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Properties of q-holonomic functions

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In a similar manner as in the papers by W. Koepf, D. Schmersau, Spaces of functions satisfying simple differential equations, Konrad-Zuse-Zentrum Berlin (ZIB), Technical Report TR 94-2 (1994) and Salvy, B., Zimmermann, P., GFUN: A package for the manipulation of generating and holonomic functions in one variable, *ACM Transactions on Mathematical Software*, (1994), pp. 163–177, where explicit algorithms for finding the differential equations satisfied by holonomic functions were given, in this paper we deal with the space of the *q-holonomic* functions which are the solutions of linear q-differential equations with polynomial coefficients. The sum, product and the composition with power functions of q-holonomic functions are also q-holonomic and the resulting q-differential equations can be computed algorithmically.

Keywords: q-derivative; q-differential equation; Algorithm; Algebra of q-holonomic functions

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1. Preliminaries

The purpose of this paper is to continue the research exposed in Refs [7,8]. There, the authors discussed *holonomic* functions which are the solutions of homogeneous linear differential equations with polynomial coefficients.

In the present investigation, we consider a similar problem from the point of view of q-calculus. As general references for q-calculus see Refs [2,4]. We begin with a few definitions.

Let $q \in \mathbb{R}, q \neq 1$[#]. The q-complex number $[a]_q$ is given by

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}.$$

Of course

$$\lim_{q\to 1}[a]_q = a.$$

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#Actually, in all the algorithms developed, we will consider q as an indeterminate.

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The q-factorial $[n]_q$ of a positive integer n and the q-binomial coefficient are defined by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-Pochhammer symbol is given as

$$\begin{aligned} &(a;q)_0 = 1, \\ &(a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \quad k = 1, 2, \ldots, \\ &(a;q)_\infty = \lim_{k\to\infty}(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}) \quad (|q|<1) \end{aligned}$$

and

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty} \quad (|q|<1, \lambda \in \mathbb{C}).$$

The *q-derivative* of a function $f(x)$ is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx} \; (x \neq 0), \quad D_q f(0) := \lim_{x\to 0} D_q f(x), \tag{1}$$

and higher order q-derivatives are defined recursively

$$D_q^0 f := f, \quad D_q^n f := D_q D_q^{n-1} f, \quad n = 1, 2, 3, \ldots. \tag{2}$$

Of course, if f is differentiable at x, then

$$\lim_{q\to 1} D_q f(x) = f'(x).$$

The next four lemmas are well-known in q-calculus and their proofs can be found, for example, in [3,4].

LEMMA 1.1. *For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathbb{C}$ and $q \neq 1$, we have linearity and product rules*

$$\begin{aligned} D_q(\alpha u(x) + \beta v(x)) &= \alpha D_q u(x) + \beta D_q v(x), \\ D_q(u(x)\cdot v(x)) &= u(qx) D_q v(x) + v(x) D_q u(x) \\ &= u(x) D_q v(x) + v(qx) D_q u(x). \end{aligned}$$

LEMMA 1.2. *The Leibniz rule for the higher order q-derivatives of a product of functions is given as*

$$D_q^n(u(x)\cdot v(x)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^{n-k} u(q^k x) D_q^k v(x).$$

LEMMA 1.3. *For an arbitrary function $u(x)$ and for $t(x) = cx^k$ ($c \in \mathbb{C}, k \in \mathbb{N}, q^k \neq 1$) we have for the composition with $t(x)$*

$$D_q(u \circ t)(x) = D_{q^k} u(t) \cdot D_q t(x).$$

LEMMA 1.4. *The values of the function for the shifted argument and for higher q-derivatives are connected by the two relations:*

$$f(q^n x) = \sum_{k=0}^{n} (-1)^k (1-q)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x^k D_q^k f(x), \tag{3}$$

$$D_q^n f(x) = \frac{1}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^k x). \tag{4}$$

For our further work, it is useful to write the product rule in slightly different form.

