INNER BOUNDS FOR THE EXTREME ZEROS OF THE HAHN, CONTINUOUS HAHN, CONTINUOUS DUAL HAHN, MEIXNER-POLLACZEK AND PSEUDO-JACOBI POLYNOMIALS

A. JOOSTE*, P. NJIONOU SADJANG†, AND W. KOEPF‡

Abstract. Let \( \{p_n\}_{n=0}^{\infty} \) be a sequence of polynomials, orthogonal with respect to the measure \( \mu(x) \) on an interval \((a, b)\) with respect to the measure \( \sigma_k(x) \mu(x) \) where \( \sigma_k(x) > 0 \). Relations of finite-type involving \( g_{n-2,k}, P_n \) and \( p_{n-1} \) only exist for specific values of \( k \) and can be used to identify inner bounds for the extreme zeros of \( p_n \). In this paper, we identify the most precise upper bounds for the smallest and lower bounds for the largest zeros of the Hahn, Continuous Hahn, Continuous Dual Hahn, Meixner-Pollaczek and Pseudo-Jacobi polynomials, that can be obtained from this method. Algorithmic methods are used to generate the necessary mixed three-term recurrence relations and numerical examples are provided to illustrate the quality of the new bounds.

Key words. Orthogonal polynomials, extreme zeros, bounds for zeros.

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1. Introduction. A sequence of real polynomials \( \{p_n\}_{n=0}^{\infty} \), where \( p_n \) is of exact degree \( n \), is orthogonal with respect to a positive measure \( \mu(x) > 0 \) on an interval \((a, b)\), if the scalar product

\[
\langle p_m, p_n \rangle = \int_{a}^{b} p_m(x) p_n(x) \, d\mu(x) = 0, \quad m \neq n.
\]

If \( \mu(x) \) is absolutely continuous, then it can be represented by a real weight function \( w(x) > 0 \) so that \( d\mu(x) = w(x) \, dx \). If \( \mu(x) \) is discrete with support in \( \mathbb{N}_{\geq 0} \), then it can be represented by a discrete weight \( w(x) \geq 0 \) \((x \in \mathbb{N}_{\geq 0})\), and the scalar product is given by

\[
\langle p_m, p_n \rangle = \sum_{x=0}^{\infty} p_m(x) p_n(x) \, w(x).
\]

The orthogonal polynomial families under consideration in this paper are the following ones (see [10]):

- Hahn polynomials: discrete weight \( w(x) = \left(\frac{\alpha + x}{x}\right)\left(\frac{\beta + N - x}{N - x}\right) \) in \( \{0, 1, \ldots, N\} \) and

\[
Q_n(x; \alpha, \beta, N) = \binom{N}{x} \binom{N}{x} \binom{\alpha + x}{\alpha + 1, \beta + 1, -x}^N F_2 \left( \begin{array}{c} -n, n + \alpha + \beta + 1, -x \\ \alpha + 1, -N \end{array} \right).
\]

- Continuous Hahn polynomials: continuous weight \( w(x) = \Gamma(a + ix) \Gamma(b + ix) \Gamma(c - ix) \Gamma(d - ix) \) in the interval \((-\infty, \infty)\) and

\[
p_n(x; a, b, c, d) = x^n \frac{(a + c)_n (a + d)_n}{n!} \binom{N}{x} \binom{\alpha + x}{\alpha + 1, \beta + 1, -x}^N F_2 \left( \begin{array}{c} -n, n + a + c + b + d - 1, a + ix \\ a + c, a + d \end{array} \right).
\]

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Continuous Dual Hahn polynomials: continuous weight

\[ w(x) = \left| \frac{\Gamma(a + ix)\Gamma(b + ix)\Gamma(c + ix)}{\Gamma(2ix)} \right|^2 \]

in the interval \((0, \infty)\) and

\[ S_n(x^2; a, b, c) = (a + b)_n(a + c)_n \cdot {}_3F_2 \left( \begin{array}{c} -n, a + ix, a - ix \\ a + b, a + c \end{array} \bigg| 1 \right). \]

Meixner-Pollaczek polynomials: continuous weight \( w(x) = |\Gamma(\lambda + ix)|^2 e^{(2\phi - \pi)x} \)
in the interval \((-\infty, \infty)\) and

\[ p_n^{(\lambda)}(x; \phi) = \frac{(2\lambda)_n}{n!} e^{i\phi} \cdot _2F_1 \left( \begin{array}{c} -n, \lambda + ix \\ 2\lambda \end{array} \bigg| 1 - e^{-2i\phi} \right). \]

Pseudo-Jacobi polynomials: continuous weight \( w(x) = (1 + x^2)^a e^{2b \arctan x} \)
in the interval \((-\infty, \infty)\) and

\[ P_n(x; a, b) = (-1)^n \frac{(a + ib + 1)_n}{n!} \cdot _2F_1 \left( \begin{array}{c} -n, 2a + n + 1 \\ a + bi + 1 \end{array} \bigg| 1 - i\frac{x}{2} \right). \]

Here,

\[ \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \]
denotes the Gamma function,

\[ {}_pF_q \left( \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{array} \bigg| x \right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdots (\alpha_p)_k \cdot (\beta_1)_k \cdots (\beta_q)_k}{k!} x^k \]
denotes the hypergeometric series and \((a)_k = a(a + 1) \cdots (a + k - 1)\) denotes the shifted factorial (Pochhammer symbol), as usual.

Every sequence of orthogonal polynomials satisfies a three-term recurrence relation of the form [15, Theorem 3.2.1]

\[ (1.1) \quad p_n(x) = (A_n x + B_n) p_{n-1}(x) - C_n p_{n-2}(x), \quad n \in \mathbb{N}_{\geq 1} \]

where \(A_n, B_n\) and \(C_n\) do not depend on \(x\), \(p_{-1} \equiv 0\), \(p_0(x) = 1\) and \(A_n, C_n > 0\). An important consequence of (1.1) is that for each \(n \in \mathbb{N}_{\geq 1}\), the polynomial \(p_n\) has exactly \(n\) real, simple zeros in \((a, b)\) [15, Theorem 3.3.1]. Furthermore, each open interval with endpoints at successive zeros of \(p_n\) contains exactly one zero of \(p_{n-1}\). Stieltjes [15, Theorem 3.3.3] extended this interlacing property by proving that if \(m < n - 1\), provided \(p_m\) and \(p_n\) are co-prime (i.e., they do not have any common zeros), there exist \(m\) open intervals with endpoints at successive zeros of \(p_n\), each of which contains exactly one zero of \(p_m\). Beardon [2, Theorem 5] provided additional insight into the Stieltjes interlacing process by proving that for every \(m < n - 1\), if \(p_m\) and \(p_n\) are co-prime, there exists a real polynomial \(S_{n-m}\) of degree \(n - m - 1\) in \(x\), whose real simple zeros, together with those of \(p_m\), interlace with the zeros of \(p_n\), a phenomenon that will be called completed Stieltjes interlacing, of which
a direct consequence is that the zeros of the polynomial $S_{n-r}$ act as “inner” bounds for the extreme zeros (i.e., upper (lower) bounds for the smallest (largest) zeros) of the polynomial $p_n$.

