On a connection between formulas about \(q\)-gamma functions

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In this short communication, we want to pay attention to a few wrong formulas which are unfortunately cited and used in a dozen papers afterwards. We prove that the provided relations and asymptotic expansion about the \(q\)-gamma function are not correct. This is illustrated by numerous concrete counterexamples. The error came from the wrong assumption about the existence of a parameter which does not depend on anything. Here, we apply a similar procedure and derive a correct formula for the \(q\)-gamma function.

Keywords: \(q\)-Gamma function; asymptotic expansion; boundary functions.

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1. Introduction

Since J. Thomae (1869) and F. H. Jackson (1904) defined the \(q\)-gamma function, it plays an important role in the theory of the basic hypergeometric series [4] and its applications [7]. Its properties and different representations were discussed in numerous papers, such as in [3], [11] and [10]. A few successful algorithms for its numerical evaluation are introduced in [6] and [5] and [1]. An asymptotic expansion of the \(q\)-gamma function was provided in [2].

Here, we will make observations on the asymptotic expansions given in [8, 9].

Let \(q \in [0, 1)\). A \(q\)-number \([a]_q\) is

\[
[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.
\]

The factorial of a positive integer number \([n]_q\) is given by

\[
[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q\cdots[1]_q, \quad (n \in \mathbb{N}).
\]
An important role in $q$–calculus plays the $q$-Pochhammer symbol defined by

$$(a; q)_0 = 1, \quad (a; q)_n = \prod_{i=0}^{n-1} (1 - aq^i) \quad (n \in \mathbb{N} \cup \{+\infty\}),$$

and

$$(a; q)_\lambda = \frac{(a; q)_\infty}{(aq^{\lambda}; q)_\infty} \quad (|q| < 1, \ \lambda \in \mathbb{C}).$$

The $q$-gamma function

$$\Gamma_q(z) = (q; q)_{z-1} (1 - q)^{1-z} = \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z} \quad (0 < q < 1, \ z \notin \mathbb{Z}^-) \quad (1.1)$$

has the following properties:

$$\Gamma_q(z + 1) = [z]_q \Gamma_q(z) \quad (z \in \mathbb{C}), \quad \Gamma_q(n + 1) = [n]_q! \quad (n \in \mathbb{N}_0).$$

In particular,

$$\lim_{q \to 1-} \Gamma_q(z) = \Gamma(z).$$

The exact $q$–Gauss multiplication formula can be found in [4] or [3]:

$$\Gamma_q(nx) \prod_{k=1}^{n-1} \Gamma_q\left(\frac{k}{n}\right) = [n]_q^{nx-1} \prod_{k=0}^{n-1} \Gamma_q\left(x + \frac{k}{n}\right) \quad (x > 0; \ n \in \mathbb{N}).$$

Equivalently, substituting $z = nx$, it can be written in the form

$$\Gamma_q\left(z\right) \prod_{k=1}^{n-1} \Gamma_q\left(\frac{k}{n}\right) = [n]_q^{z-1} \prod_{k=0}^{n-1} \Gamma_q\left(\frac{z+k}{n}\right) \quad (z > 0; \ n \in \mathbb{N}). \quad (1.2)$$

2. Our corrections to the paper [8]

Starting from the definition

$$\Gamma_q(x) = (q; q)_\infty (1 - q)^{1-x}(q^x; q)_\infty^{-1},$$

we can write

$$\Gamma_q(x) = (q; q)_\infty (1 - q)^{1/2-x}(1 - q)^{1/2-x}e^{-\log(q^x; q)\infty}.$$  

Hence the function $\Gamma_q(x)$ can be written in the form

$$\Gamma_q(x) = a(q) \cdot (1 - q)^{1/2-x} e^{\mu(x)} \quad (a(q) \in \mathbb{R}), \quad (2.1)$$

where

$$0 < a(q) = (q; q)_\infty (1 - q)^{1/2} < 1, \quad \mu(x, q) = -\log(q^x; q)_\infty. \quad (2.2)$$

Let

$$\psi(x, q) = \frac{q^x}{(1 - q)(1 - q^x)}.$$
From the estimate
$$0 < \mu(x, q) < \psi(x, q) \quad (0 < q < 1, x > 0),$$
it exists \( \theta(x, q) \in (0, 1) \) such that
$$\mu(x, q) = \theta(x, q) \cdot \psi(x, q).$$

Therefore, relation (2.1) becomes
$$\Gamma_q(x) = a(q) \cdot (1 - q)^{1/2-x} e^{\theta(x, q) \cdot \psi(x, q)}.$$  \hspace{1cm} (2.3)

