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On the Fekete-Szegö problem for close-to-convex functions II

By

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Let $C(\beta)$, $\beta \geqq 0$, denote the family of normalized close-to-convex functions of order β. For $\beta = 1$ this is the usual set of close-to-convex functions, which had been defined by Kaplan.

In a previous paper [3] we solved the Fekete-Szegö problem of maximizing $|a_3 - \lambda a_2^2|$, $\lambda \in [0, 1]$, for close-to-convex functions. The largest number λ_0 for which $|a_3 - \lambda_0 a_2^2|$ is maximized by the Koebe function $z/(1-z)^2$ is $\lambda_0 = 1/3$.

Now we generalize this result to $C(\beta)$, $\beta \geqq 1$, showing that the largest number $\lambda_0(\beta)$ for which $|a_3 - \lambda_0(\beta) a_2^2|$ is maximized over $C(\beta)$ by k_β with

$$k_\beta(z) = \frac{1}{2(\beta+1)}\left(\left(\frac{1+z}{1-z}\right)^{\beta+1} - 1\right)$$

is $\lambda_0(\beta) = \dfrac{2}{3}\dfrac{\beta}{\beta+1}$.

On the other hand, for all $\beta \geqq 0$, $|a_3 - \frac{2}{3} a_2^2|$ is maximized over $C(\beta)$ by the odd function h_β with

$$h'_\beta(z) = \frac{(1+z^2)^\beta}{(1-z^2)^{\beta+1}}, \quad h_\beta(0) = 0,$$

similarly as in the case $\beta = 1$.

Another interesting case is $\lambda = 1$, where we get

$$|a_3 - a_2^2| \leqq \begin{cases} \frac{1}{3}(1 + 2\beta) & \text{if} \quad \beta \leqq 1 \\ \frac{1}{3}(\beta^2 + 2\beta) & \text{if} \quad \beta \geqq 1, \end{cases}$$

with equality for h_β and k_β if $\beta \leqq 1$ and $\beta \geqq 1$ respectively, generalizing the well-known elementary inequality for $\beta = 1$, which is a consequence of the univalence.

From these results we conclude that

$$\big||a_3| - |a_2|\big| \leqq \frac{\beta}{3}(1 + 2\beta)$$

if $\beta \geqq 1$ with equality for k_β, and

$$\big||a_3| - |a_2|\big| \leqq \tfrac{1}{3}(1 + 2\beta)$$

if $\beta \leqq 1/2$ with equality for h_β.

In the last section we give the sharp bound of a Nehari type condition for $C(\beta)$, generalizing therefore Neharis result that convex functions satisfy the Nehari univalence criterion.

1. Introduction. Let S denote the family of univalent functions f of the unit disk $\mathbb{D}$, normalized by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots. \tag{1}$$

Let St denote the subset of starlike functions, i.e. functions that have a starlike range with respect to the origin. A function f, normalized by (1), is called close-to-convex of order β, $\beta \geqq 0$, if there exist a starlike function g and a real number α, so that

$$|\arg(e^{i\alpha} z f'(z)/g(z))| \leqq \beta \frac{\pi}{2}.$$

Let $C(\beta)$ denote the family of close-to-convex functions of order β. For $\beta \leqq 1$ it turns out that a function is close-to-convex of order β, if and only if it maps the unit disk univalently onto a domain whose complement E is the union of rays, which are pairwise disjoint up to their tips, so that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which also lies in E (see [7], p. 176).

In a previous paper [3] we solved the Fekete-Szegö problem of maximizing $|a_3 - \lambda a_2^2|$, $\lambda \in [0, 1]$, for close-to-convex functions of order 1. Now we generalize this result to $C(\beta)$. For $\beta > 1$ we are able to show that there is a number $\lambda_0 > 0$ such that $|a_3 - \lambda_0 a_2^2|$ is maximized over $C(\beta)$ by the function k_β with

$$k_\beta(z) = \frac{1}{2(\beta+1)}\left(\left(\frac{1+z}{1-z}\right)^{\beta+1} - 1\right) = \sum_{n=1}^{\infty} A_n z^n.$$

The largest number with this property is $\lambda_0 = \dfrac{2}{3}\dfrac{\beta}{\beta+1}$. For all $\beta \geqq 0$ we show that $|a_3 - \frac{2}{3} a_2^2|$ is maximized over $C(\beta)$ by the odd function h_β with

$$h_\beta(z) = \int_0^z \frac{(1+\zeta^2)^\beta}{(1-\zeta^2)^{\beta+1}}\, d\zeta = \sum_{n=1}^{\infty} B_n z^n,$$

similarly as in the case $\beta = 1$.
If $\lambda = 1$, then the result splits:

$$|a_3 - a_2^2| \leqq \begin{cases} B_3 - B_2^2 = B_3 & \text{if } \beta \leqq 1 \\ A_2^2 - A_3 & \text{if } \beta \geqq 1. \end{cases}$$

This seems to be the most interesting result of this paper, especially in the case $\beta \leqq 1$, because it generalizes the well-known estimate for $\beta = 1$, which is a consequence of the univalence.