LEMMA 1.5. *The product rule for the q-derivative can be written in the form*

$$D_q(u(x) \cdot v(x)) = u(x) D_q v(x) + v(x) D_q u(x) - (1-q) x D_q u(x) D_q v(x). \tag{5}$$

In the same manner, higher q-derivatives can be expressed by

$$D_q^n(u(x) \cdot v(x)) = \sum_{\nu=0}^{n} \sum_{\mu=0}^{n} \alpha_{\nu,\mu}^{(n)}(x) D_q^\nu u(x) D_q^\mu v(x),$$

where the coefficients $\alpha_{\nu,\mu}^{(n)}(x)$ are symmetric

$$\alpha_{\nu,\mu}^{(n)}(x) = \alpha_{\mu,\nu}^{(n)}(x) \quad (\nu, \mu = 1, \ldots, n)$$

and can be computed recursively:

$$\begin{aligned}
\alpha_{0,0}^{(n+1)}(x) &= 0, \\
\alpha_{0,n+1}^{(n+1)}(x) &= \alpha_{0,n}^{(n)}(qx), \\
\alpha_{n+1,n+1}^{(n+1)}(x) &= -(1-q) x \alpha_{n,n}^{(n)}(qx), \\
\alpha_{0,\mu}^{(n+1)}(x) &= D_q \alpha_{0,\mu}^{(n)}(x) + \alpha_{0,\mu-1}^{(n)}(qx), \\
\alpha_{n+1,\mu}^{(n+1)}(x) &= \alpha_{n,\mu}^{(n)}(qx) - (1-q) x \alpha_{n,\mu-1}^{(n)}(qx), \\
\alpha_{\nu,\mu}^{(n+1)}(x) &= D_q \alpha_{\nu,\mu}^{(n)}(x) + \alpha_{\nu-1,\mu}^{(n)}(qx) + \alpha_{\nu,\mu-1}^{(n)}(qx) - (1-q) x \alpha_{\nu-1,\mu-1}^{(n)}(qx),
\end{aligned}$$

with initial values

$$\alpha_{0,0}^{(1)} = 0, \quad \alpha_{0,1}^{(1)} = 1, \quad \alpha_{1,1}^{(1)} = -(1-q)x.$$

Let us finally recall that the q-hypergeometric series is given by Refs [2,6]

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} \middle| q, x\right) := \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{r} (a_j; q)_k}{\prod_{j=1}^{s} (b_j; q)_k} \frac{x^k}{(q;q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}.$$

2. On q-holonomic functions

For every function $f(x)$ which is a solution of a *polynomial homogeneous linear q-differential equation*

$$\sum_{k=0}^{n} \tilde{p}_k(x;f) D_q^k f(x) = 0 \quad (\tilde{p}_k \in \mathbb{K}(q)[x], n \in \mathbb{N}) \tag{6}$$

we say that $f(x)$ is a *q-holonomic function*. The smallest n such that $\tilde{p}_n \not\equiv 0$ is not the zero polynomial is called the *holonomic order* of $f(x)$. Here $\mathbb{K}$ is a field, typically $\mathbb{K} = \mathbb{Q}(a_1, a_2, \ldots)$ or $\mathbb{K} = \mathbb{C}(a_1, a_2, \ldots)$ where $a_1, a_2, \ldots$ denote some parameters. An equation of type (6) is called *a q-holonomic equation*.

Although the following examples of q-holonomic functions of first order are well-known, we state them with complete proofs so that the paper is self-contained.

Example 2.1. Since

$$D_q x^s = [s]_q x^{s-1} \quad (x, \alpha, s \in \mathbb{R}),$$

we have

$$f(x) = x^s \Rightarrow x D_q f(x) - [s]_q f(x) = 0,$$

or

$$(q-1) x D_q f(x) - (q^s - 1) f(x) = 0,$$

i.e. the power function is (for integer s) a q-holonomic function of first order.

