COEFFICIENTS OF SYMMETRIC FUNCTIONS
OF BOUNDED BOUNDARY ROTATION

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Abstract. The well-known inclusion relation between functions with bounded boundary rotation and close-to-convex functions of some order is extended to m-fold symmetric functions. This leads solving the corresponding result for close-to-convex functions to the sharp coefficient bounds for m-fold symmetric functions of bounded boundary rotation at most $k\pi$ when $k \geq 2m$. Moreover it shows that an m-fold symmetric function of bounded boundary rotation at most $(2m + 2)\pi$ is close-to-convex and thus univalent.

1. Introduction

We consider functions which are analytic in the unit disk $D$. By $P$ we denote the family of functions $p$ which have the normalization

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

and have positive real part; by $\tilde{P}$ we denote the family of functions $p$ which are normalized by (1) and there exists a complex number $a$ such that the rotated function $ap$ has positive real part.

We consider functions $f$ which have the usual normalization

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots.$$ 

A function is called m-fold symmetric if it has the special form ($m \in \mathbb{N}$),

$$f(z) = z + a_{m+1} z^{m+1} + a_{2m+1} z^{2m+1} + \cdots.$$ 

By $K_m, S_{m}, C_m(\beta)$ and $V_m(k)$ respectively we denote the families of m-fold symmetric convex, starlike, close-to-convex functions of order $\beta$ and functions of bounded boundary rotation at most $k\pi$, respectively. A function is called convex or starlike if it maps the unit disk univalently onto a convex or starlike domain respectively.

A function $f$ is called close-to-convex of order $\beta$, $\beta \geq 0$, if there is a convex function $\varphi$ such that $f'/\varphi' = p^\beta$ for some function $p \in \tilde{P}$. For $\beta \leq 1$ it turns out that a function is close-to-convex or order $\beta$, if and only if it maps...
D univalently onto a domain whose complement $E$ is the union of rays, which are pairwise disjoint up to their tips, such that every ray is the bisector of a sector of angle $(1 - \beta)\pi$ which wholly lies in $E$ (see e.g. [2], and [12, p. 176]). By means of the introductory paper of Kaplan [7], it is easily verified that for an $m$-fold symmetric function $f$ the corresponding function $\varphi$ can be chosen also to be $m$-fold symmetric. This observation is due to Pommerenke [11], who studied coefficient problems in $C_m(\beta)$. His asymptotic results give support to the conjecture that if $\beta > 1 - 2/m$, then the coefficients of a function $f \in C_m(\beta)$ given by (2) are dominated in modulus by the corresponding coefficients of the function $g$ given by

\begin{equation}
(3) \quad g'(z) = \frac{(1 + z^m)^\beta}{(1 - z^m)^\beta + 2/m}, \quad g(0) = 0.
\end{equation}

Coefficient domination is denoted by $f \ll g$.

The above statement had been settled for $m = 1$ by Brannan, Clunie and Kirwan [4] and the final step by Aharnov and Friedland [1] and independently by Brannan [3], (see e.g. [14, Chapter 2]), and for $\beta = 1$ by Pommerenke [11, Theorem 3]. This latter statement includes the truth of the Littlewood-Paley conjecture (see e.g. [6, §3.8]) for odd close-to-convex functions (of order one).

In §2 we give a proof of the above statement for $\beta \geq 1 - 1/m$, whereas for $0 < \beta < 1 - 1/m$ the statement is false as examples show, so that the number $1 - 1/m$ is sharp. However, for $\beta = 0$, i.e. for convex functions, the statement is again true, as was shown by Robertson [13, p. 380].

The boundary rotation of a function $f$ is defined by

$$\sup_{0 < r < 1} \int_0^{2\pi} \left| \text{Re} \left( 1 + \frac{zf''}{f'} \right)(re^{i\theta}) \right| d\theta.$$ 

Paatero [10] showed that $f \in V_1(k)$, if and only if

$$1 + \frac{zf''}{f'} = \left( \frac{k}{4} + \frac{1}{2} \right) \cdot p_1 - \left( \frac{k}{4} - \frac{1}{2} \right) \cdot p_2$$

for some $p_1, p_2 \in P$. An inspection of Paatero’s proof shows that for an $m$-fold symmetric function, $p_1$ and $p_2$ can be chosen to have the form

\begin{equation}
(4) \quad p_{1,2}(z) = 1 + c_m z^m + c_{2m} z^{2m} + \cdots.
\end{equation}

It is well known [4], (see e.g. [17, Theorem 2.26]) that functions of bounded boundary rotation are close-to-convex of some order, namely

$$V_1(k) \subset C_1(k/2 - 1).$$

In §3 we give an improvement of this result for $m$-fold symmetric functions:

$$V_m(k) \subset C_m((k/2 - 1)/m),$$

which leads to the solution of the coefficient problem for $m$-fold symmetric functions of bounded boundary rotation when $k \geq 2m$. This result includes the
truth of the Littlewood-Paley conjecture for odd functions of bounded boundary rotation $6\pi$.