The study of completed Stieltjes interlacing between polynomials of different orthogonal sequences, where the different sequences are obtained by integer shifts of the parameters of the appropriate polynomials, leads to even more precise inner bounds for the extreme zeros of the polynomials under consideration, as was done for the Gegenbauer, Laguerre and Jacobi polynomials [3], the Meixner and Krawtchouk polynomials [6] and the Pseudo-Jacobi polynomials in [7]. Mixed three-term recurrence relations satisfied by the polynomials under consideration and obtained from the connection between the appropriate polynomials, their hypergeometric representations, as well as contiguous function relations satisfied by these polynomials, are used to obtain these bounds. A computer package [16] for computing contiguous relations of exclusively $2F_1$ series is helpful in this regard.

In this paper, we provide inner bounds for the extreme zeros of different sequences of orthogonal polynomials that lie on the $3F_2$ and $2F_1$ planes of the Askey scheme, namely the Hahn, continuous Hahn, continuous dual Hahn polynomials, Meixner-Pollaczek and Pseudo-Jacobi polynomials. In order to find the identities needed to prove our results, we use a (different) algorithmic method. Zeilberger’s algorithm (see [11], command sumrecursion of the Maple package hsum.mpl accompanying [11]) generates recurrence equations for hypergeometric sums. It clearly finds—given their hypergeometric representations—the three-term recurrences of the families considered. But Zeilberger’s algorithm is much more flexible as is shown in [11]. Similarly as the commands sumdiffseq ([11], Session 10.5) and sumdiffrule ([11], Session 10.7) are slight variations of sumrecursion by just changing the setting in the computation, one can write Maple routines to compute the desired identities such as (2.2). The only obstacles might be that the results can be very complicated symbolic expressions or that the computation is unfeasible by time or memory constraints.

In the Hahn case, we compare the quality of our newly found bounds with results obtained in [12]. An intensive study on the location of the zeros of the Hahn polynomials is made in [13] and lower (upper) bounds for the smallest (largest) zeros of Hahn polynomials are provided in [1, 12]. Hahn polynomials play a role in stochastic processes in genetics [9], as well as in quantum mechanics [8] where Clebsch-Gordon coefficients are expressed as Hahn polynomials, and a generalisation of the continuous dual Hahn polynomials is associated with symmetric birth and death processes with quadratic rates [4].

The following result provides the conditions necessary for the mixed three-term recurrence relations to hold and will be used to prove our results.

**Theorem 1.1.** [6] Let $\{p_n\}_{n=0}^\infty$ be a sequence of polynomials orthogonal on the (finite or infinite) interval $(a, b)$ with respect to $d\mu(x) > 0$. Let $k \in \mathbb{N}_{\geq 0}$ be fixed and suppose $\{g_{n,k}\}_{n=0}^\infty$ is a sequence of polynomials orthogonal with respect to $\sigma_k(x)d\mu(x) > 0$ on $(a, b)$, where $\sigma_k(x)$ is a polynomial of degree $k$, that satisfies

$$
(x - B_n)p_{n-1}(x) = a_{k-2}(x)p_n(x) + A_n\sigma_k(x)g_{n-2,k}(x), \quad n \in \mathbb{N}_{\geq 1},
$$

with $g_{-1,k} = 0$, $A_n$, $B_n$, $a_{-1}$, $a_{-2}$ constants and $a_{k-2}$ a polynomial of degree $k - 2$ defined on $(a, b)$ whenever $k \in \{2, 3, \ldots \}$. Then

(i) $k \in \{0, 1, 2, 3, 4\}$;

(ii) the $n - 1$ real, simple zeros of $(x - B_n)g_{n-2,k}$ interlace with the zeros of $p_n$ and $B_n$ is an upper bound for the smallest, as well as a lower bound for the largest zero of $p_n$ if $g_{n-2,k}$ and $p_n$ are co-prime;

(iii) if $g_{n-2,k}$ and $p_n$ are not co-prime,
(a) they have one common zero that is equal to $B_n$ and this common zero cannot be the largest or smallest zero of $p_n$;
(b) the $n - 2$ zeros of $g_{n-2,k}(x)$ interlace with the $n - 1$ non-common zeros of $p_n$;
(c) $B_n$ is an upper bound for the smallest as well as a lower bound for the largest zero of $p_n$.

All relevant contiguous relations that we need in the next sections were computed automatically using the Maple package hsum.mpl accompanying [11] and procedures that are specifically adapted for each family using the corresponding hypergeometric representation. These computations with their complete results can be downloaded from http://www.mathematik.uni-kassel.de/~koepf/Publikationen. In the given article, however, we have only included the necessary bounds deduced from the much more complicated contiguous relations since the full relations do not contribute to our results.

2. Inner bounds for the extreme zeros of Hahn polynomials. For $n \in \{0, 1, 2, \ldots, N\}$, the Hahn polynomials are orthogonal for $\alpha, \beta > -1$ with respect to the discrete weight

$$w(x) = \left(\frac{\alpha + x}{x}\right)\left(\frac{\beta + N - x}{N - x}\right) \text{ at } x \in \{0, 1, 2, \ldots, N\} \text{ (cf. [10])}.$$

For $\alpha, \beta > -1$, the parameter shifted Hahn polynomials $Q_n(x; \alpha + k, \beta + m, N)$ are orthogonal at the points $x \in \{0, 1, 2, \ldots, N\}$ with respect to

$$\left(\frac{\alpha + k + x}{x}\right)\left(\frac{\beta + m + N - x}{N - x}\right) = (x+\alpha+1)_k(-x+\beta+N+1)_m \omega(x) = \sigma_{k,m}(x)w(x) > 0$$

and together with $Q_n(x; \alpha, \beta, N)$, they satisfy the mixed three-term recurrence relations

$$(2.1) \quad (x - B_n^{\alpha,\beta}(k, m))Q_{n-1}(x; \alpha, \beta, N)
= a_{k+m-2}(x)Q_n(x; \alpha, \beta, N)(x) + A_n\sigma_{k,m}(x)Q_{n-2}(x; \alpha + k, \beta + m, N),$$

where $a_0(x)$, $a_1(x)$, $A_n$ and $B_n^{\alpha,\beta}(k, m)$ are constants and $a_{k+m-2}(x)$ is a polynomial of degree $k + m - 2$ for $k + m \in \{2, 3, \ldots\}$.

