INNER BOUNDS FOR THE EXTREME ZEROS OF THE HAHN, CONTINUOUS HAHN AND CONTINUOUS DUAL HAHN POLYNOMIALS

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ABSTRACT. We apply Zeilberger's algorithm to generate relations of finite-type necessary to obtain inner bounds for the extreme zeros of orthogonal polynomial sequences with ${}_3F_2$ hypergeometric representations. Using this method, we improve previously obtained upper bounds for the smallest and lower bounds for the largest zeros of the Hahn polynomials and we identify inner bounds for the extreme zeros of the Continuous Hahn and Continuous Dual Hahn polynomials. Numerical examples are provided to illustrate the quality of the new bounds.

1. INTRODUCTION

A sequence of real polynomials $\{p_n\}_{n=0}^{\infty}$, where p_n is of exact degree n, is orthogonal with respect to a positive measure $\mu(x) > 0$ on an interval (a, b), if the scalar product

$$\langle p_m, p_n \rangle = \int_a^b p_m(x) p_n(x) d\mu(x) = 0, \quad m \neq n.$$

If $\mu(x)$ is absolutely continuous, then it can be represented by a real weight function $w(x) > 0$ so that $d\mu(x) = w(x)\,dx$. If $\mu(x)$ is discrete with support in $\mathbb{N}_{\geqq 0}$, then it can be represented by a discrete weight $w(x) \geqq 0$ ($x \in \mathbb{N}_{\geqq 0}$) and the scalar product is given by

$$\langle p_m, p_n \rangle = \sum_{x=0}^{\infty} p_m(x)\, p_n(x)\, w(x)\ .$$

The orthogonal polynomial families under consideration in this paper are the following ones (see [10]):

- Hahn polynomials: discrete weight $w(x) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$ in $\{0, 1, \ldots, N\}$ and
$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\left. \begin{array}{c} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{array} \right| 1 \right).$$
- Continuous Hahn polynomials: continuous weight $w(x) = \Gamma(a+ix)\Gamma(b+ix)$ $\Gamma(c-ix)\Gamma(d-ix)$ in the interval $(-\infty, \infty)$ and
$$p_n(x; a, b, c, d) = i^n \frac{(a+c)_n (a+d)_n}{n!} {}_3F_2\left(\left. \begin{array}{c} -n, n+a+c+b+d-1, a+ix \\ a+c, a+d \end{array} \right| 1 \right).$$

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- Continuous Dual Hahn polynomials: continuous weight

$$w(x)=\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2$$

in the interval $(0,\infty)$ and

$$S_n(x^2;a,b,c)=(a+b)_n(a+c)_n\cdot {}_3F_2\left(\begin{array}{c}-n,a+ix,a-ix\\ a+b,a+c\end{array}\middle|\,1\right).$$

Here,

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt$$

denotes the Gamma function,

$${}_pF_q\left(\begin{array}{c}\alpha_1,\alpha_2,\dots,\alpha_p\\ \beta_1,\beta_2,\dots,\beta_q\end{array}\middle|\,x\right)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\cdot(\alpha_2)_k\cdots(\alpha_p)_k}{(\beta_1)_k\cdot(\beta_2)_k\cdots(\beta_q)_k}\frac{x^k}{k!}$$

denotes the hypergeometric series and $(a)_k=a(a+1)\cdots(a+k-1)$ denotes the shifted factorial (Pochhammer symbol), as usual.

Every sequence of orthogonal polynomials satisfies a three-term recurrence relation of the form [14, Theorem 3.2.1]

$$p_n(x)=(A_nx+B_n)p_{n-1}(x)-C_np_{n-2}(x),\quad n\in\mathbb{N}_{\geqq 1} \tag{1.1}$$

where A_n, B_n and C_n do not depend on x, $p_{-1}\equiv 0$, $p_0(x)=1$ and $A_n, C_n>0$. An important consequence of (1.1) is that for each $n\in\mathbb{N}_{\geqq 1}$, the polynomial p_n has exactly n real, simple zeros in (a,b) [14, Theorem 3.3.1]. Furthermore, each open interval with endpoints at successive zeros of p_n contains exactly one zero of p_{n-1}. Stieltjes [14, Theorem 3.3.3] extended this interlacing property by proving that if $m<n-1$, provided p_m and p_n are co-prime (i.e., they do not have any common zeros), there exist m open intervals with endpoints at successive zeros of p_n, each of which contains exactly one zero of p_m. Beardon [2, Theorem 5] provided additional insight into the Stieltjes interlacing process by proving that for every $m<n-1$, if p_m and p_n are co-prime, there exists a real polynomial S_{n-m} of degree $n-m-1$ in x, whose real simple zeros, together with those of p_m, interlace with the zeros of p_n, a phenomenon that will be called *completed Stieltjes interlacing,* of which a direct consequence is that the zeros of the polynomial S_{n-m} act as "inner" bounds for the extreme zeros (i.e., upper (lower) bounds for the smallest (largest) zeros) of the polynomial p_n.