On the other hand, formula (1.2) can be written in the form
$$a_p(q) \Gamma_q(x) = [p] q^{\sum_{k=0}^{n-1} \Gamma_q^p \left( \frac{x+k}{p} \right)} \quad (x > 0; \ p \in \mathbb{N}),$$  \hspace{1cm} (2.4)

where
$$a_p(q) = [p] \Gamma_q^p \left( \frac{1}{p} \right) \Gamma_q^p \left( \frac{2}{p} \right) \cdots \Gamma_q^p \left( \frac{p}{p} \right).$$

Substituting \( q \to q^p \) and \( x \to k/p \) into the definition (1.1) of the \( q \)-gamma function, we have
$$\Gamma_q^p \left( \frac{k}{p} \right) = \frac{(q^p; q^p)^n}{(q^k; q^p)^n} \ (1 - q^p)^{1-k/p} = (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^k; q^p)_n}.$$  

Using moreover
$$\prod_{k=1}^{p} (1 - q^p)^{1-k/p} = (1 - q^p)^{n-1},$$
the following holds:
$$a_p(q) = [p] \prod_{k=1}^{p} \Gamma_q^p \left( \frac{k}{p} \right) = [p] \prod_{k=1}^{p} (1 - q^p)^{1-k/p} \lim_{n \to \infty} \frac{(q^p; q^p)_n}{(q^k; q^p)_n}$$

$$= [p] \prod_{k=1}^{p} (1 - q^p)^{1-k/p} \lim_{n \to \infty} \prod_{k=1}^{p} \frac{(q^p; q^p)_n}{(q^k; q^p)_n}$$
$$= [p] (1 - q^p)^{n-1} \lim_{n \to \infty} \prod_{k=1}^{p} \frac{(q^p; q^p)_n}{(q^k; q^p)_n}.$$  

The following identity is valid
$$\prod_{k=1}^{p} (q^k; q^p)_n = (q; q)_n^p.$$  

Using estimate (2.3), we get
$$\Gamma_q^p(n+1) = a(q^p) \cdot (1 - q^p)^{-n-1/2} e^{\theta(n+1,q^p) \cdot \psi(n+1,q^p)}$$

Since
$$\frac{(q^p; q^p)_n}{(1 - q^p)^{n+1}} = \Gamma_q^p(n+1) = a(q^p) \cdot (1 - q^p)^{p(-1/2-n)} e^{p \cdot \theta(n+1,q^p) \cdot \psi(n+1,q^p)},$$
and
\[
\frac{\prod_{k=1}^{p}(q^{k}; q^{p})_{n}}{(1-q)^{np}} = \frac{(q; q)_{np}}{(1-q)^{np}} = \Gamma_q(np + 1) = a(q) \cdot (1-q)^{-1/2-np} \cdot e^{\theta(np+1,q) \cdot \psi(np+1,q)},
\]
we have
\[
a_p(q) = \frac{a^p(q^p)}{a(q)} [p]^q \frac{\lim_{n \to \infty} e^{p \cdot \theta(n+1,q^p) \cdot \psi(n+1,q^p)}}{e^{\theta(np+1,q) \cdot \psi(np+1,q)}}.
\]

From
\[
\lim_{n \to \infty} \psi(n+1,q^p) = \lim_{n \to \infty} \psi(np+1,q) = 0 \quad (0 < q < 1; \ p \in \mathbb{N}),
\]
we find
\[
a_p(q) = \frac{[p]^q a^p(q^p)}{a(q)}.
\]
In that manner, the parameter \(a_p(q)\) from formula (2.4) is expressed via the parameter \(a(q)\) from formula (2.3).

3. Faults in paper [8]

In the very beginning, the author has supposed that \(\Gamma_q(x)\) for \(0 < q < 1; \ x > 0\), can be written in the form
\[
\Gamma_q(x) = a \cdot (1-q)^{1/2-x} \cdot e^{\mu(x)} \quad (a \in \mathbb{R}),
\]
where
\[
\mu(x, q) = -\log(q^x; q) > 0.
\]

His efforts in looking for \(\mu(x)\) we shortened a lot by starting from the definition of \(\Gamma_q(x)\). From the fact that
\[
0 < \mu(x) < \frac{q^x}{(1-q)(1-q^x)},
\]
and
\[
(1-q)(1-q^x) = 1 - q - q^x + q^{x+1} > 1 - q - q^x,
\]
the author in [8] concluded wrongly that
\[
0 < \mu(x) < \frac{q^x}{(1-q) - q^x}.
\]
But, expression \(1 - q - q^x\) is not positive for all \(q \in (0, 1)\) and \(x > 0\). Indeed,
\[
1 - q - q^x \leq 0 \iff 1 - q \leq q^x \iff x \cdot \log q \geq \log(1-q) \iff x \leq \frac{\log(1-q)}{\log q}.
\]

