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On Two Conjectures of M. S. Robertson

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In this note we disprove two conjectures made by M. S. Robertson.

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In his work on certain subordination classes Robertson conjectured the following.

Conjecture (A): let

$$W(z,x) := \left(\frac{\left(\frac{1+z}{1-z}\right)^x - 1}{2xz}\right)^{1/2} = \sum_{n=0}^{\infty} B_n(x) z^n,$$

then $B_n(x) = \sum_{k=0}^{n} \alpha_k^{(n)} x^k$.

It is conjectured that $\alpha_k^{(n)} \geq 0$ $(k = 0, \ldots, n(n \in \mathbb{N}_0))$ [3].

Conjecture (B): let

$$S(z) = \left(\frac{e^z - 1}{z}\right)^{1/2} = \sum_{n=0}^{\infty} b_n z^n.$$

Then it is conjectured that $b_n \geq 0 (n \in \mathbb{N})$ ([4]).

In [4] Robertson proved the equivalence of Conjecture (A) and Conjecture (B).

Using Computer Algebra systems we are able to show that both conjectures fail to be true.

Let us consider Conjecture (B). It turns out that the first negative coefficient is $b_{13} = -20287103/43878270659198976000$. In detail one has

$$S(z) = 1 + \frac{1}{4}z + \frac{5}{96}z^2 + \frac{1}{128}z^3 + \frac{79}{92160}z^4 + \frac{3}{40960}z^5 + \frac{71}{12386304}z^6$$
$$+ \frac{113}{247726080}z^7 + \frac{3053}{118908518400}z^8 + \frac{1}{22649241600}z^9$$

$$+\frac{17}{930128855040}z^{10}+\frac{19}{744103084032}z^{11}+\frac{935917}{1218840851644416000}z^{12}$$
$$-\frac{20287103}{4387827065919897600}z^{13}-\frac{2452337}{21061569916415508480}z^{14}$$
$$+\frac{5053}{52100358482165760}z^{15}+\frac{141886453}{6547139448302878064640}z^{16}$$
$$-\frac{233110081}{113160434908938633216000}z^{17}$$
$$-\frac{55660769987}{135036261999581083824291840}z^{18}$$
$$+\frac{3128088241183}{70218856239782163588631756800}z^{19}+O(z^{20}).$$

These calculations had been done independently by the three Computer Algebra systems MACSYMA [1], MATHEMATICA [5] and MAPLE [2].

With MACSYMA (Version 415) one is also able to disprove Conjecture (A) directly. It turns out that

$$W(z,x)=1+\frac{x}{2}z+\left(\frac{1}{6}+\frac{5x^2}{24}\right)z^2+\left(\frac{x}{4}+\frac{x^3}{16}\right)z^3$$
$$+\left(\frac{31}{360}+\frac{25x^2}{144}+\frac{79x^4}{5760}\right)z^4+\left(\frac{41x}{240}+\frac{7x^3}{96}+\frac{3x^5}{1280}\right)z^5$$
$$+\left(\frac{863}{15120}+\frac{85x^2}{576}+\frac{79x^4}{3840}+\frac{71x^6}{193536}\right)z^6$$
$$+\left(\frac{3961x}{30240}+\frac{427x^3}{5760}+\frac{11x^5}{2560}+\frac{113x^7}{1935360}\right)z^7$$
$$+\left(\frac{76813}{1814400}+\frac{9355x^2}{72576}+\frac{5609x^4}{230400}+\frac{923x^6}{1161216}+\frac{3053x^8}{464486400}\right)z^8$$
$$+\left(\frac{129127x}{1209600}+\frac{503x^3}{6912}+\frac{297x^5}{51200}+\frac{113x^7}{774144}+\frac{x^9}{44236800}\right)z^9$$
$$+\left(\frac{572813}{17107200}+\frac{200201x^2}{1741824}+\frac{770803x^4}{29030400}\right.$$
$$\left.+\frac{11999x^6}{9953280}+\frac{51901x^8}{2786918400}+\frac{17x^{10}}{908328960}\right)z^{10}$$

$$+\left(\frac{7214563x}{79833600}+\frac{58567x^3}{829440}+\frac{14971x^5}{2150400}\right.$$
$$\left.+\frac{11413x^7}{46448640}+\frac{19x^9}{265420800}+\frac{19x^{11}}{363331584}\right)z^{11}$$

$$+\left(\frac{6016720439}{217945728000}+\frac{3988297x^2}{38320128}+\frac{4625213x^4}{165888000}+\frac{1540487x^6}{975421440}\right.$$
$$\left.+\frac{1920337x^8}{55738368000}+\frac{17x^{10}}{259522560}+\frac{935917x^{12}}{297568567296000}\right)z^{12}$$

$$+\left(\frac{102672775873x}{1307674368000}+\frac{3735911x^3}{54743040}+\frac{2031271x^5}{258048000}+\frac{2042249x^7}{5852528640}\right.$$
$$\left.\frac{2299x^9}{15925248000}+\frac{437x^{11}}{2179989504}-\frac{20287103x^{13}}{5356234211328000}\right)z^{13}$$

$$+\left(\frac{6128434777}{261534873600}+\frac{59857944737x^2}{627683696640}+\frac{62808871x^4}{2189721600}\right.$$
$$+\frac{224011177x^6}{117050572800}+\frac{74062727x^8}{1404606873600}+\frac{2227x^{10}}{15571353600}$$
$$\left.+\frac{935917x^{12}}{71416456151040}-\frac{2452337x^{14}}{128549621071187200}\right)z^{14}+O(z^{15})$$

from which one sees that the first negative coefficient is that one of x^{13} in the polynomial $B_{13}(x)$, i.e. $\alpha_{13}^{(13)}$.

This result is also available using MAPLE and MATHEMATICA. But these two languages are not similarly effective in this special situation than MACSYMA. (Often MACSYMA turns out to be slower than MATHEMATICA and MAPLE in solving the same problem. This is not true in our case.) MACSYMA's taylor command arrives at the above representation immediately, whereas MATHEMATICA's Series command and MAPLE's taylor commands do not simplify intermediate results.

This implies that the complexity of the calculation of the polynomials $B_n(x)$ is then exponential in the order n. In fact, after calculating the truncated Taylor series of W with MAPLE (SUN-3-UNIV-Version IV) we had to simplify each polynomial $B_n(x)$ separately. The non-simplified output for $B_n(x)$ roughly doubled increasing n by one and $B_{14}(x)$ finally consisted of more than 600,000 tokens, so that we had to apply MAPLE's simplify procedure separately to seven summands for $n = 14$ and three summands for $n = 13$. Finally this gave the result.

The situation is quite similar with MATHEMATICA (386-MS-DOS-Version 1.2), which ran out of memory when applying the command chain

```
w = ((((1 + z)/(1 - z))^x - 1)/(2*x*z))^(1/2)
Series[w, {z, 0, 15}]
Simplify[%]
```

and the intermediate result after the second step had a similar size as MAPLE's. On the other hand MATHEMATICA solved the composite command Simplify[Series [w,{z,0,15}]], but needed much more time than MACSYMA.

References

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