CONVEX FUNCTIONS AND THE NEHARI UNIVALENCE CRITERION [1]

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Nehari [4] showed that a convex function, i.e. a function which maps the unit disk $\mathbb{D}$ univalently onto a convex domain, satisfies the Nehari univalence criterion, i.e.

$$(1-|z|^2)^2|S_f(z)| \leq 2, \ \ z \in \mathbb{D},$$

where $S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ denotes the Schwarzian derivative, and that this result is sharp, as the function $f(z) = \frac{1}{2} \log \frac{1+z}{1-z}$ shows. The method of the proof does not give all sharp functions. We shall show that the sharp function is essentially unique using another approach implying Nehari's result.

This shows furthermore that all convex domains except of parallel strip domains are Jordan domains in $\hat{\mathbb{C}}$, using a result of Gehring and Pommerenke [3].

Moreover we give a geometrical description of convex domains whose corresponding convex functions satisfy the stronger relation

$$\sup_{z\in\mathbb{D}} (1-|z|^2)^2|S_f(z)| < 2. \tag{1}$$

This result generalizes Nehari's [4] that bounded convex functions satisfy (1).

Without loss of generality we may assume that $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ is normalized. The following result is well-known.

Theorem 1 (see e.g. [6]). *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *is convex, then*

$$|a_3 - a_2^2| \leq \frac{1-|a_2|^2}{3}.$$

As a consequence we have

[1] This paper is in final form and no version of it will be submitted for publication elsewhere.

Corollary 1. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *is convex, then*

$$|a_3 - a_2^2| \leq \frac{1}{3}$$

with equality if and only if

$$f(z) = \frac{1}{2x} \log \frac{1 + xz}{1 - xz}, \quad x \in \partial\mathbb{D}. \tag{2}$$

Proof. The inequality obviously follows from the theorem. Equality can only occur if $a_2 = 0$ and $|a_3| = 1/3$. Now we have to show that in this case f is of the form (2). If $f(z) = z + a_3z^3 + \cdots$ is convex, then $1 + z\frac{f''(z)}{f'(z)} = 1 + 6a_3z^2 + \cdots$ has positive real part (see e.g. [5], Theorem 2.7). So the second coefficient of this function is bounded by 2 (see e.g. [5], Corollary 2.3), and it follows that $|a_3| \leq 1/3$. Equality holds if and only if

$$1 + z\frac{f''(z)}{f'(z)} = \frac{1 + x^2z^2}{1 - x^2z^2}, \quad x \in \partial\mathbb{D},$$

(see e.g. [5], Corollary 2.3) , which is equivalent to (2). □

Hence we get

Theorem 2. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots$ *is convex, then*

$$(1 - |a|^2)^2 |S_f(a)| \leq 2, \tag{3}$$

for all $a \in \mathbb{D}$, *with equality if and only if* $f(\mathbb{D})$ *is a parallel strip domain, i.e.*

$$f(z) = \frac{1}{x + y} \log \frac{1 + xz}{1 - yz}, \quad x, y \in \partial\mathbb{D}, \ y \neq -x.$$

Proof. The theorem is true for $a = 0$, as Corollary 1 shows. Let now $a \in \mathbb{D}\backslash\{0\}$. Then the function $g(z) = z + b_2z^2 + b_3z^3 + \cdots$, defined by

$$g(z) = \frac{f\left(\frac{z + a}{1 + \bar{a}z}\right) - f(a)}{(1 - |a|^2)\, f'(a)}, \tag{4}$$

is also convex. On the other hand it is easily seen that

$$(1 - |a|^2)^2 S_f(a) = S_g(0) = 6(b_3 - b_2^2),$$

so that inequality (3) follows. If equality holds, then g has the form (2), and because of (4) f must have a similar range, which gives the result. □

As a consequence one has

Corollary 2. *If f is convex, and if $f(\mathbb{D})$ is no parallel strip domain, then $f(\mathbb{D})$ is a Jordan domain in $\hat{\mathbb{C}}$.*

Proof. This follows from [3], Theorem 1. □

Finally we shall give a complete geometrical description for convex domains whose corresponding convex functions satisfy the stronger relation (1).