Example 2.2. For $0 < |q| < 1$, $\lambda \in \mathbb{R}$, $x \neq 0, 1$, we have

$$D_q((x;q)_\lambda) = -[\lambda]_q (qx;q)_{\lambda-1} = \frac{-[\lambda]_q}{1-x} (x;q)_\lambda.$$

Hence

$$f(x) = (x;q)_\lambda \Rightarrow (x-1) D_q f(x) - [\lambda]_q f(x) = 0$$

or

$$(q-1)(x-1) D_q f(x) - (q^\lambda - 1) f(x) = 0.$$

Therefore, the q-Pochhammer symbol is (for integer λ) also q-holonomic of first order.

Similarly, from

$$D_q((x;q)_\infty) = -(1-q)^{-1} (qx;q)_\infty = -\frac{1}{1-q} \frac{1}{1-x} (x;q)_\infty,$$

we get

$$f(x) = (x;q)_\infty \Rightarrow (1-x) D_q f(x) + \frac{1}{1-q} f(x) = 0.$$

Example 2.3. The small q-exponential function

$$e_q(x) = {}_1\phi_0\left(\begin{matrix}0\\ -\end{matrix}\middle| q, x\right) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} x^n, \quad |x| < 1, 0 < |q| < 1, \tag{7}$$

has q-derivative

$$\begin{aligned}
D_q e_q(x) &= \frac{e_q(x) - e_q(qx)}{x - qx}\\
&= \frac{1}{x - qx}\left(\sum_{n=0}^{\infty} \frac{1}{(q;q)_n} x^n - \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}(qx)^n\right)\\
&= \frac{1}{x - qx}\sum_{n=0}^{\infty} \frac{x^n - (qx)^n}{(q;q)_n}\\
&= \frac{1}{x - qx}\left\{x + \sum_{n=2}^{\infty} \frac{1 - q^n}{(1-q)(1-q^2)\cdots(1-q^{n-1})(1-q^n)} x^n\right\}\\
&= \frac{x}{x - qx}\left\{1 + \sum_{k=1}^{\infty} \frac{1}{(1-q)(1-q^2)\cdots(1-q^k)} x^k\right\}\\
&= \frac{1}{1-q} e_q(x),
\end{aligned}$$

i.e. the small q-exponential function is q-holonomic of first order:

$$f(x) = e_q(x) \Rightarrow (1-q) D_q f(x) - f(x) = 0.$$

Note that this q-differential equation as well the resulting q-differential equations of the next four examples and similar ones can be obtained completely automatically by the `qsumdiffeq` command of the Maple package `qsum` by Böing and Koepf [1] using the q-version of Zeilberger's algorithm [6]. The above equation, e.g. is obtained using the q-hypergeometric representation (7) and the command

```
qsumdiffeq(1/qpochhammer(q,q,n)*x^n,q,n, f(x))}
```

Example 2.4. The big q-exponential function

$$E_q(x) = {}_0\phi_0\left(\begin{matrix}-\\ -\end{matrix}\middle| q, -x\right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n, \quad 0 < |q| < 1$$

has q-derivative

$$D_q E_q(x) = \frac{1}{x - qx}\left(\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} x^n - \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n}(qx)^n\right) = \frac{1}{1-q} E_q(qx).$$

which can be obtained in a similar way as in Example 2.3. Since

$$f(qx) = f(x) - (1-q)x(D_q f)(x),$$

we conclude that the big q-exponential function is also q-holonomic of first order:

$$f(x) = E_q(x) \Rightarrow (1-q)(x+1) D_q f(x) - f(x) = 0.$$

Example 2.5. For $0 < |q| < 1$, both the q-sine and q-cosine functions

$$\sin_q(x) = \frac{e_q(ix) - e_q(-ix)}{2i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q;q)_{2n+1}} x^{2n+1},$$

$$\cos_q(x) = \frac{e_q(ix) + e_q(-ix)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q;q)_{2n}} x^{2n},$$

satisfy

$$(1-q)^2 D_q^2 f(x) + f(x) = 0$$

and are therefore q-holonomic of second order.