The study of completed Stieltjes interlacing between polynomials of *different* orthogonal sequences, where the different sequences are obtained by integer shifts of the parameters of the appropriate polynomials, leads to even more precise inner bounds for the extreme zeros of the polynomials under consideration, as was done for the Gegenbauer, Laguerre and Jacobi polynomials [3], the Meixner and Krawtchouk polynomials [6] and the Pseudo-Jacobi polynomials in [7]. Mixed three-term recurrence relations satisfied by the polynomials under consideration and obtained from the connection between the appropriate polynomials, their hypergeometric representations, as well as contiguous function relations satisfied by these polynomials, are used to obtain these bounds and a Maple computer package [15] for computing contiguous relations of exclusively ${}_2F_1$ series is helpful in this regard.

In order to obtain similar identities for polynomial sequences that lie on the ${}_3F_2$ plane of the Askey scheme, different Maple routines are necessary. In this paper, we apply Zeilberger's algorithm (see [11], command `sumrecursion` of the Maple package `hsum.mpl` accompanying [11]) to generate the three-term recurrences of the ${}_3F_2$ hypergeometric families under consideration. Zeilberger's algorithm, however, is much more flexible as is shown in [11]. Similarly as the commands `sumdiffeq` ([11], Session 10.5) and `sumdiffrule` ([11], Session 10.7) are slight variations of `sumrecursion` by just changing the setting in the computation, one can write Maple routines to compute the desired identities such as (2.2). Some of the equations obtained in this way are very complicated symbolic expressions resulting in poor performance due to time or memory constraints.

In this paper, we provide inner bounds for the extreme zeros of different sequences of the Hahn, continuous Hahn and continuous dual Hahn polynomials. In the Hahn case, we compare the quality of our newly found bounds with results obtained in [12]. An intensive study on the location of the zeros of the Hahn polynomials is made in [13] and lower (upper) bounds for the smallest (largest) zeros of Hahn polynomials are provided in [1, 12]. Hahn polynomials play a role in stochastic processes in genetics [9], as well as in quantum mechanics [8] where Clebsch-Gordon coefficients are expressed as Hahn polynomials, and a generalisation of the continuous dual Hahn polynomials is associated with symmetric birth and death processes with quadratic rates [4].

The following result provides the conditions necessary for the mixed three-term recurrence relations to hold and will be used to prove our results.

Theorem 1.1. [6] *Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the (finite or infinite) interval (a,b) with respect to $d\mu(x) > 0$. Let $k \in \mathbb{N}_{\geqq 0}$ be fixed and suppose $\{g_{n,k}\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to $\sigma_k(x)d\mu(x) > 0$ on (a,b), where $\sigma_k(x)$ is a polynomial of degree k, that satisfies*

$$(x - B_n)p_{n-1}(x) = a_{k-2}(x)p_n(x) + A_n\sigma_k(x)g_{n-2,k}(x),\ n \in \mathbb{N}_{\geqq 1}, \tag{1.2}$$

with $g_{-1,k} = 0$, A_n, B_n, a_{-1}, a_{-2} constants and a_{k-2} a polynomial of degree $k-2$ defined on (a,b) whenever $k \in \{2,3,\dots\}$. Then

- (i) *$k \in \{0,1,2,3,4\}$;*
- (ii) *the $n-1$ real, simple zeros of $(x-B_n)g_{n-2,k}$ interlace with the zeros of p_n and B_n is an upper bound for the smallest, as well as a lower bound for the largest zero of p_n if $g_{n-2,k}$ and p_n are co-prime;*
- (iii) *if $g_{n-2,k}$ and p_n are not co-prime,*
  - (a) *they have one common zero that is equal to B_n and this common zero cannot be the largest or smallest zero of p_n;*
  - (b) *the $n-2$ zeros of $g_{n-2,k}(x)$ interlace with the $n-1$ non-common zeros of p_n;*
  - (c) *B_n is an upper bound for the smallest as well as a lower bound for the largest zero of p_n.*

All relevant contiguous relations that we need in the next sections were computed automatically using the Maple package `hsum.mpl` accompanying [11] and procedures that are specifically adapted for each family using the corresponding hypergeometric representation. These computations with their complete results can be downloaded from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen`. In the given article, however, we have only included the necessary bounds deduced from the much more complicated contiguous relations since the full relations do not contribute to our results.

2. Inner bounds for the extreme zeros of Hahn polynomials

For $n \in \{0,1,2,\dots,N\}$, the Hahn polynomials are orthogonal for $\alpha,\beta > -1$ with respect to the discrete weight $w(x) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$ at $x \in \{0,1,2,\dots,N\}$ (cf. [10]).