2. On the Fekete-Szegö problem. The following notations will be used throughout the present section. For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$ there is representation of the form

$$f'(z) = \frac{g(z)}{z} \cdot \tilde{q}(z). \tag{2}$$

with some function $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in St$ and some function

$$\tilde{q}(z) = 1 + \tilde{q}_1 z + \tilde{q}_2 z^2 + \cdots$$

with $\tilde{q}(z) = (\tilde{p}(z))^{\beta}$, $\tilde{p}(z) = 1 + \tilde{p}_1 z + \tilde{p}_2 z^2 + \cdots$ and $\operatorname{Re}(e^{i\alpha}\tilde{p}(z)) > 0$, $\alpha \in]-\pi/2, \pi/2[$. Note that

$$\begin{aligned} \tilde{q}_1 &= \beta \tilde{p}_1, \\ \tilde{q}_2 &= \beta\left(\tilde{p}_2 + \frac{\beta - 1}{2}\tilde{p}_1^2\right). \end{aligned} \tag{3}$$

The function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, defined by

$$\tilde{p}_n = \cos\alpha \cdot e^{-i\alpha} \cdot p_n, \quad n \in \mathbb{N} \tag{4}$$

has positive real part. Comparing coefficients in (2) one gets

$$\begin{aligned} 3a_3 &= b_3 + \tilde{q}_1 b_2 + \tilde{q}_2, \\ 2a_2 &= b_2 + \tilde{q}_1, \end{aligned}$$

so that with aid of (3) it follows that

$$\begin{aligned} a_3 - \lambda a_2^2 = \frac{1}{3}\left(b_3 - \frac{3}{4}\lambda b_2^2\right) + \frac{\beta}{3}\left(\tilde{p}_2 - \left(\frac{3}{4}\lambda\beta - \frac{\beta - 1}{2}\right)\tilde{p}_1^2\right) \\ + \beta \tilde{p}_1 b_2\left(\frac{1}{3} - \frac{\lambda}{2}\right). \end{aligned} \tag{5}$$

Now we consider the case $\lambda = 2/3$.

Theorem 1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$. Then*

$$|a_3 - \tfrac{2}{3}a_2^2| \leqq B_3 - \tfrac{2}{3}B_2^2 = \tfrac{1}{3}(1 + 2\beta).$$

Proof. From (5) and the triangle inequality it follows that

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leqq \frac{1}{3}\left|b_3 - \frac{1}{2}b_2^2\right| + \frac{\beta}{3}\left|\tilde{p}_2 - \frac{1}{2}\tilde{p}_1^2\right|.$$

Now $|b_3 - \frac{1}{2}b_2^2| \leqq 1$ (see e.g. [3], Lemma 3). Further in the proof of Theorem 2 in [3] it was shown that

$$\left|\tilde{p}_2 - \frac{1}{2}\tilde{p}_1^2\right| \leqq \cos\alpha\left(2 - \frac{|p_1|^2}{2}(1 - |\sin\alpha|)\right),$$

using the inequality $|p_2 - p_1^2/2| \leqq 2 - |p_1|^2/2$ (see e.g. [3], Lemma 2), so that

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leqq \frac{1}{3} + \frac{2\beta}{3}\cos\alpha - \beta\cos\alpha\frac{|p_1|^2}{2}(1 - |\sin\alpha|) \leqq \frac{1}{3}(1 + 2\beta). \quad \square$$

Next we choose $\lambda = \frac{2}{3}\frac{\beta}{\beta+1}$.

Theorem 2. *Let $\beta \geqq 1$ and $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$. Then*

$$\left| a_3 - \frac{2}{3}\frac{\beta}{\beta+1} a_2^2 \right| \leqq A_3 - \frac{2}{3}\frac{\beta}{\beta+1} A_2^2 = 1 + \frac{2}{3}\beta.$$

Proof. Let be $\lambda \in [0, 2/3]$ and consider Eq. (5). Then we have the estimate $|b_3 - \frac{3}{4} b_2^2| \leqq 3(1-\lambda)$, which is a consequence of [3], Lemma 3, further Eqs. (4) and $|b_2| \leqq 2$, getting

$$|a_3 - \lambda a_2^2| \leqq 1 - \lambda + \frac{\beta}{3}\cos\alpha \left| p_2 - \left(\frac{3}{4}\lambda\beta - \frac{\beta-1}{2}\right)\cos\alpha \cdot e^{-i\alpha} p_1^2 \right|$$
$$+ \beta\cos\alpha\left(\frac{2}{3} - \lambda\right)|p_1|.$$