From Theorem 1.1 it follows that the mixed three-term recurrence relations (2.1) only exist if $k + m \in \{0, 1, 2, 3, 4\}$ and each of the points $B_n^{\alpha,\beta}(k, m)$, such that $k + m \in \{0, 1, 2, 3, 4\}$ is an upper bound for the smallest as well as a lower bound for the largest zero of $Q_n(x; \alpha, \beta, N)$, moreover, the relations that involve the largest possible parameter difference, are found to be particularly useful to obtain sharp bounds.

When $k = 4$ and $m = 0$, we obtain the relation

$$(2.2) \quad a_1(x)Q_{n-1}(x; \alpha, \beta, N) = a_2(x)Q_n(x; \alpha, \beta, N) + a_n(x + \alpha + 1)_4 Q_{n-2}(x; \alpha + 4, \beta, N),$$

where $a_2(x)$ is a polynomial with $2$ real zeros,

$$a_n = (n - 1)(\beta + n - 1)(\alpha + n + 1)_2(\alpha + \beta + n + 1)_2(\alpha + \beta + 2n)$$
and \( u_1 \) is a polynomial of degree 1 in \( x \), such that \( u_1(B_{n,0}^α(4,0)) = 0 \) for

\[
B_{n,0}^α(4,0) = \left( n^6 + (3α + 3β + 3)n^5 + (Nα + 3α^2 + 8αβ + 3β^2 + N + 4α + 8β - 1)n^4 \\
+ (α + β + 1)(2Nα + α^2 + 6αβ + β^2 + 2N - 2α + 6β - 7)n^3 \\
+ (N^2α^2 + Nα^3 + 4αβ^2 + 2αβ^2 + 2αβ + 3β^3 \\
+ 3α^2 + 2Nαβ + Nβ^2 - 12α^3 - α^2 + 7αβ^2 + 2β^3 \\
+ 2N^2 - 3Nα + 5Nβ - 42α^2 - 15β - β^2 - 3N - 56α - 12β - 24)n^2 \\
+ (α + β + 1)(N^2α^2 + 2Nαβ + α^2 + α^2 + 3Nα - 2Nα^2 + 5α + 10α^3 \\
- 3α^2 + αβ + 2N^2 - 6Nα + 3Nβ - 34α^2 - 9αβ - β^2 - 4N - 46α - 6β \\
- 20n + (α + 1)2(Nα + 3N + α - β + 4)(α + 2 + N + β)2) / \right)
\]

\[
(2N + 3α + β + 8)n^4 + 2(α + β + 1)(2N + 3α + β + 8)n^3 \\
+ (2N^2 + 8Nα^2 + 8Nαβ + 2Nβ^2 + 7α^3 + 13α^2β + 7αβ^2 + 3β^3 \\
+ 4N^2 + 30Nα + 10Nβ + 41α^2 + 48αβ + 13β^2 + 30N + 79α + 39β + 52)n^2 \\
+ (α + β + 1)(2N^2α + 6Nαβ + 4Nαβ + 4α^2 + 6α^2β + 2αβ^2 + 4N^2 \\
+ 26Nα + 6Nβ + 27α^2 + 24αβ + 3β^2 + 28N + 60α + 22β + 44)n \)
\]

\[(2.3) + (\alpha + 1)2(α + β + 2)(α + 2 + N + β)2). \]

The weight function \( w \) satisfies the symmetry property \( w(α, β, x) = w(β, α, N - x) \) from which the symmetry relation

\[
(\alpha + n)Q_n(N - x; α, β, N) = (-1)^n(β + n)nQ_n(x; β, α, N)
\]

can be proved (cf. \([9]\)) and we can deduce that when \( x \) is a zero of \( Q_n(x; α, β, N) \), then \( N - x \) will be a zero of \( Q_n(x; β, α, N) \). Likewise, the extra interlacing points obtained from the mixed three-term recurrence relations satisfied by \( Q_n(x; α, β, N) \), \( Q_{n-1}(x; α, β, N) \) and \( Q_{n-2}(x; α + m, β + k, N) \), are

\[(2.4) B_{n,0}^α(0, 4) = N - B_{n,0}^β(0, 4) \]

for all values of \( k \) and \( m \) in \( \mathbb{N}_{\geq 2} \) such that \( k + m \in \{0, 1, 2, 3, 4\} \).

Letting \( k = 0 \) and \( m = 4 \) in \((2.1)\), we obtain

\[(2.5) v_1(x)Q_{n-1}(x; α, β, N) = a_2(x)Q_n(x; α, β, N) + b_n(-x + β + N + 1)Q_{n-2}(x; α, β + 4, N), \]

where \( a_2(x) \) is a polynomial of degree 2, \( b_n = (n - 1)(α + β + n + 1)2(α + β + 2n) \), \( v_1(x) \) is a polynomial of degree 1 with zero \( B_{n,0}^α(0, 4) \), obtained from \((2.4)\), i.e.,

\[(2.6) B_{n,0}^α(0, 4) = N - B_{n,0}^β(0, 4). \]

From \([5, \text{Theorem 7.1.1}]\) it follows that the zeros of the Hahn polynomials increase with \( α \) and decrease with \( β \) which implies that the points \( B_{n,0}^α(4, 0) \) and \( B_{n,0}^β(0, 4) \) satisfy

\[0 < x_{n,1} < B_{n,0}^α(4, 0) < B_{n,0}^β(0, 4) < x_{n,n} < N, \]

for all values of \( α, β > -1 \) and \( n \in \{0, 1, 2, \ldots, N\} \) and we conclude that \( B_{n,0}^α(4, 0) \) is the best upper bound for the smallest zero and \( B_{n,0}^β(0, 4) \) is the most precise lower bound for the largest zero of \( Q_n(x; α, β, N) \) obtained by this method.
In [12, Lemma 9], the following inner bounds for the extreme zeros of the Hahn polynomials are provided for $\alpha \geq \beta > -1$ or $\alpha \leq \beta \leq -N - 1$:

\[
 x_{n,1} < \frac{(n + \alpha)(N - n + 1)}{\alpha + \beta + 1} < \frac{N(\alpha + n) + (\beta + n)(n - 1)}{\alpha + \beta + 2n} < x_{n,n}.
\]  

(2.7)  

(2.8)

In the tables below we compare these bounds, together with the bounds obtained from relations (2.2) and (2.5), to the actual values of the extreme zeros. In each case, the more precise bound is printed in bold.