**Example 3.1.** We examined the sign changes of the function \(h_q(x) \equiv 1 - q - q^x\) for different \(q\) and \(x\). Notice that \(x \to +\infty\) if \(q \to 1^-\).
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Table 1. Unique real zero of the function \(h_q(x)\) and the sign changes for random values of \(q\) and \(x\)

<table>
<thead>
<tr>
<th>(q)</th>
<th>(x \cdot 1 - q - q^x = 0)</th>
<th>(x)</th>
<th>(q)</th>
<th>(1 - q - q^x)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.045758</td>
<td>1.10500</td>
<td>0.9</td>
<td>0.592727</td>
</tr>
<tr>
<td>0.3</td>
<td>0.296248</td>
<td>2.27287</td>
<td>0.752038</td>
<td>0.275286</td>
</tr>
<tr>
<td>0.5</td>
<td>1.0000</td>
<td>6.47584</td>
<td>0.816692</td>
<td>0.0861563</td>
</tr>
<tr>
<td>0.7</td>
<td>3.37555</td>
<td>43.2362</td>
<td>0.946066</td>
<td>0.0370453</td>
</tr>
<tr>
<td>0.9</td>
<td>21.8543</td>
<td>60.1635</td>
<td>0.954814</td>
<td>0.0167368</td>
</tr>
</tbody>
</table>

This estimate should be written in the form

\[
0 < \mu(x) < \frac{q^x}{(1 - q) - q^x}, \quad \left(0 < q < 1; \ x > \frac{\log(1 - q)}{\log q}\right).
\]

Furthermore, from the estimate

\[
0 < \mu(x) < \frac{q^x}{(1 - q) - q^x},
\]

the author in [8] concluded wrongly that

\[
\mu(x) = \frac{\theta q^x}{(1 - q) - q^x},
\]

where \(\theta\) is a number independent of \(x\) between 0 and 1.

**Example 3.2.** We find counterexamples which show that \(\theta\) depends on \(x\) and \(q\). At the first table, we fixed \(q = 0.9\) and take a few random values for \(x\). In another we changed the rule of variables.

Table 2. The dependence of parameter \(\theta\) from \(x\) and \(q\)

<table>
<thead>
<tr>
<th>(x)</th>
<th>(q)</th>
<th>(\theta)</th>
<th>(x)</th>
<th>(q)</th>
<th>(\theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.78377</td>
<td>0.9</td>
<td>-7.27980</td>
<td>10.5</td>
<td>0.63920</td>
<td>1.00000</td>
</tr>
<tr>
<td>13.2554</td>
<td>0.9</td>
<td>-1.58344</td>
<td>10.5</td>
<td>0.234682</td>
<td>1.00000</td>
</tr>
<tr>
<td>20.6473</td>
<td>0.9</td>
<td>-0.139893</td>
<td>10.5</td>
<td>0.494904</td>
<td>0.99898</td>
</tr>
<tr>
<td>25.7471</td>
<td>0.9</td>
<td>0.342512</td>
<td>10.5</td>
<td>0.618621</td>
<td>0.98504</td>
</tr>
<tr>
<td>32.2948</td>
<td>0.9</td>
<td>0.673069</td>
<td>10.5</td>
<td>0.806515</td>
<td>0.473541</td>
</tr>
<tr>
<td>43.8850</td>
<td>0.9</td>
<td>0.904181</td>
<td>10.5</td>
<td>0.915828</td>
<td>-4.19862</td>
</tr>
</tbody>
</table>

In continuation, the author in [8] got the wrong formulas (2.21)-(2.27). He concluded that

\[
a_p = \sqrt{[2]_q \Gamma_q(1/2)},
\]

and

\[
\Gamma_q(x) = \sqrt{[2]_q \Gamma_q(1/2)(1-q)^{1/2-x}} e^{\theta \frac{x^q}{x-q}} \quad (0 < \theta < 1).
\]

The following wrong version of the \(q\)-Gauss multiplication formula was provided

\[
[n]_q^{1/2-x} [2]_q^{(n-1)/2} \Gamma_q^{-1} (1/2) \Gamma_q(x) = \prod_{k=0}^{n-1} \Gamma_q q^x \left(\frac{x+k}{n}\right) \quad (x > 0; \ n \in \mathbb{N}).
\]
In a special case, for \( n = 2 \), it agrees with the exact \( q \)-Legendre relation. Also, when \( q \to 1 \), it reduces to well-known formulas for gamma-function.