For $\alpha,\beta > -1$, the parameter shifted Hahn polynomials $Q_n(x;\alpha+k,\beta+m,N)$ are orthogonal at the points $x \in \{0,1,2,\dots,N\}$ with respect to

$$\binom{\alpha+k+x}{x}\binom{\beta+m+N-x}{N-x} = (x+\alpha+1)_k(-x+\beta+N+1)_m\omega(x) = \sigma_{k,m}(x)w(x) > 0$$

and together with $Q_n(x;\alpha,\beta,N)$, they satisfy the mixed three-term recurrence relations

$$\begin{aligned}(x - B_n^{\alpha,\beta}(k,m))&Q_{n-1}(x;\alpha,\beta,N)\\ &= a_{k+m-2}(x)Q_n(x;\alpha,\beta,N)(x) + A_n\sigma_{k,m}(x)Q_{n-2}(x;\alpha+k,\beta+m,N),\end{aligned} \tag{2.1}$$

where $a_0(x)$, $a_1(x)$, A_n and $B_n^{\alpha,\beta}(k,m)$ are constants and $a_{k+m-2}(x)$ is a polynomial of degree $k+m-2$ for $k+m \in \{2,3,\dots\}$.

From Theorem 1.1 it follows that the mixed three-term recurrence relations (2.1) only exist if $k+m \in \{0,1,2,3,4\}$ and each of the points $B_n^{\alpha,\beta}(k,m)$, such that $k+m \in \{0,1,2,3,4\}$ is an upper bound for the smallest as well as a lower bound for the largest zero of $Q_n(x;\alpha,\beta,N)$, moreover, the relations that involve the largest possible parameter difference, are found to be particularly useful to obtain sharp bounds.

When $k = 4$ and $m = 0$, we obtain the relation

$$u_1(x)Q_{n-1}(x;\alpha,\beta,N) = a_2(x)Q_n(x;\alpha,\beta,N) + a_n(x+\alpha+1)_4\,Q_{n-2}(x;\alpha+4,\beta,N), \tag{2.2}$$

where $a_2(x)$ is a polynomial with 2 real zeros,

$$a_n = (n-1)(\beta+n-1)(\alpha+n+1)_2(\alpha+\beta+n+1)_2(\alpha+\beta+2n)$$

and u_1 is a polynomial of degree 1 in x, such that $u_1(B_n^{\alpha,\beta}(4,0)) = 0$ for

$$\begin{aligned}
B_n^{\alpha,\beta}(4,0) \;=\; & \Big(n^6 + (3\,\alpha+3\,\beta+3)\,n^5 + (N\,\alpha+3\,\alpha^2+8\,\alpha\,\beta+3\,\beta^2+N+4\,\alpha+8\,\beta-1)\,n^4\\
+\; & (\alpha+\beta+1)\,(2\,N\,\alpha+\alpha^2+6\,\alpha\,\beta+\beta^2+2\,N-2\,\alpha+6\,\beta-7)\,n^3\\
+\; & (N^2\,\alpha^2+N\,\alpha^3+4\,N\,\alpha^2\,\beta+N\,\alpha\,\beta^2-\alpha^4+2\,\alpha^3\,\beta+5\,\alpha^2\,\beta^2+2\,\alpha\,\beta^3\\
+\; & 3\,N^2\,\alpha+N\,\alpha^2+9\,N\,\alpha\,\beta+N\,\beta^2-12\,\alpha^3-\alpha^2\,\beta+7\,\alpha\,\beta^2+2\,\beta^3\\
+\; & 2\,N^2-3\,N\,\alpha+5\,N\,\beta-42\,\alpha^2-15\,\alpha\,\beta-\beta^2-3\,N-56\,\alpha-12\,\beta-24)\,n^2\\
+\; & (\alpha+\beta+1)(N^2\,\alpha^2+2\,N\,\alpha^2\,\beta-\alpha^4+\alpha^2\,\beta^2+3\,N^2\,\alpha-2\,N\,\alpha^2+5\,N\,\alpha\,\beta\\
-\; & 10\,\alpha^3-3\,\alpha^2\,\beta+\alpha\,\beta^2+2\,N^2-6\,N\,\alpha+3\,N\,\beta-34\,\alpha^2-9\,\alpha\,\beta-\beta^2-4\,N\\
-\; & 46\,\alpha-6\,\beta-20)\,n+(\alpha+1)_2(N\,\alpha+3\,N+\alpha-\beta+4)\,(\alpha+2+N+\beta)_2\Big)\Big/\\
& \Big((2\,N+3\,\alpha+\beta+8)\,n^4+2\,(\alpha+\beta+1)\,(2\,N+3\,\alpha+\beta+8)\,n^3\\
+\; & (2\,N^2\,\alpha+8\,N\,\alpha^2+8\,N\,\alpha\,\beta+2\,N\,\beta^2+7\,\alpha^3+13\,\alpha^2\,\beta+7\,\alpha\,\beta^2+\beta^3+4\,N^2\\
+\; & 30\,N\,\alpha+10\,N\,\beta+41\,\alpha^2+48\,\alpha\,\beta+13\,\beta^2+30\,N+79\,\alpha+39\,\beta+52)\,n^2\\
+\; & (\alpha+\beta+1)(2\,N^2\,\alpha+6\,N\,\alpha^2+4\,N\,\alpha\,\beta+4\,\alpha^3+6\,\alpha^2\,\beta+2\,\alpha\,\beta^2+4\,N^2\\
+\; & 26\,N\,\alpha+6\,N\,\beta+27\,\alpha^2+24\,\alpha\,\beta+3\,\beta^2+28\,N+60\,\alpha+22\,\beta+44)\,n\\
+\; & (\alpha+1)_2\,(\alpha+\beta+2)\,(\alpha+2+N+\beta)_2\Big).
\end{aligned} \tag{2.3}$$

