}^{n+m-l} \frac{(-1)^{n+m}\begin{bmatrix} k+l \\ l \end{bmatrix}_q}{(q^4;q)_{k+l}(\frac{q(1+q)}{q^n+q^m};q)_{k+l}} (\frac{q^2(1+q)}{q^n+q^m};q)_{k+l}(q^{-n-m};q)_{k+l} q^N,$$

*where* $N = (k+l)(m+n+7) - m^2 - n^2 + 3\binom{k+l}{2} - \frac{k}{2}(4l+k+1)$.

*In the particular cases where* $P_n$, $P_m$ *and* $P_l$ *have the same parameters, these relations can be reduced to:*

(1) ***Alternative q-Charlier***

$\tilde{K}_n(x,\alpha;q)\tilde{K}_m(x,\alpha;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{K}_l(x,\alpha;q)$ *with*

$$E_l(m,n) = \frac{(-1)^{n+m+l} q^{7l+\binom{n}{2}+\binom{n}{2}} \left(q^{-m-n}, -\alpha q^{m-n}, -\alpha q^{n-m}, \frac{\alpha(q+1)q^{n+m+2}}{q^{n+m}(\alpha q^m+\alpha q^n+q+q^2)-q^n-q^m}, \alpha^2 q^{n+m}; q\right)_l}{(-\alpha q^m;q)_m (-\alpha q^n;q)_n \left(q^4, \frac{\alpha(q+1)q^{n+m+1}}{q^{n+m}(\alpha q^m+\alpha q^n+q+q^2)-q^n-q^m}; q\right)_l} \times$$

$$ {}_6\phi_5\left( \begin{matrix} q^{l+1}, q^{l-m-n}, \alpha^2 q^{m+n+l}, -\alpha q^{m+l-n}, -\alpha q^{l+n-m}, \frac{\alpha(q+1)q^{n+m+l+2}}{q^{n+m}(\alpha q^m+\alpha q^n+q+q^2)-q^n-q^m} \\ q^{l+4}, -\alpha q^{2l+1}, \frac{\alpha(q+1)q^{n+m+l+1}}{q^{n+m}(\alpha q^m+\alpha q^n+q+q^2)-q^n-q^m}, 0, 0 \end{matrix} \middle| q; -q^7 \right).$$

*(2)* ***Little** $q$**-Laguerre-Wall***

$\tilde{p}_n(x,\alpha;q)\tilde{p}_m(x,\alpha;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{p}_l(x,\alpha;q)$ *with*

$$E_l(m,n) = \frac{(-1)^{n+m+l} q^{\binom{n}{2}+\binom{m}{2}+7l}\left(q^{-m-n}, \frac{\alpha q^4(q^n+q^m)}{q^{n+m+1}(1+\alpha+q+\alpha q)-q^n-q^m};q\right)_l (\alpha;q)_n(\alpha;q)_m(1-\alpha q^m)(1-\alpha q^n)}{\left(\alpha q^3, q^4, \alpha^2 q^4, \alpha q^4, \frac{\alpha q^3(q^n+q^m)}{q^{n+m+1}(1+\alpha+q+\alpha q)-q^n-q^m};q\right)_l (1-\alpha)^2} \times$$

$$ {}_6\phi_5\left(\begin{array}{c} q^{l-m-n}, q^{l+1}, \alpha q^{l+1}, \frac{\alpha q^{l+4}(q^n+q^m)}{q^{n+m+1}(1+\alpha+q+\alpha q)-q^n-q^m}, 0, 0 \\ q^{l+4}, \alpha q^{l+3}, \alpha^2 q^{l+4}, \alpha q^{l+4}, \frac{\alpha q^{l+3}(q^n+q^m)}{q^{n+m+1}(1+\alpha+q+\alpha q)-q^n-q^m} \end{array}\middle| q; -q^7\right).$$