Writing

$$\left(\frac{3}{4}\lambda\beta - \frac{\beta-1}{2}\right)\cos\alpha \cdot e^{-i\alpha} = \frac{1}{2} - \mu,$$

we have $|2\mu|^2 = 1 - \cos^2\alpha(1 - \beta^2(1 - \frac{3}{2}\lambda)^2)$, which implies with aid of [3], Lemma 2, and the triangle inequality

$$\left| p_2 - \left(\frac{3}{4}\lambda\beta - \frac{\beta-1}{2}\right)\cos\alpha \cdot e^{-i\alpha} p_1^2 \right|$$
$$\leqq 2 + \frac{|p_1|^2}{2}\left(\sqrt{1 - \cos^2\alpha\left(1 - \beta^2\left(1 - \frac{3}{2}\lambda\right)^2\right)} - 1\right),$$

so that – using the notations $y := \cos\alpha$, $p := |p_1|$ and $\gamma := 2 - 3\lambda$ – it follows that

$$3|a_3 - \lambda a_2^2| \leqq 1 + \gamma + \beta y\left(2 - \frac{p^2}{2}\left(1 - \sqrt{1 - y^2\left(1 - \frac{\gamma^2\beta^2}{4}\right)}\right)\right) + \beta\gamma y p. \tag{6}$$

Now we specialize, substituting $\lambda = \frac{2}{3}\frac{\beta}{\beta+1}$, i.e. $\gamma = \frac{2}{\beta+1}$, and have finally

$$3\left| a_3 - \frac{2}{3}\frac{\beta}{\beta+1} a_2^2 \right|$$
$$\leqq 1 + \frac{2}{\beta+1}(1 + \beta y p) + \beta y\left(2 - \frac{p^2}{2}\left(1 - \sqrt{1 - y^2\left(1 - \frac{\beta^2}{(\beta+1)^2}\right)}\right)\right)$$
$$=: F_\beta(p, y).$$

Now we shall show that F_β attains its maximal value for $(p, y) \in [0, 2] \times [0, 1]$ at the point $(2, 1)$. Observe that

$$F_\beta(2, 1) = 3 + 2\beta, \tag{7}$$

which leads to the statement of the theorem.

Suppose now, that F_β attains its maximal value at an interior point $(p_0, y_0) \in]0, 2[\times]0, 1[$. Then the partial derivative $\partial F_\beta/\partial p$ vanishes at (p_0, y_0), which implies the relation

$$p_0\left(1 - \sqrt{1 - y_0^2\left(1 - \frac{\beta^2}{(\beta+1)^2}\right)}\right) = \frac{2}{\beta+1}.$$

Therefore at (p_0, y_0) the value of F_β becomes

$$\begin{aligned} F_\beta(p_0, y_0) &= 1 + \frac{2}{\beta+1} + \frac{2\beta y_0 p_0}{\beta+1} + 2\beta y_0 - \frac{\beta y_0 p_0}{2}\cdot\frac{2}{\beta+1} \\ &= 1 + \frac{2}{\beta+1} + \frac{\beta y_0 p_0}{\beta+1} + 2\beta y_0 \\ &< 1 + \frac{2}{\beta+1} + \frac{2\beta}{\beta+1} + 2\beta = 3 + 2\beta, \end{aligned}$$

contradicting our assumption that the value is maximal. So F_β attains its maximal value at a boundary point. In both cases $y = 0$ or $p = 0$ an easy computation shows that the value (7) is not attained. If $y = 1$ we have

$$\begin{aligned} F_\beta(p, 1) &= 1 + \frac{2}{\beta+1} + 2\beta + \frac{\beta}{\beta+1}\cdot\frac{p}{2}(4-p) \leqq 1 + \frac{2}{\beta+1} + 2\beta + \frac{2\beta}{\beta+1} \\ &= 3 + 2\beta. \end{aligned}$$

If $p = 2$, then

$$F_\beta(2, y) =: H_\beta(y) = 1 + \frac{2}{\beta+1}(1 + 2\beta y) + 2\beta y\sqrt{1 - y^2\left(1 - \frac{\beta^2}{(\beta+1)^2}\right)}.$$

A calculation shows that H_β is increasing for $y \in [0, 1]$, which implies the result. □

Remark. We note that we conjecture the truth of the statement of Theorem 2 also for $\beta < 1$, but our method does not work in this case, because it is only sharp if $y = 1$, and the maximal value of F_β is not attained at such a point if $\beta \in]0, 1[$.