The weight function w satisfies the symmetry property $w(\alpha,\beta,x) = w(\beta,\alpha,N-x)$ from which the symmetry relation

$$(\alpha+n)_n Q_n(N-x;\alpha,\beta,N) = (-1)^n(\beta+n)_n Q_n(x;\beta,\alpha,N)$$

can be proved (cf. [9]) and we can deduce that when x is a zero of $Q_n(x;\alpha,\beta,N)$, then $N-x$ will be a zero of $Q_n(x;\beta,\alpha,N)$. Likewise, the extra interlacing points obtained from the mixed three-term recurrence relations satisfied by $Q_n(x;\alpha,\beta,N), Q_{n-1}(x;\alpha,\beta,N)$ and $Q_{n-2}(x;\alpha+m,\beta+k,N)$, are

$$B_n^{\alpha,\beta}(m,k) = N - B_n^{\beta,\alpha}(k,m) \tag{2.4}$$

for all values of k and m in $\mathbb{N}_{\geqq 0}$ such that $k+m \in \{0,1,2,3,4\}$.

Letting $k=0$ and $m=4$ in (2.1), we obtain

(2.5)

$$v_1(x)Q_{n-1}(x;\alpha,\beta,N) = a_2(x)Q_n(x;\alpha,\beta,N)+b_n(-x+\beta+N+1)_4\, Q_{n-2}(x;\alpha,\beta+4,N),$$

where $a_2(x)$ is a polynomial of degree 2, $b_n = (n-1)(\alpha+\beta+n+1)_2(\alpha+\beta+2n)$, $v_1(x)$ is a polynomial of degree 1 with zero $B_n^{\alpha,\beta}(0,4)$, obtained from (2.4), i.e.,

$$B_n^{\alpha,\beta}(0,4) = N - B_n^{\beta,\alpha}(4,0). \tag{2.6}$$

From [5, Theorem 7.1.1] it follows that the zeros of the Hahn polynomials increase with α and decrease with β which implies that the points $B_n^{\alpha,\beta}(4,0)$ and $B_n^{\alpha,\beta}(0,4)$ satisfy

$$0 < x_{n,1} < B_n^{\alpha,\beta}(4,0) < B_n^{\alpha,\beta}(0,4) < x_{n,n} < N,$$

for all values of $\alpha,\beta > -1$ and $n \in \{0,1,2,\dots,N\}$ and we conclude that $B_n^{\alpha,\beta}(4,0)$ is the best upper bound for the smallest zero and $B_n^{\alpha,\beta}(0,4)$ is the most precise lower bound for the largest zero of $Q_n(x;\alpha,\beta,N)$ obtained by this method.

In [12, Lemma 9], the following inner bounds for the extreme zeros of the Hahn polynomials are provided for $\alpha \geqq \beta > -1$ or $\alpha \leqq \beta \leqq -N-1$:

$$x_{n,1} < \frac{(n+\alpha)(N-n+1)}{\alpha+\beta+1} \tag{2.7}$$

$$< \frac{N(\alpha+n)+(\beta+n)(n-1)}{\alpha+\beta+2n} < x_{n,n}. \tag{2.8}$$

In tables 1 to 4 we compare these bounds, together with the bounds obtained from relations (2.2) and (2.5), to the actual values of the extreme zeros. In each case, the more precise bound is printed in bold.

TABLE 1. Comparison of bounds for the extreme zeros of $Q_5(x;10,2,N)$ for different values of N.

N	$x_{5,1}$	$B_5(4,0)$ in (2.3)	Bound in (2.7)	Bound in (2.8)	$B_5(0,4)$ in (2.6)	$x_{5,5}$
5	0.1659	1.4108	**0.6818**	**4.6818**	2.7262	4.9975
10	1.5604	**3.4837**	4.0909	**8.0909**	7.8550	9.9130
50	15.8455	**20.4292**	31.3636	35.3636	**46.1970**	47.8746
100	34.2895	**41.7837**	65.4545	69.4545	**93.0772**	94.9150
500	182.5365	**212.9863**	338.1820	342.1820	**466.2995**	470.6930

TABLE 2. Comparison of bounds for the extreme zeros of $Q_5(x;\alpha,2,30)$ for different values of α.