<table>
<thead>
<tr>
<th>Table 2.1</th>
<th>Comparison of bounds for the extreme zeros of $Q_5(x; 10, 2, N)$ for different values of $N$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$x_{5,1}$</td>
</tr>
<tr>
<td>5</td>
<td>0.1659</td>
</tr>
<tr>
<td>10</td>
<td>1.5604</td>
</tr>
<tr>
<td>50</td>
<td>15.8455</td>
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<tr>
<td>100</td>
<td>34.2895</td>
</tr>
<tr>
<td>500</td>
<td>182.5365</td>
</tr>
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<table>
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<th>Table 2.2</th>
<th>Comparison of bounds for the extreme zeros of $Q_5(x; \alpha, 2, 30)$ for different values of $\alpha$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$x_{5,1}$</td>
</tr>
<tr>
<td>-0.5</td>
<td>0.2966</td>
</tr>
<tr>
<td>5</td>
<td>5.0673</td>
</tr>
<tr>
<td>10</td>
<td>8.5443</td>
</tr>
<tr>
<td>50</td>
<td>18.3546</td>
</tr>
<tr>
<td>200</td>
<td>23.2194</td>
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</tbody>
</table>

<table>
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<th>Table 2.3</th>
<th>Comparison of bounds for the extreme zeros of $Q_5(x; 10.5, \beta, 30)$ for different values of $\beta$.</th>
</tr>
</thead>
<tbody>
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<td>$\beta$</td>
<td>$x_{5,1}$</td>
</tr>
<tr>
<td>-0.5</td>
<td>11.2191</td>
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<tr>
<td>5</td>
<td>6.9636</td>
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<tr>
<td>50</td>
<td>1.0722</td>
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<tr>
<td>200</td>
<td>0.0657</td>
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<table>
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<tr>
<th>Table 2.4</th>
<th>Comparison of bounds for the extreme zeros of $Q_{100}(x; 3, -0.5, N)$ for different values of $N$.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$</td>
<td>$x_{100,1}$</td>
</tr>
<tr>
<td>1000</td>
<td>0.0361</td>
</tr>
<tr>
<td>10 000</td>
<td>7.4941</td>
</tr>
<tr>
<td>100 000</td>
<td>96.2846</td>
</tr>
<tr>
<td>500 000</td>
<td>489.3568</td>
</tr>
</tbody>
</table>
3. Inner bounds for extreme zeros of Continuous Hahn polynomials. Let \( n \in \mathbb{N}_{\geq 0} \).

The continuous Hahn polynomials are orthogonal on \( \mathbb{R} \) with respect to the weight function

\[
w(a, b, c, d, x) = \Gamma(a + ix)\Gamma(b + ix)\Gamma(c - ix)\Gamma(d - ix)
\]

if the real parts of \( a, b, c \) and \( d \) are positive and \( c = \bar{a}, d = \bar{b} \) (cf. [10]).

The conditions necessary to obtain orthogonality force us to simultaneously shift both parameters \( a \) and \( c \), as well as \( b \) and \( d \) and the parameter shifted polynomial \( p_n(x; a + k, b + m, c + k, d + m) \), which is orthogonal on \( \mathbb{R} \) with respect to

\[
\Gamma(a + k + ix)\Gamma(b + m + ix)\Gamma(c + k - ix)\Gamma(d + m - ix)
\]

\[
= (a + ix)_k(b + ix)_m(c - ix)_k(d - ix)_m w(a, b, c, d, x)
\]

\[
= \sigma_{k,m}(x) w(a, b, c, d, x) > 0,
\]

together with the polynomial \( p_n(x; a, b, c, d) \), satisfy the mixed three-term recurrence relations

\[
(x - B_n(k, m))p_{n-1}(x; a, b, c, d) = a_{k-m-2}(x)p_n(x; a, b, c, d) - d_n\sigma_{k,m}(x)p_{n-2}(x; a + k, b + m, c + k, d + m),
\]

where \( a_{k-m-2}(x) \) is a polynomial of degree \( k + m - 2 \) when \( k + m \in \{2, 3, \ldots\} \), and \( a_0, a_1, d_n \) and \( B_n(k, m) \) are constants. From Theorem 1.1 we deduce that each point \( B_n(k, m) \) such that \( k + m \in \{0, 1, 2, 3, 4\} \) will be an upper (lower) bound for the smallest (largest) zero of \( p_n(x; a, b, c, d) \).

The mixed three-term recurrence relation that involves the largest parameter shift is

\[
(x - B_n(0, 2))p_{n-1}(x; a, b, c, d) = a_2(x) = \frac{1}{n(b - d)(a + b + c + d + n - 1) + (b + d)(ab + a - b - cd - c + d)}p_n(x; a, b, c, d)
\]

\[
- \frac{1}{n(b - d)(a + b + c + d + n - 1) + (b + d)(ab + a - b - cd - c + d)}\times a_0, a_1, d_n \text{ and } B_n(k, m) \text{ are constants. From Theorem 1.1 we deduce that each point } B_n(k, m) \text{ such that } k + m \in \{0, 1, 2, 3, 4\} \text{ will be an upper (lower) bound for the smallest (largest) zero of } p_n(x; a, b, c, d). \]

The mixed three-term recurrence relation that involves the largest parameter shift is

\[
(x - B_n(0, 2))p_{n-1}(x; a, b, c, d) = a_2(x) = \frac{1}{n(b - d)(a + b + c + d + n - 1) + (b + d)(ab + a - b - cd - c + d)}p_n(x; a, b, c, d)
\]

and, when \( a = p + iq = \bar{c} \) and \( b = r + is = \bar{d} \) where \( p, q, r, s \in \mathbb{R} \), and \( p, r > 0 \), we have

\[
\sigma_{0,2}(x) = (b + ix)_2(d - ix)_2 = r^2 + (r + 1)^2 + 2(s + x)^2 > 0,
\]

\[
B_n(0, 2) = \frac{1}{n(b - d)(a + b + c + d + n - 1) + (b + d)(ab + a - b - cd - c + d)}\times a_0, a_1, d_n \text{ and } B_n(k, m) \text{ are constants. From Theorem 1.1 we deduce that each point } B_n(k, m) \text{ such that } k + m \in \{0, 1, 2, 3, 4\} \text{ will be an upper (lower) bound for the smallest (largest) zero of } p_n(x; a, b, c, d). \]

Furthermore, from the fact that \( w(a, b, c, d, x) = w(b, a, d, c, x) \) and consequently \( p_n(x; a, b, c, d) = p_n(x; b, a, d, c) \), shifting the parameters \( a \) and \( c \) yields the bound

\[
B_n(2, 0) = -\frac{2ps(n + r) + q(n - 1)(n + 2r) + 2qr(r + 1)}{n(p + 2r - 1) + 2r(p + r)}
\]
It is easy to prove that \( B_n(0, 2) \leq B_n(2, 0) \) for all \( n \geq 2 \) when \( q \leq s \) (i.e., \( \text{Im} (a) \leq \text{Im} (b) \)) and thus, applying Theorem 1.1, the point \( B_n(0, 2) \) in (3.1) will be the most precise upper bound for the smallest zero and \( B_n(2, 0) \) in (3.2) will be the best lower bound for the largest zero of \( p_n(x; a, b, c, d) \) obtained by this method, provided that \( \text{Im} (a) \leq \text{Im} (b) \). The order of these bounds will reverse when \( \text{Im} (a) \geq \text{Im} (b) \).