Example 2.6. The q-hypergeometric series ${}_r\phi_s$ is q-holonomic. The `qsumdiffeq` command computes in particular for

$$f(x) = {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix} \middle| q, x\right)$$

the q-holonomic equation

$$\begin{aligned} 0 = {} & (xabq - c)x(q-1)^2 D_q^2 f(x) \\ & + (-xb - xa + 1 + xabq - c + xab)(q-1) D_q f(x) \\ & + (-1+a)(-1+b) f(x). \end{aligned}$$

Example 2.7. Most q-orthogonal polynomials are q-holonomic. The Big q-Jacobi polynomials (see e.g. [5], 3.5) are given by

$$f(x) = P_n(x; a, b, c; q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q, q\right).$$

They satisfy the q-holonomic equation

$$\begin{aligned} 0 = {} & q^n a(bqx - c)(q-1)^2 (1 - qx) D_q^2 f(x) \\ & + (q-1)(abq^{n+1} + abq^{2n+1}x + x - q^n a - q^n c - abq^{n+1}x - abq^{n+2}x \\ & + q^{n+1}ac) D_q f(x) + (q^n - 1)(abq^{n+1} - 1) f(x) \end{aligned}$$

which is again easily determined by the `qsumdiffeq` command. The following lemma will be the crucial tool for the investigations of the next section.

LEMMA 2.1. *If $f(x)$ is a function satisfying a holonomic equation* (6) *of order n, then the functions $D_q^l f(x)\,(l = n, n+1, \ldots)$ can be expressed as*

$$D_q^l f(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x;f) D_q^k f(x), \tag{8}$$

where $p_k^{(l)}(x)$ are rational functions defined by

$$p_k^{(l)}(x) = \begin{cases} \delta_{kl}, & 0 \le l < n-1, \\ -\frac{\tilde{p}_k(x)}{\tilde{p}_n(x)}, & l = n \\ p_{k-1}^{(l-1)}(qx) + D_q p_k^{(l-1)}(x) + p_{n-1}^{(l-1)}(qx) p_k^{(n)}(x), & l > n, \end{cases}$$

for $0 \le k \le n-1$ and 0 for other k's.

Proof. The representations (8) and the corresponding coefficients are evident by equation (6) for $l = 0, 1, \ldots, n$. By q-deriving and using Lemma 1.1, from

$$D_q^n f(x) = \sum_{k=0}^{n-1} p_k^{(n)}(x) D_q^k f(x)$$

we get

$$\begin{aligned} D_q^{n+1} f(x) &= \sum_{k=0}^{n-1} D_q\left(p_k^{(n)}(x) D_q^k f(x)\right) \\ &= \sum_{k=0}^{n-1} p_k^{(n)}(qx) D_q^{k+1} f(x) + \sum_{k=0}^{n-1} D_q\left(p_k^{(n)}(x)\right) D_q^k f(x) \\ &= \sum_{k=0}^{n-1} \left(p_{k-1}^{(n)}(qx) + D_q\left(p_k^{(n)}(x)\right) D_q^k f(x)\right) + p_{n-1}^{(n)}(x) D_q^n f(x) \\ &= \sum_{k=0}^{n-1} p_k^{(n+1)}(x) D_q^k f(x), \end{aligned}$$

with

$$p_k^{(n+1)}(x) = p_{k-1}^{(n)}(qx) + D_q p_k^{(n)}(x) + p_{n-1}^{(n)}(qx) p_k^{(n)}(x) \quad (0 \le k \le n-1).$$

Repeating the procedure, we get the representation and coefficients for arbitrary $l > n$. □

We finish this section by noticing that there are functions which are not q-holonomic.

LEMMA 2.2. *The exponential function $f(x) = a^x (a > 0, a \neq 1)$ is not q-holonomic.*

Proof. Taking successive q-derivatives of $f(x) := a^x$ up to order n generates iteratively the functions of the list $L := \{a^x, a^{qx}, a^{q^2x}, \ldots, a^{q^nx}\}$. Since the members of L are linearly independent over $\mathbb{K}(q)[x]$ (by mathematical induction), and since L contains $n+1$ elements, no q-holonomic equation for $f(x)$ of order n exists. □

3. Operations with q-holonomic functions

In this section, we will formulate and prove a few theorems about q-holonomic functions provided by derivation, addition or multiplication of the given q-holonomic functions.

