ON THE FEKETE-SZEGÖ PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS
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ABSTRACT. Let $S$ be the familiar class of normalized univalent functions in the unit disk. Fekete and Szegö proved the well-known result
\[ \max_{f \in S} |a_3 - \lambda a_2^2| = 1 + 2e^{-2\lambda/(1-\lambda)} \]
for $\lambda \in [0, 1]$. We consider the corresponding problem for the family $C$ of close-to-convex functions and get
\[ \max_{f \in C} |a_3 - \lambda a_2^2| = \begin{cases} 3 - 4\lambda & \text{if } \lambda \in [0, 1/3], \\ 1/3 + 4/(9\lambda) & \text{if } \lambda \in [1/3, 2/3], \\ 1 & \text{if } \lambda \in [2/3, 1]. \end{cases} \]
As an application it is shown that $||a_3| - |a_2|| \leq 1$ for close-to-convex functions, in contrast to the result in $S$
\[ \max_{f \in S} |a_3| - |a_2| = 1.029\ldots . \]

1. Introduction. Let $S$ denote the family of univalent functions $f$ of the unit disk, normalized by
\[ f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \]
Let $St$ denote the subset of starlike functions, i.e. functions that have a starlike range with respect to the origin. Further let $C$ denote the family of close-to-convex functions, which have been introduced by Kaplan [4]. A function $f$, normalized by (1), is called close-to-convex if there exist a starlike function $g$ and a real number $\alpha$, such that
\[ \Re(\exp(\alpha z) f'(z)/g(z)) > 0, \quad \alpha \in ] - \pi/2, \pi/2[, \]
It turns out that a function is close-to-convex if and only if it maps the unit disk univalently onto a domain whose complement is the union of half-lines, which are pairwise disjoint up to possibly equal tips (see [6-7, 1]).
A well-known function of this kind is the Koebe function $k$ with
\[ k(z) = \sum_{n=1}^{\infty} n z^n = \frac{z}{(1 - z)^2} = \frac{1}{4} \left( \left( \frac{1 + z}{1 - z} \right)^2 - 1 \right) , \]
which maps the unit disk onto the complement of the half-line $]-\infty, -1/4]$, as the last representation shows.
Many extremal problems within the class $S$ are solved by the Koebe function. On the other hand, the Koebe function satisfies
\[ |a_3 - \lambda a_2^2| = |3 - 4\lambda| , \]
whereas Fekete and Szegö showed [3]

$$\max_{f \in S} |a_3 - \lambda a_2^2| = 1 + 2e^{-2\lambda/(1-\lambda)}$$

for $\lambda \in [0, 1]$.

For $\lambda = 0, 1$ the Koebe function gives the maximum, but there is no $\lambda_0 \in [0, 1]$ such that the functional $|a_3 - \lambda_0 a_2^2|$ is maximized by $k$. We shall show that

$$\max_{f \in C} |a_3 - \lambda a_2^2| = 3 - 4\lambda$$

for $\lambda \in [0, 1/3]$, so that for close-to-convex functions the situation is quite different. This result implies furthermore that

$$\max_{f \in C} |a_3| - |a_2| = 1,$$

in contrast to the known estimate in $S$,

$$\max_{f \in S} |a_3| - |a_2| = 1.029 \ldots$$

(see e.g. [2, Theorem 3.11]). Moreover we show that

$$\max_{f \in C} |a_3 - \lambda a_2^2| = \begin{cases} 1/3 + 4/(9\lambda) & \text{if } \lambda \in [1/3, 2/3], \\ 1 & \text{if } \lambda \in [2/3, 1]. \end{cases}$$

2. Preliminary results. Here we give some lemmas which will be used in the next section to solve the main problem.

Recall that a function $f$ is called close-to-convex of order $\beta$ if there exist a starlike function $g$ and a real number $\alpha$, such that

$$|\arg(e^{i\alpha}zf'(z)/g(z))| < \beta \pi/2.$$

**Lemma 1** (see [5, Lemma 1]). Let $f \in C$. Then the function $h$, defined by

$$h'(z) = (f'(z^2))^{1/2}, \quad h(0) = 0,$$

is an odd close-to-convex function of order $1/2$.

**Lemma 2** (see [8, p. 166, formula (10)]). Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $\text{Re} \, p > 0$. Then

$$|p_2 - p_1^2/2| \leq 2 - |p_1|^2/2.$$

**Lemma 3.** Let $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in S$. Then

$$|b_3 - \lambda b_2^2| \leq \max\{1, |3 - 4\lambda|\}$$

which is sharp for the Koebe function $k$ if $|\lambda - 3/4| \geq 1/4$ and for $(k(z^2))^{1/2} = z/(1 - z^2)$ if $|\lambda - 3/4| \leq 1/4$.