α	$x_{5,1}$	$B_5(4,0)$ in (2.3)	Bound in (2.7)	Bound in (2.8)	$B_5(0,4)$ in (2.6)	$x_{5,5}$
-0.5	0.2966	0.7131	n/a	n/a	25.0283	26.3038
5	5.0673	**7.5314**	15.2941	19.2941	**26.7850**	28.3414
10	8.5443	**11.9184**	17.7273	21.7273	**27.2727**	29.0000
50	18.3546	**21.4496**	23.0645	27.0645	**27.3696**	29.9025
200	23.2194	**24.7356**	25.1415	**29.1415**	26.6385	29.9985

TABLE 3. Comparison of bounds for the extreme zeros of $Q_5(x;10.5,\beta,30)$ for different values of β.

β	$x_{5,1}$	$B_5(4,0)$ in (2.3)	Bound in (2.8)	Bound in (2.7)	$B_5(0,4)$ in (2.6)	$x_{5,5}$
-0.5	11.2191	**14.0779**	20.1500	24.1500	**29.0938**	29.9141
5	6.9636	**10.7099**	15.8039	19.8039	**24.8851**	27.6465
10	5.0265	**9.0092**	13.2131	17.2131	**21.4190**	25.3339
50	1.0722	5.2996	n/a	n/a	10.9749	15.6847
200	0.0657	4.2096	n/a	n/a	6.1001	8.8014

TABLE 4. Comparison of bounds for the extreme zeros of $Q_{100}(x;3,-0.5,N)$ for different values of N.

N	$x_{100,1}$	$B_{100}(4,0)$ in (2.3)	$B_{100}(0,4)$ in (2.6)	$x_{100,100}$
1 000	0.0361	5.8021	995.2478	999.9999
10 000	7.9491	11.6801	9 999.1817	9 999.6205
100 000	96.2846	115.4888	99 994.0984	99 994.2871
500 000	489.3568	578.8144	499 969.9708	499 970.4528

3. INNER BOUNDS FOR EXTREME ZEROS OF CONTINUOUS HAHN POLYNOMIALS

Let $n \in \mathbb{N}_{\geqq 0}$. The continuous Hahn polynomials are orthogonal on $\mathbb{R}$ with respect to the weight function

$$w(a,b,c,d,x) = \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)$$

if the real parts of a, b, c and d are positive and $c = \bar{a}$, $d = \bar{b}$ (cf. [10]).

The conditions necessary to obtain orthogonality force us to simultaneously shift both parameters a and c, as well as b and d and the parameter shifted polynomial $p_n(x; a+k, b+m, c+k, d+m)$, which is orthogonal on $\mathbb{R}$ with respect to

$$\begin{aligned}\Gamma(a+k+ix)&\Gamma(b+m+ix)\Gamma(c+k-ix)\Gamma(d+m-ix)\\ &= (a+ix)_k(b+ix)_m(c-ix)_k(d-ix)_m\, w(a,b,c,d,x)\\ &= \sigma_{k,m}(x)\, w(a,b,c,d,x) > 0,\end{aligned}$$

together with the polynomial $p_n(x;a,b,c,d)$, satisfy the mixed three-term recurrence relations

$$\begin{aligned}&(x - B_n(k,m))p_{n-1}(x;a,b,c,d)\\ &= a_{k+m-2}(x)p_n(x;a,b,c,d) - d_n\sigma_{k,m}(x)p_{n-2}(x;a+k,b+m,c+k,d+m),\end{aligned}$$

where $a_{k+m-2}(x)$ is a polynomial of degree $k+m-2$ when $k+m \in \{2,3,\dots\}$, and $a_0,\ a_1,\ d_n$ and $B_n(k,m)$ are constants. From Theorem 1.1 we deduce that each point

$B_n(k,m)$ such that $k+m \in \{0,1,2,3,4\}$ will be an upper (lower) bound for the smallest (largest) zero of $p_n(x;a,b,c,d)$.

The mixed three-term recurrence relation that involves the largest parameter shift is

$$\begin{aligned}&(x-B_n(0,2))p_{n-1}(x;a,b,c,d)\\ &= \frac{a_2(x)}{n(b-d)(a+b+c+d+n-1)+(b+d)(ab+a-b-cd-c+d)}p_n(x;a,b,c,d)\\ &-\frac{(a+b+c+d+n-1)_2(a+b+c+d+2n-2)(a+c+n-2)}{n(b-d)(a+b+c+d+n-1)+(b+d)(ab+a-b-cd-c+d)}\\ &\qquad\times\sigma_{0,2}(x)p_{n-2}(x;a,b+2,c,d+2)\end{aligned}$$

and, when $a=p+iq=\bar{c}$ and $b=r+is=\bar{d}$ where $p,q,r,s\in\mathbb{R}$, and $p,r>0$, we have

$$\sigma_{0,2}(x)=(b+ix)_2(d-ix)_2=r^2+(r+1)^2+2(s+x)^2>0,$$

$$\begin{aligned}B_n(0,2) &= i\,\frac{n(b-d)(a+b+c+d+n-1)+(b+d)(ab+a-b-cd-c+d)}{2n(a+b+c+d+n-1)+(b+d)(a+b+c+d)}\\ &= -\frac{2ps(n+r)+s(n-1)(n+2r)+2qr(r+1)}{n(n+2p+2r-1)+2r(p+r)} \qquad (3.1)\end{aligned}$$

and $a_2(x)$ is a polynomial of degree 2.