THEOREM 3.1. *If $f(x)$ is a q-holonomic function of order n, then the function $h_m(x) = D_q^m f(x)$ is a q-holonomic function of order at most n for every $m \in \mathbb{N}$.*

Proof. If we prove the statement for $m = 1$, the final conclusion follows by mathematical induction.

Let $h(x) = D_q f(x)$, where the function $f(x)$ satisfies (6). If $\tilde{p}_0(x) \equiv 0$ is the zero polynomial, then obviously $h(x)$ is a q-holonomic function of order $n-1$.

Hence, let $\tilde{p}_0(x) \not\equiv 0$. Then, by Lemma 2.1, we have

$$D_q^n f(x) = \sum_{k=0}^{n-1} p_k^{(n)}(x) D_q^k f(x),$$

wherefrom

$$\begin{aligned} f(x) &= \frac{1}{p_0^{(n)}(x)} \left(D_q^n f(x) - \sum_{k=1}^{n-1} p_k^{(n)}(x) D_q^k f(x) \right) \\ &= \frac{1}{p_0^{(n)}(x)} \left(D_q^{n-1} h(x) - \sum_{k=0}^{n-2} p_{k+1}^{(n)}(x) D_q^k h(x) \right). \end{aligned}$$

Also, by q-deriving, we get

$$\begin{aligned} D_q^n h(x) \quad &= D_q^{n+1} f(x) = \sum_{k=0}^{n-1} p_k^{(n+1)}(x) D_q^k f(x) \quad = p_0^{(n+1)}(x) f(x) + \sum_{k=1}^{n-1} p_k^{(n+1)}(x) D_q^{k-1} h(x) \\ &= \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)} \left(D_q^{n-1} h(x) - \sum_{k=0}^{n-2} p_{k+1}^{(n)}(x) D_q^k h(x) \right) + \sum_{k=0}^{n-2} p_{k+1}^{(n+1)}(x) D_q^k h(x). \end{aligned}$$

Hence,

$$D_q^n h(x) = \sum_{k=0}^{n-1} P_k(x;h) D_q^k h(x),$$

where

$$P_k(x;h) = p_{k+1}^{(n+1)}(x) - \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)} p_{k+1}^{(n)}(x), \quad k = 0, 1, \ldots n-2, \quad P_{n-1}(x;h) = \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)}.$$

By multiplying with the common denominator of the rational functions $\{P_k(x;h), k = 0, 1, \ldots, n-1\}$, we can conclude that $h(x)$ satisfies the equation

$$\sum_{k=0}^{n} \tilde{p}_k(x;h) D_q^k h(x) = 0,$$

i.e. $h(x) = D_q f(x)$ is a q-holonomic function of order $\leq n$. □

We note that the proof of Theorem 3.1 provides an (iterative) algorithm to compute the corresponding q-differential equation for $D_q^m f(x)$.

Example 3.1. In Example 2.2, for the q-Pochhammer symbol we proved that it satisfies

$$f(x) = (x;q)_\infty \Rightarrow (1-x)D_q f(x) + \frac{1}{1-q} f(x) = 0.$$

Hence, we have

$$h_m(x) = D_q^m((x;q)_\infty) \Rightarrow (1-q^m x)D_q h_m(x) + \frac{q^m}{1-q} h_m(x) = 0 \quad (m \in \mathbb{N}_0).$$

THEOREM 3.2. *If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, then the function $u(x)+v(x)$ is q-holonomic of order at most $m+n$.*

Proof. If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, they satisfy holonomic equations

$$\sum_{k=0}^{n} \tilde{p}_k(x) D_q^k u(x) = 0, \qquad \sum_{j=0}^{m} \tilde{r}_j(x) D_q^j v(x) = 0, \tag{9}$$

where $\tilde{p}_k(x)$ and $\tilde{r}_j(x)$ are polynomials and $\tilde{p}_n \not\equiv 0$, $\tilde{r}_m \not\equiv 0$. According to Lemma 2.1, $D_q^l u(x)$ and $D_q^l v(x)$ can be represented as

$$D_q^l u(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) D_q^k u(x), \qquad D_q^l v(x) = \sum_{j=0}^{m-1} r_j^{(l)}(x) D_q^j v(x), \tag{10}$$

where $p_k^{(l)}(x)$ and $r_j^{(l)}(x)$ are rational functions given by Lemma 2.1.