**Proof.** Because $g \in S$, the function

$$zg'(z)/g(z) = 1 + b_2 z + (2b_3 - b_2^2) z^2 + \cdots = 1 + p_1 z(3) + p_2 z^2 + \cdots$$

has positive real part, so that $|p_2 - 1/2 p_1^2| \leq 2 - |p_1|^2/2$ by Lemma 2. Let now $\lambda \in C$. Then by (3) we have

$$|b_3 - \lambda b_2^2| = \frac{1}{2} |p_2 - (1 - 2\lambda)p_1^2| = \frac{1}{2} |p_2 - \frac{1}{2} p_1^2 + (\frac{3}{2} - 2\lambda)p_1^2|$$

$$\leq \frac{1}{2} \left(2 - \frac{1}{2} |p_1|^2 + |\frac{3}{2} - 2\lambda| |p_1|^2\right)^2.$$
If now $|\lambda - 3/4| \leq \frac{1}{4}$, then
$$|b_3 - \lambda b_2^2| \leq \frac{1}{2} \left( 2 - \frac{1}{2}|p_1|^2 + \frac{1}{2}|p_1|^2 \right) = 1.$$  
On the other hand, if $|\lambda - 3/4| \geq \frac{1}{4}$, then we use $|p_1| \leq 2$ (see e.g. [8, Corollary 2.3]), and get
$$|b_3 - \lambda b_2^2| \leq 1 + \frac{1}{2} \left( \left| \frac{3}{2} - 2\lambda \right| - \frac{1}{2} \right) |p_1|^2$$
$$\leq 1 + |3 - 4\lambda| - 1 = |3 - 4\lambda|.$$

3. **Main results.** The first step of the solution of the Fekete-Szegö problem for close-to-convex functions is the special case $\lambda = 1/3$.

**Theorem 1.** Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C$. Then $|a_3 - \lambda a_2^2| \leq \frac{5}{3}$.

**Proof.** Let $f \in C$. Then by Lemma 1 the function $h$, defined by (2), is an odd close-to-convex function of order $1/2$.

For such functions, the author gave sharp bounds on the coefficients (see [5, Theorem 1]), in particular, the fifth coefficient of $h$ is bounded in modulus by $1/2$. On the other hand the fifth coefficient of $h$ is given by $\frac{3}{10}(a_3 - \frac{1}{3}a_2^2)$, which implies the result.

The next corollary follows easily from the theorem using $|a_2| \leq 2$ (see e.g. [2, Theorem 2.2]).

**Corollary 1.** Let $A \in [0,1/3]$. Then
$$\max_{f \in C} |a_3 - Aa_2^2| = 3 - 4A.$$  
The maximum is attained by the Koebe function.

Another consequence of the theorem is the following result about successive coefficients of close-to-convex functions.

**Corollary 2.** Let $f \in C$. Then $|a_3| - |a_2| \leq 1$.

**Proof.** It is well known that $|a_2| - |a_3| \leq 1$ for all $f \in S$ (see e.g. [2, Theorem 3.11]). Moreover, if $|a_2| \leq 1$, then also $|a_3| - |a_2| \leq 1$ (see e.g. [2, proof of Theorem 3.11]). Now let $f \in C$ and $|a_2| \in [1,2]$. Then Theorem 1 implies that
$$|a_3| - |a_2| \leq |a_3 - \frac{1}{3}a_2^2| + \frac{1}{2}|a_2|^2 - |a_2|$$
$$\leq \frac{5}{3} + \frac{1}{2}|a_2|^2 - |a_2| \leq 1,$$
as $|a_2|$ is in the above range.