Furthermore, we observe that

\[
\lim_{n \to \infty} B_n(2, 0) = -\text{Im} (a) \quad \text{and} \quad \lim_{n \to \infty} B_n(0, 2) = -\text{Im} (b)
\]

and the bounds are thus less sharp for larger values of \( n \), and, when \( \text{Im} (a) = \text{Im} (b) \), \( B_n(2, 0) = B_n(0, 2) = -\text{Im} (a) \) for all values of \( n \). We refer the reader to Table 3.1 for examples that illustrate these results.

**Table 3.1**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( a )</th>
<th>( b )</th>
<th>( x_{0.1} )</th>
<th>( B_n(0, 2) ) from (3.1)</th>
<th>( B_n(2, 0) ) from (3.2)</th>
<th>( x_{n,n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>( 1 ) (- 2 i )</td>
<td>( 3 ) (- 20 i )</td>
<td>17.9415</td>
<td>20</td>
<td>20</td>
<td>22.0598</td>
</tr>
<tr>
<td>5</td>
<td>( 1 ) (- 20 i )</td>
<td>( 1 + 155 i )</td>
<td>-140.1610</td>
<td>-139.0909</td>
<td>4.0909</td>
<td>5.1612</td>
</tr>
<tr>
<td>5</td>
<td>( 1 ) (- 15 i )</td>
<td>( 1 + 15 i )</td>
<td>-12.5501</td>
<td>-12.2727</td>
<td>12.2727</td>
<td>12.5501</td>
</tr>
<tr>
<td>5</td>
<td>( 10 + i )</td>
<td>( 1 + 15 i )</td>
<td>-15.4565</td>
<td>-14.6316</td>
<td>-9.8000</td>
<td>-7.8590</td>
</tr>
<tr>
<td>35</td>
<td>( 1 ) (- i )</td>
<td>( 2 + 0.2 i )</td>
<td>-15.3222</td>
<td>-0.1898</td>
<td>0.9966</td>
<td>12.5501</td>
</tr>
<tr>
<td>35</td>
<td>( 1 + 0.9 i )</td>
<td>( 1 + i )</td>
<td>-16.1997</td>
<td>-0.9997</td>
<td>-0.9003</td>
<td>14.3098</td>
</tr>
</tbody>
</table>

4. Inner bounds for the extreme zeros of Continuous Dual Hahn polynomials. Consider the Continuous Dual Hahn polynomials

\[
\tilde{S}_n(x^2; a, b, c) = \frac{S_n(x^2; a, b, c)}{(a + b)_n(a + c)_n,}
\]

which are orthogonal in the interval \((0; \infty)\) with respect to the continuous weight function

\[
w(x; a, b, c) = \left| \frac{\Gamma (a + i x) \Gamma (b + i x) \Gamma (c + i x)}{\Gamma (2ix)} \right|^2.
\]

We are interested in inner bounds for the extreme zeros of the polynomials \( \tilde{S}_n(x^2; a, b, c) \) and from the three-term recurrence relation satisfied by the Continuous Dual Hahn polynomials (cf. [10, eqn. (9.3.4)]), we obtain the point

\[
B_n(0, 0, 0) = (a + b + n - 1)(a + c + n - 1) + (n - 1)(b + c + n - 2) - a^2
\]

which is an upper (lower) bound for the smallest (largest) zero of these polynomials.

The parameter shifted Continuous Dual Hahn polynomials \( \tilde{S}_n(x^2; a + k, b + l, c + m) \), are orthogonal in the interval \((0; \infty)\) with respect to \( \sigma_{k,l,m}(x^2)w(x; a, b, c) \), where

\[
\sigma_{k,l,m}(x^2) = \left| (a + i x)_k(b + i x)_l(c + i x)_m \right|^2
\]

and satisfy the mixed three-term recurrence relations

\[
(x^2 - B_n(k, l, m)) \tilde{S}_{n-1}(x^2; a, b, c) = a_{k-l+m-2}(x^2) \tilde{S}_n(x^2; a, b, c) - d_n \sigma_{k,l,m}(x^2) \tilde{S}_{n-2}(x^2; a + k, b + l, c + m),
\]
where \( a_{k+l+m-2} \) is a polynomial of degree \( k+l+m-2 \) in \( x^2 \) when \( k+l+m \in \{2, 3, \ldots \} \), and \( a_0, a_1, d_n, \) and \( B_n(k, l, m) \) are constants. From Theorem 1.1 we deduce that each point \( B_n(k, l, m) \) such that \( k+l+m \in \{0, 1, 2, 3, 4\} \) will be an upper (lower) bound for the smallest (largest) zero of \( \tilde{S}_n(x^2; a, b, c) \).

The mixed three-term recurrence relations that provide us with relatively good bounds are those that involve a total parameter shift of 4 units, i.e., when we shift

(i) one parameter by 4 units;
(ii) two of the parameters by 2 units each;
(iii) one parameter by 3 units and another one by 1 unit;
(iv) two parameters by 1 unit and the third one by 2 units.

Neither the weight function, nor the zeros of the polynomial \( \tilde{S}_n(x^2; a, b, c) \) depend on the order in which the parameters \( a, b \) and \( c \) occur and shifting \( a \) by 4 units leads to exactly the same bound as shifting \( b \) or \( c \) by 4 units, provided that the parameter with the smallest numerical value is the one that is shifted. We list the three relations that involve the parameter shifts mentioned in (i) – (iii) above, as numerical evidence confirms that these lead to the best possible upper bounds for the smallest zeros of the Continuous Dual Hahn polynomials.