From Theorem 2 and the well-known bound $|a_2| \leqq 1 + \beta = A_2$ (see e.g. [1]) it follows immediately

Corollary 1. *Let* $\beta \geqq 1, 0 \leqq \lambda \leqq \frac{2}{3}\frac{\beta}{\beta+1}$. *Then*

$$\max_{f \in C(\beta)} |a_3 - \lambda a_2^2| = A_3 - \lambda A_2^2.$$

We remark that for fixed β the number $\lambda_0 = \frac{2}{3}\frac{\beta}{\beta+1}$ is best possible in the sense that for each $\lambda \in]\lambda_0, 2/3[$ there is a function $f \in C(\beta)$ with

$$|a_3 - \lambda a_2^2| > A_3 - \lambda A_2^2.$$

Examples come from the choice

$$g(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots$$

and ($t \in [0, 1]$)

$$\tilde{p}(z) = t\left(\frac{1+z}{1-z}\right) + (1-t)\left(\frac{1+z^2}{1-z^2}\right) = 1 + 2tz + z^2 + \cdots,$$

for which – writing again $\gamma := 2 - 3\lambda$ – it follows that

$$3(a_3 - \lambda a_2^2) = 1 + \gamma + 2\beta + 2\beta\gamma t + t^2\beta(\gamma\beta - 2) =: F(t).$$

The relation $F'(t_0) = 0$ implies that

$$t_0 = \frac{\gamma}{2 - \gamma\beta}.$$

If now $\lambda \in]\lambda_0, 2/3[$, i.e. $\gamma \in]0, 2/(\beta+1)[$, then this number t_0 lies between 0 and 1 which shows that F has a local maximum at t_0, so that k_β does not give the maximum.

These arguments hold also if $\beta < 1$.

Now we come to the case $\lambda = 1$, which seems to be the most interesting part of this paper, because it generalizes the well-known estimate $|a_3 - a_2^2| \leqq 1$ for close-to-convex functions, which is a consequence of the univalence. Here we shall also study the question which functions give equality. Therefore we shall use the following

Lemma. *Let be $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in St$. Then*

$$\left|b_3 - \frac{3}{4}b_2^2\right| \leqq 1 - \frac{|b_2|^2}{4}.$$

P r o o f. This follows from the fact that for each $g \in St$ the function $\int_0^z (g(\zeta)/\zeta)\, d\zeta$ is convex and from a corresponding result for convex functions (see e.g. [9]). □

We get

Theorem 3. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$. Then*

$$|a_3 - a_2^2| \leqq \begin{cases} B_3 - B_2^2 = \frac{1}{3}(1 + 2\beta) & \text{if} \quad \beta \in [0, 1] \\ A_2^2 - A_3 = \frac{1}{3}(\beta^2 + 2\beta) & \text{if} \quad \beta \geqq 1. \end{cases}$$

Equality holds if and only if

$$f(z)=\begin{cases} h_\beta(xz), & |x|=1 \quad \text{if} \quad \beta \leqq 1\\ k_\beta(xz), & |x|=1 \quad \text{if} \quad \beta \geqq 1. \end{cases}$$

Proof. From (5) we deduce by the same procedure as in Theorem 2, using the Lemma, that

$$3|a_3-a_2^2| \leqq 1-\frac{b^2}{4}+\beta y\left(2-\frac{p^2}{2}(1-\sqrt{1-y^2(1-\beta^2/4)})\right)+\frac{\beta y p b}{2}$$
$$=: F_\beta(p,b,y),$$

where $p:=|p_1|$, $b:=|b_2|$ and $y:=\cos\alpha$.

From this the result follows at once if $\beta=0$. (The case of equality is considered at the end of the proof.)

We shall show that F_β takes its maximal value in $Q:=[0,2]^2\times[0,1]$ at the point $(2,2,1)$ if $\beta\geqq 1$ and at the point $(0,0,1)$ if $\beta\leqq 1$, which gives the result.

Let firstly $\beta\geqq 2$. Then the coefficient of p^2 is nonnegative so that F_β is maximized at $p=2$. Furthermore

$$F_\beta(2,b,y)=1-\frac{b^2}{4}+2\beta y\sqrt{1+y^2(\beta^2/4-1)}+\beta y b$$
$$\leqq 1-\frac{b^2}{4}+\beta^2+\beta b,$$

which takes its maximal value for $b\in[0,2]$ at $b=2$, as is easily veryfied.