The following notation will be used throughout the paper. For $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in C$ there is a representation of the form

$$f'(z) = \frac{g(z)}{z} \cdot \tilde{p}(z)$$

with some function $g(z) = z + b_2z^2 + b_3z^3 + \cdots \in S_{\infty}$ and some function $\tilde{p}(z) = 1 + \tilde{p}_1z + \tilde{p}_2z^2 + \cdots$ such that $\text{Re}(e^{i\alpha} \tilde{p}(z)) > 0$, $\alpha \in -\pi/2, \pi/2]$. Then the function $p(z) = 1 + p_1z + p_2z^2 + \cdots$, defined by

$$\tilde{p}_n = \cos \alpha \cdot e^{-i\alpha} \cdot p_n, \quad n \in \mathbb{N},$$
has positive real part. Comparing coefficients in (4) we get
$$3a_3 = b_3 + \tilde{p}_1b_2 + \tilde{p}_2, \quad 2a_2 = b_2 + \tilde{p}_1,$$
so that

\( a_3 - \lambda a_2^2 = \frac{1}{3}(b_3 - \frac{3}{2}\lambda b_2^2) + \frac{1}{3}(\tilde{p}_2 - \frac{3}{4}\lambda \tilde{p}_1^2) + \tilde{p}_1 b_2 (\frac{1}{3} - \lambda/2). \)

Now we consider the case \( \lambda = 2/3. \)

**THEOREM 2.** Let \( f(z) = z = a_2 z^2 + a_3 z^3 + \cdots \in C. \) Then \( |a_3 - \frac{2}{3}a_2^2| \leq 1. \)

**PROOF.** From (6) it follows that

\[ |a_3 - \frac{2}{3}a_2^2| \leq \frac{1}{3}|b_3 - \frac{1}{2}b_2^2| + \frac{1}{3}|	ilde{p}_2 - \frac{1}{2}p_1^2|. \]

From (5) we get

\[ \tilde{p}_2 - \frac{1}{2}p_1^2 = \cos \alpha \cdot e^{-i\alpha}(p_2 - \frac{1}{2} \cos \alpha \cdot e^{-i\alpha}p_1^2) \]

\[ = \cos \alpha \cdot e^{-i\alpha}(p_2 - \frac{1}{2}p_1^2 + \mu p_1^2), \]

where \( |2\mu|^2 = |1 - \cos \alpha \cdot e^{-i\alpha}|^2 = \sin^2 \alpha. \) Now we get with the aid of Lemmas 2 and 3 that

\[ |a_3 - \frac{2}{3}a_2^2| \leq \frac{1}{3} + \frac{1}{3}\cos\alpha \left( 2 - \frac{|p_1|^2}{2} \right) + \frac{1}{3}\cos\alpha|\sin\alpha|\frac{|p_1|^2}{2} \]

\[ \leq 1 - \cos\alpha \frac{|p_1|^2}{6}(1 - |\sin\alpha|) \leq 1. \]

An easy consequence using \( |a_3 - a_2^2| \leq 1 \) is

**COROLLARY 3.** Let \( \lambda \in [2/3, 1]. \) Then

\[ \max_{f \in C} |a_3 - \lambda a_2^2| = 1. \]

The maximum is attained by the function \((k(z^2))^{1/2}. \)

We remark that Theorem 2 provides a direct proof of \( |a_3| - |a_2| \leq 1 \) for \( |a_2| \leq 3/2 \)

(compare with the proof of Corollary 2), namely

\[ |a_3| - |a_2| \leq |a_3 - \frac{2}{3}a_2^2| + \frac{2}{3}|a_2|^2 - |a_2| \]

\[ \leq 1 + \frac{2}{3}|a_2|^2 - |a_2| \leq 1 \]

if \( |a_2| \in [0, 3/2]. \)

It remains to consider the case \( \lambda \in [1/3, 2/3[. \)

**THEOREM 3.** Let \( \lambda \in ]1/3, 2/3[. \) Then

\[ \max_{f \in C} |a_3 - \lambda a_2^2| = \frac{1}{3} + \frac{4}{9\lambda}. \]

The maximum is attained by the function \( f \), which is defined by

\[ f'(z) = \frac{1}{(1-z)^2} \cdot \left( t \frac{1+z}{1-z} + (1-t) \frac{1+z^2}{1-z^2} \right), \quad f(0) = 0, \]

where \( t = 2/(3\lambda) - 1. \)

**PROOF.** Consider equation (6). We use the estimate \( |b_3 - \frac{3}{4}\lambda b_2^2| \leq 3(1 - \lambda) \),

which comes from Lemma 3, further equations (5) and \( |b_2| \leq 2, \) getting

\[ |a_3 - \lambda a_2^2| \leq 1 - \lambda + \frac{\cos \alpha}{3} \left| p_2 - \frac{3}{4}\lambda \cos \alpha \cdot e^{-i\alpha}p_1^2 \right| + \cos \alpha \left( \frac{2}{3} - \lambda \right)|p_1|. \]
Writing \( \frac{3}{4} \lambda \cos \alpha \cdot e^{-i\alpha} = \frac{1}{2} - \mu \), we have