The relation obtained when we shift \( a \) by 4 units is:

\[
\begin{align*}
\alpha_2(x^2)\tilde{S}_n(x^2; a, b, c) &= u_1(x^2)\tilde{S}_{n-1}(x^2; a, b, c) \\
&- \frac{(n-1)(b+c+n-2)(a+b+n)\sigma_{4,0,0}(x^2)}{(a+c)(a+b)}\tilde{S}_{n-2}(x^2; a+4, b, c),
\end{align*}
\]

where, from (4.2),

\[
\sigma_{4,0,0}(x^2) = (a^2 + x^2)((a + 1)^2 + x^2)((a + 2)^2 + x^2)((a + 3)^2 + x^2),
\]

\( \alpha_2 \) is a polynomial of degree 2 in \( x^2 \) and \( u_1(B_n(4, 0, 0)) = 0 \) for

\[
B_n(4, 0, 0) = \left( 5a^4 + 9a^3 + 12a^2 + 2a^3 + 5ab + 5ac + 5bc + 3ac^2 + 3ab^2 + 3bc^2 \right)
- 2a^4b^2 - 2a^4c^2 - 5a^3b^2 - 5a^3c^2 - 3a^2b^3 - 3a^2c^3 - b^3c^3 - a^5b - a^5c - a^3b^3
- a^5c^3 + (-4a^4 - 18a^2 - 22a - 6)n^2 + (-6a^4 - 6a^3 - 6a^2 - 10a^2bc - 24a^3
- 22a^2b - 22a^2c - 30abc - 22a^2 - 18ab - 18ac - 18bc)n^2 + (-2a^3 - 6a^2b - 6a^2c
- 4a^2b^2 - 16ab^2c - 4a^2c^2 - 10a^2b^2c - 10a^2bc^2 - 6ab^2c^2 - 6a^2c^3 - 3a^4 - 18a^3b - 18a^3c
- 13a^3b^2 - 42a^2bc - 13a^3c^2 - 24ab^2c - 24ab^2c - 9b^2c^2 - 16a^3 - 5a^3b - 5a^3c
- 9ab^2 - 12abc - 9ac^2 - 9b^2c - 9bc^2 + 35a^2 + 9ab + 9ac + 9bc + 22a + 6)n
- 3b^3c^2 - 3b^3c^3 - 2ab^3 - 2abc - 2bc^3 - 15abc + 15abc + 11abc + 11abc
- 6abc - 6abc - 13a^2bc - 3ab^2c^2 - 8a^2bc - 5a^2bc - 3ab^2c^2 - 8a^2bc^2
- 12a^2b^2c - 7a^3bc^3 - 3a^2b^3c + 18abc - 14a^2bc^2 - 3a^2bc^3 - 14a^2bc^2 - 7a^3bc^3
\right) / \left( (-4a - 6)n^3 + (4 - 6a^2 - 6ab - 2bc - 6a - 8b - 6c)n^2 + (-4a^3 - 6a^2b
- 6ab^2 - 2a^2bc - 2ac^2 - 2b^2c - 6a^2 - 6ab - 6ac - 2bc^2 - 6bc - 2c^2
- 2a + 2b + 2c - 2)n - a^4 - 2a^3b - 2a^3c - a^2b^2 - 4a^2bc - a^2c^2 - 2ab^2c - 2abc - 2abc
\right).
Shifting both $a$ and $b$ by 2 units leads to the mixed three-term recurrence relation

$$\frac{a_2(x^2)}{a + b + 2n}\tilde{S}_n(x^2; a, b, c) = -(x^2 - B_n(2, 2, 0))(a + b + 1)\tilde{S}_{n-1}(x^2; a, b, c)$$

$$- \frac{(n - 1)(a + b + n)_2}{(a + b + 2n)(a + b + 1)(a + c)_2}\sigma_{2,2,0}(x^2)\tilde{S}_{n-2}(x^2; a + 2, b + 2, c),$$

where $\sigma_{2,2,0}(x^2) = (a^2 + x^2)((a + 1)^2 + x^2)(b^2 + x^2)((b + 1)^2 + x^2)$, $a_2$ is a polynomial of degree 2 in $x^2$ and

(4.4) \quad B_n(2, 2, 0) = \frac{a^2(b + c) + b^2(a + c) + 2ab(c + n) + (a + b)(2c + n - 1)}{a + b + 2n}

By shifting $a$ by 3 units and $b$ by 1, we obtain

$$a_2(x^2)\tilde{S}_n(x^2; a, b, c) = v_1(x^2)\tilde{S}_{n-1}(x^2; a, b, c)$$

$$- \frac{(n - 1)(a + b + n)_2(a + c + n)}{(a + c)(a + b)_2}\sigma_{3,1,0}(x^2)\tilde{S}_{n-2}(x^2; a + 3, b + 1, c),$$

where $a_2$ is a polynomial of degree 2 in $x^2$ and $v_1(B_n(3, 1, 0)) = 0$ for

$$B_n(3, 1, 0) = \left( (a^3 + 3a^2b + 3a^2 + 6ab + 2a + 2b) n^2 ight.$$  

$$+ (a^4 + 4a^3b + 2a^3c + 3a^2b^2 + 6a^2bc + 4ab^2c + 2a^3 + 6a^2b + 6a^2c)$$  

$$+ 4ab^2 + 12abc + 4b^2c - a^2 - 2ab + 4ac + 4bc - 2a - 2b)n$$  

$$+ (a + c)(a + b)_2(abc + a^2c - a + b + 2c - 2) \right) /$$  

$$\left( (3a + b + 3)n^2 + (3a^2 + 4ab + 2ac + b^2 + 2bc + 2a + 2b + 2c - 1) n \right.$$  

$$+ (a + c)(a + b)_2 \right).$$

In Table 4.1 we provide some examples that illustrate the quality of these bounds. The best upper bound for the smallest zero obtained in each case is printed in bold. Furthermore, when only one parameter is shifted, we shift the smallest one and in the cases where 2 parameters are shifted, we shift the 2 smallest parameters and the parameter with the smallest numerical value is the one that is shifted optimally.

**Table 4.1**

Comparison of bounds for the extreme zeros of $S_n(x^2; a, b, c)$ for different values of $a, b, c$ and $n$. $B_n(4, 0, 0)$, $B_n(2, 2, 0)$, $B_n(3, 1, 0)$ and $B_n(0, 0, 0)$ are the bounds in (4.3), (4.4), (4.5) and (4.1) respectively.