Furthermore, from the fact that $w(a,b,c,d,x) = w(b,a,d,c,x)$ and consequently $p_n(x;a,b,c,d) = p_n(x;b,a,d,c)$, shifting the parameters a and c yields the bound

$$B_n(2,0) = -\frac{2qr(n+p)+q(n-1)(n+2p)+2ps(p+1)}{n(n+2p+2r-1)+2p(p+r)}. \qquad (3.2)$$

It is easy to prove that $B_n(0,2) \leqq B_n(2,0)$ for all $n \geqq 2$ when $q \leqq s$ (i.e., Im $(a) \leqq$ Im (b)) and thus, applying Theorem 1.1, the point $B_n(0,2)$ in (3.1) will be the most precise upper bound for the smallest zero and $B_n(2,0)$ in (3.2) will be the best lower bound for the largest zero of $p_n(x;a,b,c,d)$ obtained by this method, provided that Im $(a) \leqq$ Im (b). The order of these bounds will reverse when Im $(a) \geqq$ Im (b).

Furthermore, we observe that

$$\lim_{n\to\infty} B_n(2,0) = -\text{Im}\,(a) \text{ and } \lim_{n\to\infty} B_n(0,2) = -\text{Im}\,(b)$$

and the bounds are thus less sharp for larger values of n, and, when Im (a) = Im (b), $B_n(2,0) = B_n(0,2) = -$Im (a) for all values of n. We refer the reader to Table 5 for examples that illustrate these results.

TABLE 5. Comparison of bounds for the extreme zeros of $p_n(x;a,b,c,d)$, Im $(a) \leqq$ Im (b), $c=\bar{a}$, $d=\bar{b}$, for different values of a,b and n.

n	a	b	$x_{n,1}$	$B_n(0,2)$ from (3.1)	$B_n(2,0)$ from (3.2)	$x_{n,n}$
5	$1-20i$	$3-20i$	17.9415	20	20	22.0598
5	$1-20i$	$1+155i$	-140.1610	-139.0909	4.0909	5.1612
5	$1-15i$	$1+15i$	-12.5501	-12.2727	12.2727	12.5501
5	$10+i$	$1+15i$	-15.4565	-14.6316	-9.8000	-7.8590
35	$1-i$	$2+0.2i$	-15.3222	-0.1898	0.9966	12.5501
35	$1+0.9i$	$1+i$	-16.1997	-0.9997	-0.9003	14.3098

4. Inner bounds for the extreme zeros of Continuous Dual Hahn polynomials

Consider the Continuous Dual Hahn polynomials

$$\tilde{S}_n(x^2; a, b, c) = \frac{S_n(x^2; a, b, c)}{(a+b)_n (a+c)_n}$$

which are orthogonal in the interval $(0; \infty)$ with respect to the continuous weight function

$$w(x; a, b, c) = \left| \frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)} \right|^2.$$

We are interested in inner bounds for the extreme zeros of the polynomials $\tilde{S}_n(x^2; a, b, c)$ and from the three-term recurrence relation satisfied by the Continuous Dual Hahn polynomials (cf. [10, eqn. (9.3.4)]), we obtain the point

$$B_n(0,0,0) = (a+b+n-1)(a+c+n-1) + (n-1)(b+c+n-2) - a^2 \tag{4.1}$$

which is an upper (lower) bound for the smallest (largest) zero of these polynomials.

The parameter shifted Continuous Dual Hahn polynomials $\tilde{S}_n(x^2; a+k, b+l, c+m)$, are orthogonal in the interval $(0; \infty)$ with respect to $\sigma_{k,l,m}(x^2) w(x; a, b, c)$, where

$$\sigma_{k,l,m}(x^2) = |(a+ix)_k (b+ix)_l (c+ix)_m|^2 \tag{4.2}$$

and satisfy the mixed three-term recurrence relations

$$\begin{aligned}(x^2 &- B_n(k,l,m))\tilde{S}_{n-1}(x^2; a, b, c) \\ &= a_{k+l+m-2}(x^2)\tilde{S}_n(x^2; a, b, c) - d_n \sigma_{k,l,m}(x^2)\tilde{S}_{n-2}(x^2; a+k, b+l, c+m),\end{aligned}$$

where $a_{k+l+m-2}$ is a polynomial of degree $k+l+m-2$ in x^2 when $k+l+m \in \{2, 3, \dots\}$, and a_0, a_1, d_n and $B_n(k,l,m)$ are constants. From Theorem 1.1 we deduce that each point $B_n(k,l,m)$ such that $k+l+m \in \{0,1,2,3,4\}$ will be an upper (lower) bound for the smallest (largest) zero of $\tilde{S}_n(x^2; a, b, c)$.