\[
|2\mu|^2 = |1 - \frac{3}{4} \lambda \cos \alpha \cdot e^{-i\alpha}|^2 = 1 - (3\lambda - \frac{9}{4} \lambda^2) \cos^2 \alpha,
\]

which implies with the aid of Lemma 2 that

\[
\left| p_2 - \frac{3}{4} \lambda \cos \alpha \cdot e^{-i\alpha} p_1 \right| \leq 2 + \frac{|p_1|^2}{2} \left( \sqrt{1 - \left( 3\lambda - \frac{9}{4} \lambda^2 \right) \cos^2 \alpha} - 1 \right),
\]

so that—using the notations \( y := \cos \alpha \) and \( p := |p_1| \)—it follows that

\[
|a_3 - \lambda a_2^2| \leq 1 - \lambda + \frac{y^2}{6} \left( \sqrt{1 - \left( 3\lambda - \frac{9}{4} \lambda^2 \right) y^2} - 1 \right) + p \left( \frac{2}{3} - \lambda \right)
\]

\[
= : F_\lambda(p, y).
\]

For further simplification we shall use the notation \( \gamma := 2 - 3\lambda \).

Now we shall show that \( F_\lambda \) attains its maximum value for \((p, y) \in [0, 2] \times [0, 1]\) at the point \((4/(3\lambda) - 2, 1)\). Observe that

\[
F_\lambda \left( \frac{4}{3\lambda} - 2, 1 \right) = \frac{1}{3} + \frac{4}{9\lambda}.
\]

Suppose now that \( F_\lambda \) attains its maximum value at an interior point \((p_0, y_0) \in [0, 2[ \times ]0, 1[\). Then the partial derivatives \( \partial F_\lambda / \partial p \) and \( \partial F_\lambda / \partial y \) must vanish at \((p_0, y_0)\).

The equality \( (\partial F_\lambda / \partial p)(p_0, y_0) = 0 \) gives the relation

\[
\gamma p_0 = \frac{2\gamma}{p_0} - \frac{\gamma^2}{p_0^2},
\]

so that

\[
\gamma = \frac{2\gamma/p_0 - \gamma^2/p_0^2}{6(1 - \gamma/p_0)}.
\]

Now, \( (\partial F_\lambda / \partial y)(p_0, y_0) = 0 \) implies

\[
\frac{2}{3} + \frac{\gamma p_0}{6} = \frac{p_0^2(2\gamma/p_0 - \gamma^2/p_0^2)}{6(1 - \gamma/p_0)},
\]

so that, by solving the quadratic equation for \( p_0 \), we get

\[
\gamma p_0 = 2 \left( 1 - \sqrt{1 - \gamma^2} \right).
\]

Therefore, at \((p_0, y_0)\) the value of \( F_\lambda \) becomes, using (8) and (9),

\[
F_\lambda(p_0, y_0) = 1 - \lambda + y \left( \frac{2}{3} + \frac{1}{3} \left( 1 - \sqrt{1 - \gamma^2} \right) \right)
\]

\[
\leq \frac{4 + \gamma - \sqrt{1 - \gamma^2}}{3},
\]

because \( y \leq 1 \).

Since \( \lambda \in [1/3, 2/3]\), the number \( \gamma \) lies between 0 and 1 so that there is some \( \delta \in [0, \pi/2]\) with \( \gamma = \cos \delta \) and \( \sqrt{1 - \gamma^2} = \sin \delta \). The evident inequality \( 1 < \cos \delta + \sin \delta \)
implies
\[
2 - \cos \delta < 1 + \sin \delta \\
\Rightarrow (2 - \cos \delta)(1 - \sin \delta) < 1 - \sin^2 \delta = \cos^2 \delta \\
\Rightarrow (2 - \gamma) \left(1 - \sqrt{1 - \gamma^2}\right) < \gamma^2 \\
\Rightarrow (2 - \gamma) \left(4 + \gamma - \sqrt{1 - \gamma^2}\right) < 6 - \gamma \\
\Rightarrow \frac{4 + \gamma - \sqrt{1 - \gamma^2}}{3} < \frac{1}{3} + \frac{4}{3(2 - \gamma)} = \frac{1}{3} + \frac{4}{9\lambda}.
\]

Thus, using (7) and (10), we get a contradiction to our assumption that \(F_\lambda\) attains its maximum value at \((p_0, y_0)\), so that the maximum must be attained at a boundary point.