Next we show that for $\beta\in\,]0,2[$ the maximal value of F_β is attained at the boundary of Q. Suppose, the maximal value of F_β were attained at an interior point (p_0,b_0,y_0) of Q, then the partial derivatives vanish there. The equation $\frac{\partial F_\beta}{\partial b}(q_0,b_0,y_0)=0$ gives the relation

$$(8) \qquad b_0=\beta y_0 p_0,$$

and $\frac{\partial F_\beta}{\partial p}(p_0,b_0,y_0)=0$ implies

$$(9) \qquad b_0=2p_0(1-\sqrt{1-y_0^2(1-\beta^2/4)}),$$

so that, using both (8) and (9), we get

$$(10) \qquad 1-\sqrt{1-y_0^2(1-\beta^2/4)}=\frac{\beta y_0}{2}.$$

A simple calculation then gives

$$(11) \qquad y_0=\beta.$$

For $\beta \geqq 1$ this contradicts our assumption, that F_β has its maximal value at (p_0, b_0, y_0), whereas for $\beta < 1$ we get

$$\begin{aligned} F_\beta(p_0, b_0, y_0) &\overset{(8)}{=} 1 + \frac{\beta^2 y_0^2 p_0^2}{4} + \beta y_0 \left(2 - \frac{p_0^2}{2}(1 - \sqrt{1 - y_0^2(1 - \beta^2/4)})\right) \\ &= 1 + 2\beta y_0 - \frac{\beta y_0 p_0^2}{2}\left(1 - \sqrt{1 - y_0^2(1 - \beta^2/4)} - \frac{\beta y_0}{2}\right) \\ &\overset{(10)}{=} 1 + 2\beta y_0 < 1 + 2\beta, \end{aligned}$$

such that here we get a contradiction, too, and the maximal value is attained (only) at the boundary of Q.

Now we are able to get our conclusion for $\beta = 1$ (which is known as a consequence of the univalence of close-to-convex functions of order 1). Because of (11) we know that the maximum is attained at $y = 1$, where

$$F_1(p, b, 1) = 3 - \tfrac{1}{4}(b - p)^2 \leqq 3.$$

Now it remains to show that

$$F_\beta(p, b, y) \leqq \begin{cases} 1 + 2\beta & \text{for} \quad \beta \in]0, 1[\\ \beta^2 + 2\beta & \text{for} \quad \beta \in]1, 2[\end{cases} \tag{12}$$

on the boundary of Q. For $y = 0$, $p = 0$ and for $b = 0$ the easy proof of (12) is left to the reader. Thus we must show (12) for the open faces $\{y = 1\}$, $\{p = 2\}$ and $\{b = 2\}$ of Q as well as the edges $\{p = b = 2\}$, $\{p = 2, y = 1\}$ and $\{b = 2, y = 1\}$. For $\beta \neq 1$ the maximal value is not attained on the open face $\{y = 1\}$ because of (11).

Consider now the open face $\{p = 2\}$. There

$$F_\beta(2, b, y) =: G(b, y) = 1 - \frac{b^2}{4} + \beta y (b + 2\sqrt{1 - y^2(1 - \beta^2/4)}).$$

The relation $\partial G/\partial b = 0$ implies

$$b = 2\beta y. \tag{13}$$

If furthermore $\partial G/\partial y = 0$ then

$$b + 2\sqrt{1 - y^2(1 - \beta^2/4)} = 2y^2 \frac{1 - \beta^2/4}{\sqrt{1 - y^2(1 - \beta^2/4)}}, \tag{14}$$

and substituting Eq. (13) yields

$$y^2 = \frac{1}{2 + \beta} \quad \text{or} \quad y^2 = \frac{1}{2 - \beta}$$

so that

$$y\sqrt{1 - y^2(1 - \beta^2/4)} = \tfrac{1}{2} \tag{15}$$

and

$$G(b, y) \overset{(13)}{=} 1 + \beta^2 y^2 + 2\beta y \sqrt{1 - y^2(1 - \beta^2/4)}$$
$$\overset{(15)}{=} 1 + \beta^2 y^2 + \beta < 1 + \beta^2 + \beta,$$

which implies (12).

Consider the open face $\{b = 2\}$. Then

$$F_\beta(p, 2, y) =: H(p, y) = \beta y \left(2 - \frac{p^2}{2}(1 - \sqrt{1 - y^2(1 - \beta^2/4)}) + p\right).$$

The relation $\partial H/\partial p = 0$ implies

$$p(1 - \sqrt{1 - y^2(1 - \beta^2/4)}) = 1,$$

so that

$$H(p, y) = \beta y \left(2 - \frac{p}{2} + p\right) = \beta y \left(2 + \frac{p}{2}\right) < 3\beta,$$

which gives (12).