The mixed three-term recurrence relations that provide us with relatively good bounds are those that involve a total parameter shift of 4 units, i.e., when we shift

(i) one parameter by 4 units;
(ii) two of the parameters by 2 units each;
(iii) one parameter by 3 units and another one by 1 unit;
(iv) two parameters by 1 unit and the third one by 2 units.

Neither the weight function, nor the zeros of the polynomial $\tilde{S}_n(x^2; a, b, c)$ depend on the order in which the parameters a, b and c occur and shifting a by 4 units leads to exactly the same bound as shifting b or c by 4 units, provided that the parameter with the smallest numerical value is the one that is shifted. We list the three relations that involve the parameter shifts mentioned in $(i) - (iii)$ above, as numerical evidence confirms that these lead to the best possible upper bounds for the smallest zeros of the Continuous Dual Hahn polynomials.

The relation obtained when we shift a by 4 units is:

$$\begin{aligned}&a_2(x^2)\tilde{S}_n(x^2; a, b, c) = u_1(x^2)\tilde{S}_{n-1}(x^2; a, b, c) \\ &- \frac{(n-1)(b+c+n-2)(a+b+n)_2(a+c+n)_2\sigma_{4,0,0}(x^2)}{(a+c)_4(a+b)_4}\tilde{S}_{n-2}(x^2; a+4, b, c),\end{aligned}$$

where, from (4.2),

$$\sigma_{4,0,0}(x^2) = (a^2+x^2)((a+1)^2+x^2)((a+2)^2+x^2)((a+3)^2+x^2),$$

a_2 is a polynomial of degree 2 in x^2 and $u_1\big(B_n(4,0,0)\big)=0$ for

$$\begin{aligned}
B_n(4,0,0) \;=\;& \Big(5\,a^2+9\,a^4+12\,a^3+2\,a^5+5\,ab+5\,ac+5\,bc+3\,ab^2+3\,ac^2+3\,b^2c\\
+\;& 3\,bc^2-2\,a^4b^2-2\,a^4c^2-5\,a^3b^2-5\,a^3c^2-3\,a^2b^3-3\,a^2c^3-b^3c^3-a^5b\\
-\;& a^5c-a^3b^3-a^3c^3+\big(-4\,a^3-18\,a^2-22\,a-6\big)\,n^3+(-6\,a^4-6\,a^3b\\
-\;& 6\,a^3c-10\,a^2bc-24\,a^3-22\,a^2b-22\,a^2c-30\,abc-22\,a^2-18\,ab-18\,ac\\
-\;& 18\,b\,c\,)\,n^2+(-2\,a^5-6\,a^4b-6\,a^4c-4\,a^3b^2-16\,a^3bc-4\,a^3c^2-10\,a^2b^2c\\
-\;& 10\,a^2bc^2-6\,ab^2c^2-3\,a^4-18\,a^3b-18\,a^3c-13\,a^2b^2-42\,a^2bc-13\,a^2c^2\\
-\;& 24\,ab^2c-24\,abc^2-9\,b^2c^2+16\,a^3-5\,a^2b-5\,a^2c-9\,ab^2-12\,abc-9\,ac^2\\
-\;& 9\,b^2c-9\,bc^2+35\,a^2+9\,ab+9\,ac+9\,bc+22\,a+6\,)\,n-3\,b^3c^2-3\,b^2c^3\\
-\;& 2\,ab^3-2\,ac^3-2\,b^3c-2\,bc^3+15\,a^2b+15\,a^2c+11\,a^3b+11\,a^3c-6\,ab^3c\\
-\;& 6\,abc^3+13\,a^2bc-3\,ab^3c^2-8\,a^3bc-5\,a^4bc-3\,ab^2c^3-8\,a^2b^2c^2-12\,ab^2c^2\\
-\;& 7\,a^3bc^2-3\,a^2b^3c+18\,abc-14\,a^2bc^2-3\,a^2bc^3-14\,a^2b^2c-7\,a^3b^2c\Big)\Big/\\
& \Big((-4\,a-6)\,n^3+\big(4-6\,a^2-6\,ab-6\,ac-2\,bc-6\,a-8\,b-8\,c\big)\,n^2\\
+\;& (-4\,a^3-6\,a^2b\,-6\,a^2c-2\,ab^2-8\,abc-2\,ac^2-2\,b^2c-2\,bc^2\\
-\;& 6\,a^2-6\,ab-6\,ac-2\,b^2-6\,bc-2\,c^2-2\,a+2\,b+2\,c-2\,)\,n-a^4-2\,a^3b\\
-\;& 2\,a^3c-a^2b^2-4\,a^2bc-a^2c^2-2\,ab^2c-2\,a\,b\,c^2-b^2c^2-2\,a^3\\
-\;& 3\,a^2b-3\,a^2c-ab^2-4\,abc-ac^2-b^2c-bc^2-a^2-ab-ac-bc\Big).
\end{aligned}\tag{4.3}$$