In both cases \(y = 0\) and \(p = 0\) an easy computation shows that the maximal value (7) is not attained. If \(y = 1\) we have
\[
F_\lambda(p, 1) = G_\lambda(p) = \frac{5}{3} - \lambda + \left(\frac{2}{3} - \lambda\right)p - \frac{\lambda}{4} p^2.
\]

Because \(G_\lambda(2) = 3 - 4\lambda\) is not maximal, the local maximum at \(p = 4/(3\lambda) - 2\)—given by \(dG_\lambda(p)/dp = 0\)—is global. This leads to the maximal value (7).

Now it remains to prove that
\[
F_\lambda(p, y) \leq \frac{1}{3} + \frac{4}{9\lambda}
\]
for \(p = 2, \ y \in [0, 1[\). This statement is equivalent to
\[
H_\gamma(y) := 2y \left(\sqrt{1 - \left(1 - \frac{\gamma^2}{4}\right)y^2} + \gamma\right) \leq \frac{4}{2 - \gamma} - \gamma
\]
for \(\gamma = 2 - 3\lambda \in [0, 1[\). Because we already know that \(H_\gamma(y) \leq 4/(2 - \gamma) - \gamma\) when \(y \in \{0, 1\}\), it suffices to show (11) for points with \(dH_\gamma(y)/dy = 0\). This leads to
\[
\left(1 - \frac{\gamma^2}{4}\right)y^2 = \frac{4 - \gamma^2 + \gamma\sqrt{8 + \gamma^2}}{8}.
\]

Observe that \(0 \leq y \leq 1\) when (12) is satisfied. Squaring inequality (11) and substituting (12) gives the following inequality:
\[
4 \left(\frac{4 - \gamma^2 + \gamma\sqrt{8 + \gamma^2}}{8}\right) \left(\frac{\sqrt{8 + \gamma^2} - \gamma}{4}\right) \leq \left(1 - \frac{\gamma^2}{4}\right) \left(\frac{4}{2 - \gamma} - \gamma\right)^2.
\]

It remains to prove (13). A lengthy calculation gives—after multiplying with the number \((2 - \gamma)\), which is positive—the equivalent version
\[
\gamma(2 - \gamma)(8 + \gamma^2)^{3/2} \leq (4 + 2\gamma)(4 - 2\gamma + \gamma^2)^2 - (2 - \gamma)(8 + 20\gamma^2 - \gamma^4)
\]
\[
= 48 - 24\gamma - 24\gamma^2 + 28\gamma^3 - 2\gamma^4 + \gamma^5.
\]
The right-hand side turns out to be positive:
\[
48 - 24\gamma - 24\gamma^2 + 28\gamma^3 - 2\gamma^4 + \gamma^5
\]
\[
> 28\gamma^3 - 2\gamma^4 + \gamma^5 = \gamma^3(28 - 2\gamma + \gamma^2) > \gamma^3(26 + \gamma^2) \geq 0,
\]
so that equivalently, squaring again
\[
\gamma^2(2 - \gamma)^2(8 + \gamma^2)^3 \leq (48 - 24\gamma - 24\gamma^2 + 28\gamma^3 - 2\gamma^4 + \gamma^5)^2.
\]
A further lengthy computation gives the equivalent reformulation
\[
\gamma^8 - 2\gamma^7 + 17\gamma^6 - 12\gamma^5 - 70\gamma^4 + 184\gamma^3 - 118\gamma^2 - 72\gamma + 72
\]
\[
= (1 - \gamma)^2(\gamma^6 + 16\gamma^4 + 20\gamma^3 - 46\gamma^2 + 72\gamma + 72)
\]
\[
= (1 - \gamma)^2(\gamma^6 + 16\gamma^4 + 20\gamma^3 + 26\gamma^2 + 72\gamma(1 - \gamma) + 72) \geq 0,
\]
which is trivially true. This finishes the proof of the inequality
\[
|a_3 - \lambda a_2^2| \leq \frac{1}{3} + \frac{4}{9\lambda}.
\]
From our considerations it follows that equality holds if \(b_2 = 2\) and \(b_3 = 3\) (so that \(g\) is a rotation of \(k\)), and if \(\alpha = 0\), \(p_2 = 2\), and \(p_1 = 4/(3\lambda) - 2\); the function
\[
\tilde{p}(z) = t \left(\frac{1+z}{1-z}\right) + (1-t) \left(\frac{1+z^2}{1-z^2}\right), \quad t = \frac{2}{3\lambda} - 1
\]
satisfies these conditions, which makes the result sharp. ☐

REFERENCES


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