Now we study the edges of Q. For a local maximum of F_β on the edge $\{p = b = 2\}$ we get (14) so that

$$1 + \sqrt{1 - y^2(1 - \beta^2/4)} = y^2 \frac{1 - \beta^2/4}{\sqrt{1 - y^2(1 - \beta^2/4)}},$$

which implies

$$\sqrt{1 - y^2(1 - \beta^2/4)} = \tfrac{1}{2},$$

and so it follows

$$F_\beta(2, 2, y) = 2\beta y(1 + \sqrt{1 - y^2(1 - \beta^2/4)}) = 3\beta y \leqq 3\beta,$$

which gives (12).

For a local maximum of F_β on the edge $\{p = 2, y = 1\}$ we get from (8) that

$$b = 2\beta. \tag{16}$$

For $\beta > 1$ this statement is false so that there exists no local maximum on $\{p = 2, y = 1\}$. For $\beta \in]0, 1[$ we get using (16) that

$$F_\beta(2, b, 1) = 1 - \frac{b^2}{4} + \beta^2 + \beta b = 1 + 2\beta^2 < 1 + 2\beta.$$

For a local maximum of F_β on the edge $\{b = 2, y = 1\}$ we get from (9) that

$$p = \frac{2}{2 - \beta}. \tag{17}$$

For $\beta > 1$ this statement is false so that there exists no local maximum on $\{b = 2, y = 1\}$. For $\beta \in]0, 1[$ we get using (17)

$$F_\beta(p, 2, 1) = \beta\left(2 - \frac{p^2}{2}\left(1 - \frac{\beta}{2}\right) + p\right) = 2\beta + \frac{\beta}{2-\beta} < 1 + 2\beta.$$

A calculation of F_β at the corners of Q finishes the proof of inequality (12).

Now we study the case of equality. An inspection of the proof shows that equality holds if and only if $y = 1$ and

$$\beta \leqq 1 \quad \text{and} \quad |b_2| = |p_1| = 0$$

or

$$\beta \geqq 1 \quad \text{and} \quad |b_2| = |p_1| = 2.$$

Now let $|b_2| = |p_1| = 2$. Then

$$g(z) = \frac{z}{(1 - xz)^2}, \quad |x| = 1$$

and

$$\tilde{p}(z) = \frac{1 + wz}{1 - wz}, \quad |w| = 1$$

(see e.g. [8], Theorem 1.5 and Corollary 2.3). An easy computation gives now that equality implies $w = x$, and so $f(z) = k_\beta(xz)$. If $|b_2| = |p_1| = 0$, then the use of $|b_3 - \frac{3}{4}b_2^2| \leqq 1 - |b_2|^2/4$ and $|p_2 - p_1^2/2| \leqq 2 - |p_1|^2/2$ establishes that equality implies $|b_3| = 1$ and $|p_2| = 2$. So $|p_1| = 0$ and $|p_2| = 2$, which gives

$$\tilde{p}(z) = \frac{1 + wz^2}{1 - wz^2}, \quad |w| = 1$$

(see [8], Corollary 2.3). Also from $|b_2| = 0$ and $|b_3| = 1$ it follows elementarily (using again [8], Corollary 2.3) that

$$g(z) = \frac{z}{1 - xz^2}, \quad |x| = 1.$$

A computation shows now that equality implies $w = x$, and so $f(z) = h_\beta(xz)$. □

From Theorem 1 and Theorem 3 it follows with the triangle inequality

Corollary 2. *Let* $\beta \in [0, 1]$, $\lambda \in [2/3, 1]$. *Then*

$$\max_{f \in C(\beta)} |a_3 - \lambda a_2^2| = B_3 - \lambda B_2^2 = \tfrac{1}{3}(1 + 2\beta).$$

3. On successive coefficients. From the results of Sect. 2 it follows

Theorem 4. *Let* $\beta \geqq 1$, $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$. *Then*

$$||a_3| - |a_2|| \leqq A_3 - A_2 = \frac{\beta}{3}(1 + 2\beta).$$

Proof. At first we show

$$|a_3| - |a_2| \leqq \tfrac{1}{3}(2\beta^2 + \beta). \tag{18}$$

We use Corollary 1, choosing $\lambda = 1/3$, and get

$$\begin{aligned} |a_3| - |a_2| &\leqq |a_3 - \tfrac{1}{3}a_2^2| + \tfrac{1}{3}|a_2|^2 - |a_2| \\ &\leqq \tfrac{1}{3}(\beta^2 + 2\beta + 2) + \tfrac{1}{3}|a_2|^2 - |a_2| =: F(|a_2|). \end{aligned}$$

Because F defines a convex parabola, it takes its maximum at the boundary of its interval of definition. Furthermore the relation

$$F(|a_2|) = \tfrac{1}{3}(2\beta^2 + \beta)$$

implies that $|a_2| = 2 - \beta$ or $|a_2| = 1 + \beta$, so that

$$f(|a_2|) \leqq \tfrac{1}{3}(2\beta^2 + \beta) \quad \text{for} \quad |a_2| \in [2 - \beta, 1 + \beta]. \tag{19}$$