Theorem 3. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is convex, then the following conditions are equivalent:*

(a) $\sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |S_f(z)| = 2$;

(b) *there is a sequence of domains G_k which are similar to $f(\mathbb{D})$, such that the Carathéodory kernel of (G_k) (see e.g. [5], p. 28) is a parallel strip;*

(c) *$f(\mathbb{D})$ is a parallel strip or is unbounded such that $\partial f(\mathbb{D})$ has an angle $\alpha = 0$ at ∞;*

(d) *$\partial f(\mathbb{D})$ is not a quasicircle in $\hat{\mathbb{C}}$.*

Proof. Suppose that $f(\mathbb{D})$ is a parallel strip. Then all conditions are true, (a) by Theorem 2 and (b) by choosing the constant sequence. In the sequel let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ be convex and $f(\mathbb{D})$ not a parallel strip. Then by Corollary 2 it is a Jordan domain in $\hat{\mathbb{C}}$.

(a) $\Leftrightarrow$ (b). Suppose, condition (a) holds. If the supremum is attained at an interior point $a \in \mathbb{D}$, then Theorem 2 implies that $f(\mathbb{D})$ is a parallel strip, which we assumed not to be the case. So there is a sequence (z_k) of numbers $z_k \in \mathbb{D}$ with

$$(1 - |z_k|^2)^2 |S_f(z_k)| \to 2$$

as $k \to \infty$. We define functions g_k by

$$g_k(z) := \frac{f\left(\dfrac{z + z_k}{1 + \bar{z}_k z}\right) - f(z_k)}{(1 - |z_k|^2)\ f'(z_k)} = z + b_{2k} z^2 + b_{3k} z^3 + \cdots,$$

and get

$$(1 - |z_k|^2)^2 |S_f(z_k)| = 6 |b_{3k} - b_{2k}^2| \to 2,$$

so that it follows with aid of Corollary 1 that the sequence (g_k) or a subsequence converges locally uniformly to some function of the form $\frac{1}{2x} \log \frac{1+xz}{1-xz}$, $x \in \partial\mathbb{D}$. By definition all functions g_k have ranges which are similar to $f(\mathbb{D})$ implying (b). Similarly one sees that (b) implies (a), too.

(a), (b) $\Rightarrow$ (c). We have to show that

(i) $f(\mathbb{D})$ is unbounded,

(ii) $\partial f(\mathbb{D})$ has an angle α at ∞,

(iii) $\alpha = 0$.

Step (i) is Nehari's result [4] that bounded convex functions satisfy (1), (ii) is a property of convex domains, and (iii) will be deduced from (b).

Because $f(\mathbb{D})$ is convex, the complement is the union of halfplanes $\mathbb{C}\backslash f(\mathbb{D}) = \bigcup_{t \in T} H_t$. Consider now the images of the corresponding lines ∂H_t on the Riemann sphere. They represent a family of circles on the sphere, having the north pole as a common point. So there exist two extremal directions, which correspond to the asymptotic directions of $\partial f(\mathbb{D})$ and give the semitangents of $\partial f(\mathbb{D})$ at ∞, and (ii) is verified.

Suppose now, $\alpha > 0$. We consider the images of the asymptotic lines on the Riemann sphere. These circles intersect at the north pole and at some finite point under the same angle α. So the same is true for the asymptotic lines theirselves, because the stereographic projection is angle-preserving. For an arbitrary similar region, i.e. the image of $f(\mathbb{D})$ under the conformal mapping $az + b$, this angle remains invariant, so that each similar domain has the same fixed angle α at ∞.

Now, let (G_k) be a given sequence of domains similar to $f(\mathbb{D})$, which converges to a parallel strip. With aid of a suitable Möbius transformation we transform ∞ into a finite point w_0. Of course this transformation does not preserve convexity, but it preserves univalence and the angle α. So we get a new sequence of regions $(\tilde{G}_k)$, which have the point w_0 as a common boundary point, having there the fixed angle α, and which converge to a region, having w_0 as a (doubly) boundary point with a zero angle there. This easily gives a contradiction (see e.g. [5], p. 31, problem 3), and so $\alpha = 0$.

(c) $\Rightarrow$ (d). As a well-known consequence of Ahlfors' intrinsic characterization [1] quasicircles don't have zero angles.

(d) $\Rightarrow$ (a). This is equivalent to the result of Ahlfors and Weill [2], who showed that (1) implies that $f(\mathbb{D})$ is a quasicircle in $\hat{\mathbb{C}}$. □

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