4. Bounds of the \( q \)-gamma function

Let
\[
g(x) = \ln \Gamma_q(x)
\]

Since
\[
g(x + 1) = \ln \Gamma_q(x + 1) = \ln ([x]_q \Gamma_q(x)) = \ln [x]_q + g(x),
\]

by induction, we get
\[
g(x + n) = \sum_{k=0}^{n-1} \ln [x + k]_q + g(x) \quad (n \in \mathbb{N}).
\]

It is known that \( g(x) \) is a convex function.

**Lemma 4.1.** If \( x \in (0, 1) \) and \( n \in \mathbb{N} \), then
\[
g(n) + x \ln [x + n - 1]_q \leq g(x + n) \leq (1 - x)g(n) + xg(n + 1)
\]

**Proof.** Since
\[
x + n = (1 - x)n + x(n + 1),
\]

we can write
\[
g(x + n) = g((1 - x)n + x(n + 1)) \leq (1 - x)g(n) + xg(n + 1).
\]

Let us find a lower bound for \( \Gamma_q(x) \). Since
\[
n = (1 - x)(x + n) + x(x + n - 1),
\]

and because of the convexity of the function \( g(x) \), we have
\[
g(n) \leq (1 - x)g(x + n) + xg(x + n - 1).
\]

Applying (4.1), for \( x \to x + n - 1 \), we can write
\[
g(x + n) = \ln [x + n - 1]_q + g(x + n - 1),
\]

wherefrom
\[
g(n) \leq (1 - x)g(x + n) + x(g(x + n) - \ln [x + n - 1]_q) = g(x + n) - x\ln [x + n - 1]_q,
\]

i.e.,
\[
g(n) + x \ln [x + n - 1]_q \leq g(x + n). \quad \Box
\]

**Theorem 4.1.** The following bounds are valid:
\[
[n - 1]_q! \ [n - 1 + x]_q^* \leq \Gamma_q(n + x) \leq [n - 1]_q! \ [n]_q^*, \quad (n \in \mathbb{N}_0; \ 0 \leq x < 1).
\]
Proof. According to the upper bound for \( g(x) \), we get e. g.
\[
\ln \Gamma_q(x+n) \leq (1-x) \ln \Gamma_q(n) + x \ln \Gamma_q(n+1).
\]
Hence
\[
\Gamma_q(x+n) \leq ([n-1]_q)!^{1-x} ([n]_q)^x,
\]
wherefrom
\[
\Gamma_q(x+n) \leq [n-1]_q! \cdot [n]_q^x.
\]
According to the lower bound for \( g(x) \), we get
\[
\ln \Gamma_q(n) + x \ln [x+n-1]_q \leq \ln \Gamma_q(x+n),
\]
i.e.,
\[
\Gamma_q(n) \cdot [x+n-1]_q^n \leq \Gamma_q(n+x).
\]

Theorem 4.2.
\[
[n - (1-x)]_q \leq \left( \frac{\Gamma_q(n+x)}{[n-1]_q!} \right)^{1/x} \leq [n]_q^x, \quad (n \in \mathbb{N}_0; 0 \leq x < 1).
\]

Theorem 4.3. For any \( n \in \mathbb{N} \) and \( x \in (0, 1) \) there exists \( \theta = \theta(n, x, q) \in (0, 1) \) such that
\[
\Gamma_q(n+x) = [n-1]_q! \cdot [n - \theta(1-x)]_q^x.
\]
Introducing \( y = n+x \ (n \in \mathbb{N}_0; 0 \leq x < 1) \) and denoting \( n = \lfloor y \rfloor \), we can write
\[
[[y]-1]_q^x \cdot [y-1]_q^{y-\lfloor y \rfloor} \leq \Gamma_q(y) \leq [[y]-1]_q! \cdot [[y]_q^{y-\lfloor y \rfloor} \quad (y > 1).
\]

Theorem 4.4. For any \( y \in (1, +\infty) \setminus \mathbb{N} \), it exists \( \theta = \theta(y, q) \in (0, 1) \) such that
\[
\Gamma_q(y) = [[y]-1]_q! \cdot [[y] - \theta(1 - (y - \lfloor y \rfloor))]_q^{y-\lfloor y \rfloor}.
\]

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Fig. 2. $\Gamma_q(x)$ and its bounds for $q = 0.9$.

References