From this (18) follows if $\beta \geqq 2$, because $|a_2| \leqq 1 + \beta$ in $C(\beta)$. Let now $\beta \in [1, 2[$. Then with aid of Theorem 1 we have furthermore that

$$\begin{aligned} |a_3| - |a_2| &\leqq |a_3 - \tfrac{2}{3}a_2^2| + \tfrac{2}{3}|a_2|^2 - |a_2| \\ &\leqq \tfrac{1}{3}(1 + 2\beta) + \tfrac{2}{3}|a_2|^2 - |a_2| =: G(|a_2|). \end{aligned} \tag{20}$$

The same procedure as above shows that

$$G(|a_2|) \leqq \tfrac{1}{3}(2\beta^2 + \beta) \quad \text{for} \quad |a_2| \in [1 - \beta, \beta + \tfrac{1}{2}],$$

which, together with (19), gives (18).

Now we shall show that

$$|a_2| - |a_3| \leqq \tfrac{1}{3}(2\beta^2 + \beta),$$

which is trivially true if $|a_2| \in [0, (2\beta^2 + \beta)/3]$. This gives the result for $\beta \geqq (\sqrt{7} + 1)/2$. For $\beta < (\sqrt{7} + 1)/2$ let now lie $|a_2|$ in the remaining interval $[(2\beta^2 + \beta)/3, 1 + \beta]$. Then Theorem 3 gives

$$\begin{aligned} |a_2| - |a_3| = |a_2|^2 - |a_3| - |a_2|^2 + |a_2| &\leqq |a_2^2 - a_3| - |a_2|^2 + |a_2| \\ &\leqq \tfrac{1}{3}(\beta^2 + 2\beta) - |a_2|^2 + |a_2| =: H(|a_2|). \end{aligned}$$

H takes its global maximum at $|a_2| = 1/2$ which does not lie in the interval considered. Thus, for $|a_2| \in [(2\beta^2 + \beta)/3, 1 + \beta]$, H is decreasing, and it remains to show that

$$H((2\beta^2 + \beta)/3) \leqq \tfrac{1}{3}(2\beta^2 + \beta),$$

i.e. $3\beta^2 + 6\beta \leqq (2\beta^2 + \beta)^2$, which obviously holds for $\beta \geqq 1$. □

For $\beta < 1$ one gets

Theorem 5. *Let $\beta \leqq 1/2$, $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$. Then*

$$|a_3| - |a_2| \leqq B_3 - B_2 = \tfrac{1}{3}(1 + 2\beta).$$

Proof. Theorem 1 implies (20). Because $\beta \leqq 1/2$ we have $|a_2| \in [0, 3/2]$, and it follows easily that $G(|a_2|) \leqq (1 + 2\beta)/3$. □

We are not able to show that $|a_4| - |a_3| \leqq A_4 - A_3$ for $\beta \geqq 1$, but give a weaker result in this direction.

Theorem 6. *Let* $\beta \geqq 1, f(z) = z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \cdots \in C(\beta)$. *Then*

$$\big||a_4| - |a_2|\big| \leqq A_4 - A_2 = \frac{(1+\beta)}{3}(\beta^2 + 2\beta).$$

Proof. We use [2], Lemma, implying that with $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C(\beta)$ the function $h(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, defined by $h'(z) = (f'(z^2))^{1/2}$, $h(0) = 0$, is an odd close-to-convex function of order $\beta/2$. Now, because $\beta \geqq 1$, we can use the coefficient domination theorem for such functions [2], Theorem 1, and get

$$|b_7| = \frac{1}{7}\left|2a_4 - \frac{3}{2}a_2 a_3 + \frac{a_2^3}{2}\right| \leqq \frac{(1+\beta)}{42}(\beta^2 + 2\beta + 6). \tag{22}$$

If we consider b_5, we get once more Corollary 1 for $\lambda = 1/3$, namely

$$|b_5| = \frac{3}{10}\left|a_3 - \frac{1}{3}a_2^2\right| \leqq \frac{\beta^2 + 2\beta + 2}{10}. \tag{23}$$

Now we get with aid of (22) and (23) that

$$\begin{aligned}
|a_4| - |a_2| &\leqq \left|a_4 - \frac{3}{4}a_2 a_3 + \frac{a_2^3}{4}\right| + \frac{3}{4}|a_2|\left|a_3 - \frac{1}{3}a_2^2\right| - |a_2| \\
&\leqq \frac{(1+\beta)}{12}(\beta^2 + 2\beta + 6) + \frac{3}{4}|a_2|\frac{(\beta^2 + 2\beta + 2)}{3} - |a_2| \\
&\leqq \frac{(1+\beta)}{12}(\beta^2 + 2\beta + 6) + \frac{|a_2|}{4}(\beta^2 + 2\beta - 2).
\end{aligned}$$