*(3)* $q$***-Laguerre***

$\tilde{L}_n^{(\alpha)}(x;q)\tilde{L}_m^{(\alpha)}(x;q) = \sum_{l=0}^{n+m} E_l(m,n)\tilde{L}_l^{(\alpha)}(x;q)$ *with*

$$E_l(m,n) = \sum_{k=0}^{n+m-l} \frac{(q^{\alpha+l+1};q)_k\left(q^{-m-n}, -\frac{q^2(q^{\alpha+n+2}+q^{\alpha+m+2}-q^{\alpha+1}-q^\alpha-q-1)}{q^n+q^m};q\right)_{k+l}}{\left(q^4, q^{\alpha+3}, q^{\alpha+4}, q^{2\alpha+4}, -\frac{q(q^{\alpha+n+2}+q^{\alpha+m+2}-q^{\alpha+1}-q^\alpha-q-1)}{q^n+q^m};q\right)_{k+l}}$$

$$\times \frac{(q^\alpha;q)_n(q^\alpha;q)_m \begin{bmatrix} k+l \\ l \end{bmatrix}_q (q^{\alpha+m}-1)(q^{\alpha+n}-1)q^\omega}{(q^\alpha-1)^2}.$$

*where* $\omega = (k+l)(n+m+4\alpha+7) + 3\binom{k+l}{2} - \frac{1}{2}k(2\alpha+k+4l+1) - m^2 - n^2 - m\alpha - n\alpha$

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## References

[1] S. Abramov, P. Paule and M. Petkovšek, *q-hypergeometric solutions of q-difference equations*, Discrete Math. 180 (1998), pp. 3-22.

[2] I. Area, E. Godoy, A. Ronveaux and A. Zarzo, *Inversion problems in the q-Hahn tableau*, Journal of Symbolic Computation 136 (1998), pp. 1-10.

[3] I. Area, E. Godoy, A. Ronveaux and A. Zarzo, *Solving connection and linearization problems within the Askey scheme and its q-analogue via inversion formulas*, Journal of Computational and Applied Mathematics 136 (2001), pp. 152-162.

[4] R. Askey and J. Wilson, *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials*, Memoirs of the American Mathematical Society 319 (1985).

[5] H. Böing and W. Koepf, *Algorithms for q-hypergeometric summation in computer algebra*, J. Symbolic Computation 11 (1999), pp. 1-23.

[6] P. Horn, *Faktorisierung in Schief-Polynomringen*, PhD Dissertation, Universität Kassel, 2008.

[7] R. Koekoek and R. Swarttouw, *The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue*, Report no. 98-17, (1998), Faculty of Information Technology and Systems, Delft University of Technology.

[8] R. Koekoek, P. A. Lesky and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer, 2010.

[9] W. Koepf, *Hypergeometric Summation – An algorithmic approach to summation and special function identities*, Vieweg, Braunschhweig/Wiesbaden, 1998.

[10] W. Koepf and D. Schmersau, *Representations of orthogonal polynomials*, Journal of Computational and Applied Mathematics 90 (1998), pp. 57-94.

[11] W. Koepf and D. Schmersau, *On a structure formula for classical q-orthogonal polynomials*, Journal of Computational and Applied Mathematics 136 (2001), pp. 99-107.

[12] W. Koepf and D. Schmersau, *Recurrence equations and their classical orthogonal polynomial solutions*, Applied Mathematics and Computation, 128 (2002), pp. 57-94.

[13] W. Hahn, *Über Orthogonalpolynome die q-Differentialgleichungen genügen*, Math. Nachr. 2 (1949), pp. 4-34.

[14] P. Lesky: *Eine Charakterisierung der klassischen kontinuierlichen, diskreten und q-Orthogonalpolynome*, Shaker, Aachen, 2005.

[15] M. Petkovšek, *Hypergeometric solutions of linear recurrences with polynomial coefficients*, J. Symbolic Computation 14 (1992), pp. 243-264.

[16] T. Sprenger, *Algorithmen für q-holonome Funktionen und q-hypergeometrische Reihen*, PhD Dissertation, Universität Kassel, 2009. Available at: urn:nbn:de:hebis:34-2009072129130.