2. The coefficients of symmetric close-to-convex functions.

Here we shall prove

**Theorem 1.** Let $m \in \mathbb{N}$, $\beta \geq 1 - 1/m$ and $f \in C_m(\beta)$. Then

$$f'(z) \ll \frac{(1 + z^m)^\beta}{(1 - z^m)^{\beta + 2/m}}.$$ 

**Proof.** Let $f$ be an $m$-fold symmetric close-to-convex function of order $\beta$. Then there exist $\phi \in K_m$ and $p \in \tilde{P}$ such that

$$f'(z) = \phi'(z) \cdot p^\beta(z^m).$$

For each $\phi \in K_m$ there is a $g \in St_m$ such that $g = z\phi'$ (see e.g. [14, Theorem 2.4]), for which there is a representation of the form (see [5, Theorem 3])

$$g(z) = \int_{|x|=1} \frac{z}{(1 - xz^m)^{2/m}} d\mu,$$

where $\mu$ is a Borel probability measure on the unit circle. Thus we have

$$f'(z) = \int_{|x|=1} \frac{d\mu}{(1 - xz^m)^{2/m}} \cdot p^\beta(z^m)$$

$$= \int_{|x|=1} \frac{d\mu}{(1 - x^2z^{2m})^{1/m}} \cdot \left(\frac{1 + xz^m}{1 - xz^m}\right)^{1/m} \cdot p^\beta(z^m).$$

For fixed $x \in \partial \mathbb{D}$ the function

$$\left(\frac{1 + xz^m}{1 - xz^m}\right)^{1/(\beta + 1/m)} =: q_x(z^m)$$

is of the form (4) and lies in $\tilde{P}$. A well-known lemma [4, 3], (see e.g. [14, Theorem 2.21]) implies that

$$q_x^{\beta + 1/m}(z^m) \ll \left(\frac{1 + z^m}{1 - z^m}\right)^{\beta + 1/m},$$

because $\beta + 1/m \geq 1$. Thus we get

$$f'(z) = \int_{|x|=1} \frac{d\mu}{(1 - x^2z^{2m})^{1/m}} \cdot q_x^{\beta + 1/m}(z^m)$$

$$= \sum_{j=0}^{\infty} \left(\begin{array}{c} j - 1 + 1/m \\ j \end{array}\right) z^{2mj} \left\{ \int_{|x|=1} x^{2j} q_x^{\beta + 1/m}(z^m) d\mu \right\}$$

$$\ll \sum_{j=0}^{\infty} \left(\begin{array}{c} j - 1 + 1/m \\ j \end{array}\right) z^{2mj} \left(\frac{1 + z^m}{1 - z^m}\right)^{\beta + 1/m} = \frac{(1 + z^m)^\beta}{(1 - z^m)^{\beta + 2/m}}.$$
because \( \mu \) has total mass one and all numbers \( \left( j^{-1+1/m} \right) \) are nonnegative. \( \square \)

We remark that the result is sharp, because the function \( g \) defined by (3) is in \( C_m(\beta) \) (see e.g. [11, p. 264]).

For \( 0 < \beta < 1 - 2/m \) Pommerenke showed [11, Theorem 2], that \( a_n = o(1/n) \) for a function \( f \in C_m(\beta) \), and that this cannot be improved [11, p. 265]. But on the other hand, for \( \beta > 1 - 2/m \),

\[
a_n = O(n^{\beta-2+2/m}),
\]

[11, Theorem 1].

Nevertheless, the statement of Theorem 1 is not true in the case \( 1 - 2/m < \beta < 1 - 1/m \), not even for the third nonvanishing coefficient \( a_{2m+1} \), as the following examples show. For \( 0 < t < 1 \) let

\[
f'(z) = \frac{1}{(1 - z^m)^{2/m}} \left( t \left( \frac{1 + z^m}{1 - z^m} \right) + (1 - t) \left( \frac{1 + z^{2m}}{1 - z^{2m}} \right) \right).
\]

Then obviously \( f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots \in C_m(\beta) \). It follows that

\[
(2m + 1)a_{2m+1} = 2\beta(1 + (\beta - 1)t^2) + \frac{4\beta t}{m} + \frac{1}{m} \left( 1 + \frac{2}{m} \right) =: F(t).
\]

The relation \( F'(t_0) = 0 \) implies that

\[
t_0 = \frac{1}{m(1 - \beta)},
\]

which lies between 0 and 1 if \( 0 < \beta < 1 - 1/m \), so that \( F \) has a local maximum at \( t_0 \), which is greater than the corresponding coefficient of \( g \), as is easily seen.