<table>
<thead>
<tr>
<th>$n$</th>
<th>${a, b, c}$</th>
<th>$x_{1,1}^2$</th>
<th>$B_n(4, 0, 0)$</th>
<th>$B_n(2, 2, 0)$</th>
<th>$B_n(3, 1, 0)$</th>
<th>$B_n(0, 0, 0)$</th>
<th>$x_{n,n}^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>${7, 7, 7}$</td>
<td>63.91</td>
<td>120.676</td>
<td><strong>112.000</strong></td>
<td>114.065</td>
<td>402</td>
<td>581.83</td>
</tr>
<tr>
<td>6</td>
<td>${7, 8, 9}$</td>
<td>85.53</td>
<td>148.143</td>
<td>143.778</td>
<td><strong>143.778</strong></td>
<td>476</td>
<td>690.30</td>
</tr>
<tr>
<td>6</td>
<td>${1, 19, 40}$</td>
<td>389.85</td>
<td><strong>504.834</strong></td>
<td>572.125</td>
<td>533.443</td>
<td>1404</td>
<td>2147.23</td>
</tr>
<tr>
<td>6</td>
<td>${7, 8, 40}$</td>
<td>312.91</td>
<td>440.851</td>
<td><strong>436.556</strong></td>
<td>436.556</td>
<td>1251</td>
<td>1828.49</td>
</tr>
<tr>
<td>6</td>
<td>${7, 39, 40}$</td>
<td>1204.09</td>
<td><strong>1613.998</strong></td>
<td>1799.724</td>
<td>1697.511</td>
<td>3018</td>
<td>4285.71</td>
</tr>
<tr>
<td>31</td>
<td>${7, 8, 9}$</td>
<td>29.34</td>
<td>98.629</td>
<td><strong>91.649</strong></td>
<td><strong>91.649</strong></td>
<td>3401</td>
<td>5829.19</td>
</tr>
<tr>
<td>31</td>
<td>${1, 19, 40}$</td>
<td>114.82</td>
<td><strong>157.852</strong></td>
<td>240.951</td>
<td>186.358</td>
<td>6189</td>
<td>10788.25</td>
</tr>
</tbody>
</table>
5. Inner bounds for extreme zeros of Meixner-Pollaczek and Pseudo-Jacobi polynomials. The following theorem can be proved in a similar manner as Theorem 1.1.

**Theorem 5.1.** Let \( \{p_n\}_{n=0}^{\infty} \) be a sequence of polynomials orthogonal on the (finite or infinite) interval \((a, b)\) with respect to \(d\mu(x) > 0\). Let \( k \in \mathbb{N}_{\geq 0} \) be fixed and suppose \( \{g_{n,k}\}_{n=0}^{\infty} \) is a sequence of polynomials orthogonal with respect to \(\sigma_{2k}(x)dx > 0\) on \((a, b)\), where \(\sigma_{2k}(x)\) is a polynomial of degree \(2k\), that satisfies

\[
(5.1) \quad (x - B_n)p_{n-1}(x) = a_{2k-2}(x)p_n(x) + A_n\sigma_{2k}(x)g_{n-2,k}(x), \quad n \in \mathbb{N}_{\geq 1},
\]

with \(g_{-1,k} = 0, A_n, B_n, a_{-2}\) constants and \(a_{2k-2}(x)\) a polynomial of degree \(k - 2\) defined on \((a, b)\) whenever \(k \in \{1, 2, \ldots\}\). Then

(i) \( k \in \{0, 1, 2\}; \)

(ii) the \(n - 1\) real, simple zeros of \((x - B_n)g_{n-2,k}\) interlace with the zeros of \(p_n\) and \(B_n\) is an upper bound for the smallest, as well as a lower bound for the largest zero of \(p_n\) if \(g_{n-2,k}\) and \(p_n\) are co-prime;

(iii) if \(g_{n-2,k}\) and \(p_n\) are not co-prime,

(a) they have 1 common zero that is equal to \(B_n\) and this common zero cannot be the largest or smallest zero of \(p_n\);

(b) the \(n - 2\) zeros of \(g_{n-2,k}(x)\) interlace with the \(n - 1\) non-common zeros of \(p_n\);

(c) \(B_n\) is an upper bound for the smallest as well as a lower bound for the largest zero of \(p_n\).

5.1. Meixner-Pollaczek polynomials. The monic Meixner-Pollaczek polynomials are defined by

\[
\tilde{P}^\lambda_n(x; \phi) = n! \left(\frac{ie^{i\phi}}{e^{2i\phi} - 1}\right)^n P^{(\lambda)}_n(x; \phi)
\]

and are orthogonal on the real line for \(n \in \mathbb{N}_{\geq 1}, \lambda > 0\) and \(0 < \phi < \pi\) with respect to the weight function \(w(x) = e^{(2k-\pi)x}|\Gamma(\lambda + ix)|^2\).

**Theorem 5.2.** Let \(\tilde{P}^\lambda_n(x; \phi), \lambda > 0, 0 < \phi < \pi, n \in \mathbb{N}_{\geq 0}\), denote a monic Meixner-Pollaczek polynomial of degree \(n\). Then, for each fixed \(k \in \{0, 1, 2\}\),

(i) the zeros of \(\tilde{P}^{\lambda+k}_n(x; \phi)\), together with the point \(B_n(k)\), interlace with the zeros of \(\tilde{P}^\lambda_n(x; \phi)\) and each \(B_n(k)\) is an upper bound for the smallest as well as a lower bound for the largest zero of \(\tilde{P}^\lambda_n(x; \phi)\), where

\[
(5.2) \quad B_n(k) = -\frac{\lambda}{(\lambda)n+k-1} \cot \phi
\]

if \(\tilde{P}^{\lambda+k}_n(x; \phi)\) and \(\tilde{P}^\lambda_n(x; \phi)\) are co-prime;

(ii) if \(\tilde{P}^{\lambda+k}_n(x; \phi)\) and \(\tilde{P}^\lambda_n(x; \phi)\) are not co-prime, then

(a) the two polynomials under consideration have one common zero located at the respective points identified in (i);

(b) the \(n - 2\) zeros of \(\tilde{P}^{\lambda+k}_{n-2}(x; \phi)\) interlace with the remaining \(n - 1\) (non-common) zeros of \(\tilde{P}^\lambda_n(x; \phi)\);

(c) \(B_n(k)\), as identified in (i), is an upper bound for the smallest, as well as a lower bound for the largest zero of \(\tilde{P}^\lambda_n\).