Let $h(x) = u(x) + v(x)$. Then, according to (10), we have

$$D_q^l h(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) D_q^k u(x) + \sum_{j=0}^{m-1} r_j^{(l)}(x) D_q^j v(x), \quad l = 0, 1, \ldots, m+n. \tag{11}$$

Taking the values for $l = 0, 1, \ldots, m+n-1$ in the above identities and expressing q-derivatives of $u(x)$ and $v(x)$ by q-derivatives of $h(x)$, we get

$$D_q^k u(x) = \sum_{l=0}^{m+n-1} a_k^{(l)}(x) D_q^l h(x), \quad k = 0, 1, \ldots, n-1,$$

$$D_q^j v(x) = \sum_{l=0}^{m+n-1} b_j^{(l)}(x) D_q^l h(x), \quad j = 0, 1, \ldots, m-1.$$

By eliminating $D_q^k u(x)$ $(k = 0, 1, \ldots, n-1)$ and $D_q^j v(x)$ $(j = 0, 1, \ldots, m-1)$ from the last identity $(l = m+n)$ of (11), we get

$$D_q^{m+n} h(x) = \sum_{l=0}^{m+n-1} c_l(x) D_q^l h(x),$$

where

$$c_l(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) a_k^{(l)}(x) + \sum_{j=0}^{m-1} r_j^{(l)}(x) b_j^{(l)}(x).$$

By multiplying with the common denominator of $\{c_l(x), l = 0, 1, \ldots m + n - 1\}$, we get the holonomic equation for $h(x)$

$$\sum_{l=0}^{m+n} \tilde{c}_l(x) D_q^l h(x) = 0.$$

This proves that the q-holonomic order of $u(x) + v(x)$ is at most $m + n$, but can be less. □

Note that the algorithm given in the proof of Theorem 3.2 finds a q-differential equation which is not only valid for $u(x) + v(x)$, but also for every linear combination $\lambda_1 u(x) + \lambda_2 v(x)$, in particular for $u(x) - v(x)$. An iterative version of the given algorithm will determine the q-holonomic equation of lowest order for $u(x) + v(x)$.

Example 3.2. The small q-exponential function from Example 2.3 is q-holonomic of first order and satisfies

$$u(x) = e_q(x) \Rightarrow D_q^k u(x) = \frac{1}{(1-q)^k} u(x) \quad (k = 0, 1, \ldots).$$

Also, the q-sine from Example 2.5 is q-holonomic of second order and satisfies

$$v(x) = \sin_q(x) \Rightarrow D_q^{k+2} v(x) = \frac{-1}{(1-q)^2} D_q^k v(x) \quad (k = 0, 1, \ldots).$$

Now, by the algorithm given in the proof of Theorem 3.2, the function $h(x) = u(x) + v(x)$ satisfies

$$D_q^3 h(x) = \frac{1}{1-q} D_q^2 h(x) - \frac{1}{(1-q)^2} D_q h(x) + \frac{1}{(1-q)^3} h(x).$$

i.e. it is q-holonomic of third order.

THEOREM 3.3. *If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, then the function $u(x) \cdot v(x)$ is q-holonomic of order at most $m \cdot n$.*

Proof. If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, they satisfy holonomic equations (9), and their q-derivatives (10).