Shifting both a and b by 2 units leads to the mixed three-term recurrence relation

$$\begin{aligned}
\frac{a_2(x^2)}{a+b+2n}\tilde S_n(x^2;a,b,c) &= -\big(x^2-B_n(2,2,0)\big)(a+b+1)\tilde S_{n-1}(x^2;a,b,c)\\
&\quad -\frac{(n-1)(a+b+n)_2}{(a+b+2n)(a+b)_4(a+c)_2}\,\sigma_{2,2,0}(x^2)\tilde S_{n-2}(x^2;a+2,b+2,c),
\end{aligned}$$

where $\sigma_{2,2,0}(x^2)=(a^2+x^2)((a+1)^2+x^2)(b^2+x^2)((b+1)^2+x^2)$, a_2 is a polynomial of degree 2 in x^2 and

$$B_n(2,2,0)=\frac{a^2(b+c)+b^2(a+c)+2ab(c+n)+(a+b)(2c+n-1)}{a+b+2n}. \tag{4.4}$$

By shifting a by 3 units and b by 1, we obtain

$$\begin{aligned}
a_2(x^2)\tilde S_n(x^2;a,b,c) &= v_1(x^2)\tilde S_{n-1}(x^2;a,b,c)\\
&\quad -\frac{(n-1)(a+b+n)_2(a+c+n)}{(a+c)_3(a+b)_4}\,\sigma_{3,1,0}(x^2)\tilde S_{n-2}(x^2;a+3,b+1,c),
\end{aligned}$$

where a_2 is a polynomial of degree 2 in x^2 and $v_1(B_n(3,1,0))=0$ for

$$\begin{aligned}
B_n(3,1,0) \;=\;& \Big(\big(a^3+3\,a^2b+3\,a^2+6\,ab+2\,a+2\,b\big)\,n^2\\
+\;& (a^4+4\,a^3b+2\,a^3c+3\,a^2b^2+6\,a^2bc+4\,ab^2c+2\,a^3+6\,a^2b+6\,a^2c\\
+\;& 4\,ab^2+12\,abc+4\,b^2c-a^2-2\,ab+4\,ac+4\,bc-2\,a-2\,b)n\\
+\;& (a+c)\,(a+b)_2\,(ab+ac+bc-a+b+2\,c-2)\Big)\Big/\\
& \Big((3\,a+b+3)\,n^2+\big(3\,a^2+4\,ab+2\,ac+b^2+2\,bc+2\,a+2\,b+2\,c-1\big)\,n\\
+\;& (a+c)\,(a+b)_2\Big).
\end{aligned}\tag{4.5}$$

In Table 6 we provide some examples that illustrate the quality of these bounds. The best upper bound for the smallest zero obtained in each case is printed in bold. Furthermore, when only one parameter is shifted, we shift the smallest one and in the cases where 2 parameters are shifted, we shift the 2 smallest parameters and the parameter with the smallest numerical value is the one that is shifted optimally. The point $B_n(0,0,0)$ in (4.1) is in each case the best lower bound for the largest zero of these polynomials.

TABLE 6. Comparison of bounds for the extreme zeros of $S_n(x^2; a, b, c)$ for different values of a, b, c and n. $B_n(4,0,0), B_n(2,2,0), B_n(3,1,0)$ and $B_n(0,0,0)$ are the bounds in (4.3), (4.4), (4.5) and (4.1) respectively.

n	$\{a,b,c\}$	$x_{n,1}^2$	$B_n(4,0,0)$	$B_n(2,2,0)$	$B_n(3,1,0)$	$B_n(0,0,0)$	$x_{n,n}^2$
6	$\{7,7,7\}$	63.91	120.676	**112.000**	114.065	402	581.83
6	$\{7,8,9\}$	85.53	148.143	**143.778**	**143.778**	476	690.30
6	$\{1,19,40\}$	389.85	**504.834**	572.125	533.443	1464	2147.23
6	$\{7,8,40\}$	312.91	440.851	**436.556**	**436.556**	1251	1828.49
6	$\{7,39,40\}$	1204.09	**1613.998**	1799.724	1697.511	3018	4285.71
31	$\{7,8,9\}$	29.34	98.629	**91.649**	**91.649**	3401	5829.19
31	$\{1,19,40\}$	114.82	**157.852**	240.951	186.358	6189	10788.25

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