**Proof.** Let \(\lambda > 0, 0 < \phi < \pi, k, n \in \mathbb{N}_{\geq 0}\). The parameter shifted polynomial \(\tilde{P}^{\lambda+k}_n(x; \phi)\) is orthogonal with respect to the weight function

\[
e^{(2\phi-\pi)x}|\Gamma(\lambda + k + ix)|^2 = |(\lambda + ix)_k|^2 w(x) > 0
\]
and the result follows directly from Theorem 5.1, the three-term recurrence relation satisfied by the Meixner-Pollaczek polynomials (cf. [10, (9.7.4)]) and the mixed three-term recurrence relations
\[
(x + \lambda \cot \phi) \tilde{P}_{n-1}^\lambda(x; \phi) = \frac{2\lambda}{2\lambda + n - 1} P_n^\lambda(x; \phi) + \frac{n - 1}{2\lambda + n - 1} (\lambda^2 + x^2) \tilde{P}_{n-2}^\lambda(x; \phi)
\]
and
\[
(x + \frac{(\lambda_2 \cot \phi}{\lambda + n}) \tilde{P}_{n-1}^\lambda(x; \phi)
\]
\[
= D_2(x) P_n^\lambda(x; \phi) + \frac{2(n - 1) \sin^2 \phi}{(2\lambda + 1)(2\lambda + n - 1)(\lambda + n)} (\lambda^2 + x^2) ((\lambda + 1)^2 + x^2) P_{n-2}^\lambda(x; \phi),
\]
where
\[
D_2(x) = \frac{(\lambda_2)(4\lambda - (n - 1) \cos 2\phi + n + 1) + (n - 1)x ((2\lambda + 1) \sin 2\phi - 2x \sin^2 \phi)}{(2\lambda + 1)(2\lambda + n - 1)(\lambda + n)}.
\]

The inner bounds for the extreme zeros of Meixner-Pollaczek polynomials, \(B_n(k), k \in \{0, 1, 2\}\), obtained in Theorem 5.2, satisfy
\[
x_{n,1} < B_n(2) < B_n(1) < B_n(0) < x_{n,n},
\]
for all values of \(\lambda > 0\) and \(0 < \phi < \frac{\pi}{2}\) and
\[
x_{n,1} < B_n(0) < B_n(1) < B_n(2) < x_{n,n},
\]
if \(\lambda > 0\) and \(\frac{\pi}{2} < \phi < \pi\).

In Table 5.1 we provide numerical examples in order to illustrate these bounds.

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(\phi)</th>
<th>(x_{30,1})</th>
<th>(B_{30}(0)) from (5.2)</th>
<th>(B_{30}(2)) from (5.2)</th>
<th>(x_{30,30})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.08</td>
<td>-650.5779</td>
<td>-367.963</td>
<td>-30.067</td>
<td>0.0100</td>
</tr>
<tr>
<td>0.5</td>
<td>1.57</td>
<td>-24.9119</td>
<td>-0.02349</td>
<td>-0.0002</td>
<td>24.8693</td>
</tr>
<tr>
<td>20</td>
<td>0.1</td>
<td>-853.2984</td>
<td>-488.3660</td>
<td>-83.7198</td>
<td>-52.4032</td>
</tr>
<tr>
<td>20</td>
<td>1.57</td>
<td>-39.1860</td>
<td>-0.0390</td>
<td>-0.0067</td>
<td>39.1131</td>
</tr>
</tbody>
</table>

### 5.2. Pseudo-Jacobi polynomials.

For \(n \in \mathbb{N}_{\geq 0}\), \(b \in \mathbb{R}\) and \(a < -n\), the Pseudo-Jacobi polynomials [7, 10] are orthogonal on \(\mathbb{R}\) with respect to the weight function \(w(x) = (1 + x^2)^a e^{2b \arctan x}\).

For \(b \in \mathbb{R}, a < -n, \) and \(k \in \mathbb{N}_{\geq 0}\), the polynomials \(P_n(x; a + k, b)\), which are orthogonal on the real line with respect to the weight function
\[
(1 + x^2)^{k+a} e^{2b \arctan x} = (1 + x^2)^k w(x) > 0,
\]
together with \(P_n(x)\) satisfy the mixed three-term recurrence relations
\[
(x - B_n(k)) P_n(x; a, b) = a_{2k-2}(x) P_n(x; a, b) + A_n(1 + x^2)^k P_{n-2}(x; a + k, b),
\]
for \(n \in \mathbb{N}_{\geq 1}\), where \(P_{-1} = 0, P_0 = 1, A_n, B_n(k), a_{-2}\) and \(a_0\) are constants and \(a_{2k-2}(x)\) is a polynomial of degree \(2k - 2\) for \(k \in \{2, 3, \ldots\}\).
From Theorem 5.1, we know that these identities will only exist for \( k \in \{0, 1, 2\} \). The mixed three-term recurrence relation that involves the largest possible parameter shift, (i.e., \( k = 2 \)), together with the bound

\[
B_n(2) = \frac{-b}{a + 1},
\]

is given in [7, Theorem 2.3] and from the three-term recurrence relation satisfied by Pseudo-Jacobi polynomials (cf. [10])

\[
(x - B_n(0)) P_{n-1}(x; a, b) = \frac{n(2a + n)P_n(x; a, b)}{(a + n)(2a + 2n - 1)} - \frac{(a + n - 2)(a + n) + b^2 + 1}{(a + n - 1)(2a + 2n - 1)} P_{n-2}(x; a, b)
\]

we obtain the point

\[
B_n(0) = \frac{-ab}{(a + n)(a + n - 1)}.
\]

It is easy to see that, for \( n \geq 2 \), we have \( B_n(2) \leq B_n(0) \) if \( b \geq 0 \) and thus

\[
x_{1,n} < B_n(2) \leq B_n(0) < x_{n,n}
\]

and the order of these bounds will reverse when \( b \leq 0 \). In Table 5.2 we illustrate the quality of these bounds when \( b \geq 0 \). In the specific case when \( b = 0 \), the weight function of Pseudo-Jacobi polynomials reduces to an even function of \( x \) and the zeros of the polynomial \( P_n(x; a, 0) \) are symmetric about the origin, which means that all polynomials of odd degree will have \( x = 0 \) as a zero and the example in Table 5.2 where \( b = 0 \) beautifully illustrates Theorem 5.1 iii (a) and (c).

| Table 5.2 | Comparison of bounds for the extreme zeros of \( P_{10}(x; a, b) \), \( b \geq 0 \), for different values of \( a \) and \( b \). |
|-----------|--|--|--|--|
| \( a \) | \( b \) | \( x_{10,1} \) | \( B_{10}(2) \) from (5.4) | \( B_{10}(0) \) from (5.5) | \( x_{10,10} \) |
| -15 | 8 | -0.1663 | 0.5714 | 4.000 | 5.6846 |
| -10.005 | 8 | 0.2100 | 0.8884 | 15928.358 | 15928.490 |
| -30 | 30 | 0.4386 | 1.0345 | 2.1429 | 3.2310 |
| -30 | 0 | -0.7861 | 0 | 0 | 0.7861 |

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