Because $(\beta^2 + 2\beta - 2) \geqq 0$ it follows now from $|a_2| \leqq 1 + \beta$ that

$$|a_4| - |a_2| \leqq \frac{(1+\beta)}{3}(\beta^2 + 2\beta).$$

On the other hand, for $\beta \geqq 1$

$$|a_2| - |a_4| \leqq |a_2| \leqq \frac{(1+\beta)}{3} \cdot 3 \leqq \frac{(1+\beta)}{3}(\beta^2 + 2\beta)$$

is trivially true. □

4. Close-to-convex functions and the Schwarzian derivative. For $\sigma \geqq 0$ let $N(\sigma)$ denote the family of analytic and locally univalent functions of $\mathbb{D}$, which are normalized by (1) and have the property that

(24) $$s(f) := \sup_{z\in\mathbb{D}} (1-|z|^2)^2\,|S_f(z)| \leqq \sigma .$$

Here $S_f := \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ denotes the Schwarzian derivative of f.

The Nehari univalence criterion [4] states that $N(2) \subset S$. On the other hand it is easily seen that $S \subset N(6)$ (see e.g. [4]).

Note the well-known transformation property that if ω is an automorphism of $\mathbb{D}$, then

(25) $$(1-|z|^2)^2\,|S_{f\circ\omega}(z)| = (1-|\omega(z)|^2)^2\,|S_f(\omega(z))| .$$

The following result generalizes the statements $K = C(0) \subset N(2)$ (see [5]), and $C(1) \subset N(6)$.

Theorem 7. *Let $f \in C(\beta)$, and let $s(f)$ defined by* (24). *Then*

(26) $$s(f) \leqq \begin{cases} 2+4\beta & \text{if } \beta \leqq 1 \\ 2\beta^2+4\beta & \text{if } \beta \geqq 1 . \end{cases}$$

Equality holds for the functions f defined by

(27) $$f(0)=0,\ f'(z) = \begin{cases} \dfrac{(1+z^2)^\beta}{(1-z^2)^{\beta+1}} & \text{if } \beta \leqq 1 \\ \dfrac{(1+z)^\beta}{(1-z)^{\beta+2}} & \text{if } \beta \geqq 1 . \end{cases}$$

P r o o f. Let $f \in C(\beta)$ and $a \in \mathbb{D}$. Then the function $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots$, defined by

$$g(z) = \frac{f\left(\dfrac{z+a}{1+\bar{a}z}\right) - f(a)}{f'(a)\cdot(1-|a|^2)},$$

also lies in $C(\beta)$ (see e.g. [1]). An application of Theorem 3 gives

(28) $$|b_3 - b_2^2| \leqq \begin{cases} \frac{1}{3}(1+2\beta) & \text{if } \beta \leqq 1 \\ \frac{1}{3}(\beta^2+2\beta) & \text{if } \beta \geqq 1 \end{cases},$$

so that the relation

$$(1-|a|^2)^2\,S_f(a) = S_g(0) = 6(b_3 - b_2^2),$$

which is a consequence of (25), implies the result, because a was arbitrary. The functions f, defined by (27), give the sharp bounds in (28), and so in (26). □

References

[1] W. KOEPF, Close-to convex functions and linear-invariant families. Ann. Acad. Sci. Fenn. Ser. A. I. Math. **8**, 349–355 (1983).

[2] W. KOEPF, Coefficients of symmetric functions of bounded boundary rotation. To appear.

[3] W. KOEPF, On the Fekete-Szegö problem for close-to-convex functions. Proc. Amer. Math. Soc. **100**, to appear.
[4] Z. NEHARI, The Schwarzian derivative and schlicht functions. Bull. Amer. Math. Soc. **55**, 545–551 (1949).
[5] Z. NEHARI, A property of convex conformal maps, J. Anal. Math. **30**, 390–393 (1976).
[6] CH. POMMERENKE, On the coefficients of close-to-convex functions. Mich. Math. J. **9**, 259–269 (1962).
[7] CH. POMMERENKE, On close-to-convex analytic functions. Trans. Amer. Math. Soc. **114**, 176–186 (1965).
[8] Ch. POMMERENKE, Univalent functions. Göttingen 1975.
[9] S. Y. TRIMBLE, A coefficient inequality for convex univalent functions. Proc. Amer. Math. Soc. **48**, 266–267 (1975).

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