3. The coefficients of symmetric functions of bounded boundary rotation.

It is well known that functions of bounded boundary rotation are close-to-convex of some order,

(5) \( V_1(k) \subset C_1(k/2 - 1) \).

We shall give now a generalized version of this statement for \( m \)-fold symmetric functions. We need the following

**Lemma.** Let \( f(z) = z + a_2z^2 + a_3z^3 + \cdots \) and \( h(z) = z + b_{m+1}z^{m+1} + b_{2m+1}z^{2m+1} + \cdots \) have the property

\[
h'(z) = (f'(z^m))^{1/m}.
\]

Then

\[
f \in V_1(k) \Rightarrow h \in V_m(k)
\]

and

\[
f \in C_1(\beta) \Rightarrow h \in C_m(\beta/m).
\]
Proof. Let \( f \in V_1(k) \). Then
\[
1 + \frac{z^m f''(z^m)}{f'(z^m)} = 1 + \frac{zh''(z)}{h'(z)},
\]
so that \( h \in V_m(k) \), and conversely.

If \( f \in C_1(\beta) \), then there are \( \varphi \in \mathcal{K}_1 \) and \( p \in \tilde{P} \) such that
\[
f'(z) = \varphi'(z) \cdot p^\beta(z).
\]
Now
\[
h'(z) = \left(f'(z^m)\right)^{1/m} = \left(\varphi'(z^m)\right)^{1/m} \cdot p^{\beta/m}(z^m)
\]
\[
= \varphi_1(z) \cdot p^{\beta/m}(z^m).
\]
The function \( \varphi_1 \) represents an \( m \)-fold symmetric convex function, because a function is convex, if and only if \( 1 + z \varphi''/\varphi' \in P \) (see e.g. [14, Theorem 2.4]), and
\[
1 + \frac{z\varphi''(z)}{\varphi_1(z)} = 1 + \frac{z^m \varphi''(z^m)}{\varphi(z^m)}.
\]
So it follows that \( h \in C_{\beta/m} \), and conversely. \( \square \)

We remark that the lemma can be used to show that Theorem 1 with \( \beta = 1/2, m = 2 \) is somewhat stronger than the case \( \beta = 1, m = 1 \). For example it leads to the estimates \( ||a_3|| - ||a_2|| \leq 1 \) and \( ||a_4|| - ||a_2|| \leq 2 \) for close-to-convex functions [8, 9].

An application of the lemma, with the aid of (5), gives

Theorem 2. Let \( m \in \mathbb{N}, k \geq 2 \). Then
\[
V_m(k) \subset C_{m}((k/2 - 1)/m).
\]

This leads to the following statements

Theorem 3. Let \( m \in \mathbb{N}, k \geq 2m \) and \( f \in V_m(k) \). Then
\[
f' \ll \frac{(1 + z^m)^{(k/2 - 1)/m}}{(1 - z^m)^{(k/2 + 1)/m}}.
\]

This follows with Theorem 1. Observe that the statement is sharp, because the functions defined by (3) with \( \beta = (k/2 - 1)/m \) are in \( V_m(k) \),
\[
1 + \frac{z^\varphi''}{\varphi'(z)} = \left(\frac{k}{4} + \frac{1}{2}\right) \cdot \frac{1 + z^m}{1 - z^m} - \left(\frac{k}{4} - \frac{1}{2}\right) \cdot \frac{1 - z^m}{1 + z^m}.
\]

For \( m = 2, k = 6 \) we have the statement of the Littlewood-Paley conjecture. Another example is \( m = 2, k = 4 \). Here one gets the sharp bounds for \( f \), normalized by (2),
\[
|a_{2n+1}| \leq \left\{ \begin{array}{ll}
\frac{1}{2n+1} \left(\begin{array}{c}
(n/2 + 1/2) \\
n/2
\end{array}\right) + \left(\begin{array}{c}
n/2 - 1/2 \\
n/2 - 1/2
\end{array}\right) & \text{if } n \text{ is even}, \\
\frac{2}{2n+1} \left(\begin{array}{c}
n/2 \\
n/2 - 1/2
\end{array}\right) & \text{if } n \text{ is odd}.
\end{array} \right.
\]
It is an open question if the statement of Theorem 3 remains true, when $k < 2m$. The close-to-convex counterexamples, given after Theorem 1, cannot be used here.

Furthermore we have

**Theorem 4.** Let $m \in \mathbb{N}$. Then $V_m(2m + 2)$ consists of close-to-convex and thus univalent functions.

**References**