Let $h(x) = u(x) \cdot v(x)$. Then, according to (1.5), we have

$$\begin{aligned} D_q^l h(x) &= \sum_{\nu=0}^{l} \sum_{\mu=0}^{l} \alpha_{\nu\mu}^{(l)}(x) D_q^\nu u(x) D_q^\mu v(x) \\ &= \sum_{\nu=0}^{l} \sum_{\mu=0}^{l} \alpha_{\nu\mu}^{(l)}(x) \left(\sum_{k=0}^{n-1} p_k^{(\nu)}(x) D_q^k u(x) \right) \left(\sum_{j=0}^{m-1} r_j^{(\mu)}(x) D_q^j v(x) \right), \end{aligned}$$

i.e.

$$D_q^l h(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \beta_{kj}^{(l)}(x) D_q^k u(x) D_q^j v(x) \qquad (l = 0, 1, \ldots, mn), \tag{12}$$

where

$$\beta_{kj}^{(l)}(x) = \sum_{\nu=0}^{l} \sum_{\mu=0}^{l} \alpha_{\nu\mu}^{(n)}(x) p_k^{(\nu)}(x) r_j^{(\mu)}(x).$$

Taking the relations (12) $l = 0, 1, \ldots, mn - 1$ and expressing the q-derivatives $D_q^k u(x) D_q^j v(x)$ by q-derivatives of $h(x)$, we get

$$D_q^k u(x) D_q^j v(x) = \sum_{l=0}^{mn-1} \gamma_{kj}^{(l)}(x) D_q^l h(x) \qquad (0 \le k \le n-1; 0 \le j \le m-1).$$

Eliminating all the products $D_q^k u(x) D_q^j v(x)$ from the last identity ($l = mn$) of (12), it becomes

$$D_q^{mn} h(x) = \sum_{l=0}^{mn-1} \sigma_l(x) D_q^l h(x),$$

where

$$\sigma_l(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \beta_{kj}^{(l)}(x) \gamma_{kj}^{(l)}(x).$$

By multiplying with the common denominator of $\{\sigma_l(x), l = 0, 1, \ldots mn - 1\}$, we get the q-holonomic equation for $h(x)$

$$\sum_{l=0}^{mn} \tilde{\sigma}_l(x) D_q^l h(x) = 0.$$

This proves that the q-holonomic order of $u(x) \cdot v(x)$ is at most mn, but can be less. □

Again, the proof of Theorem 3.3 provides an algorithm. An iterative version of the given algorithm will determine the q-holonomic equation of lowest order for $u(x) \cdot v(x)$.

Example 3.3. We use again $u(x) = e_q(x)$ and $v(x) = \sin_q(x)$. Now, by the given algorithm the function $h(x) = u(x) \cdot v(x)$ satisfies

$$(1-q)^2 D_q^2 h(x) - (1-q^2) D_q h(x) + (qx^2 - (1+q)(x-1)) h(x) = 0,$$

i.e. it is q-holonomic of second order.

THEOREM 3.4. *If $u(x)$ is a q-holonomic function of order n, then the function $w(x) = u(x^\nu)$ ($\nu \in \mathbb{N}$) is a q-holonomic function of order at most n.*

Proof. By assumption $u(t)$ satisfies a q-holonomic equation

$$\sum_{k=0}^{n} \tilde{p}_k(t) D_q^k u(t) = 0, \tag{13}$$

where $\tilde{p}_k(t)$ are polynomials and $\tilde{p}_n \not\equiv 0$. Then, by Lemma 2.1, $D_q^l u(t)$ can be represented as

$$D_q^l u(t) = \sum_{k=0}^{n-1} p_k^{(l)}(t) D_q^k u(t), \tag{14}$$

where $p_k^{(l)}(t)$ are rational functions determined by that lemma.

Let $t = x^\nu$. Using Lemma 1.3, we have

$$D_q w(x) = D_{q^\nu} u(t) D_q(x^\nu) = \frac{u(t) - u(q^\nu t)}{(1 - q^\nu)t} [\nu]_q x^{\nu - 1}.$$

According to (4), we get

$$D_q w(x) = \sum_{j=1}^{\nu} e_{j,\nu}(x) D_q^j u(t),$$

where

$$e_{j,\nu}(x) = (-1)^{j-1} (1-q)^{j-1} \begin{bmatrix} \nu \\ j \end{bmatrix}_q q^{\binom{j}{2}} x^{\nu j - 1}, \quad j = 1, 2, \ldots, \nu. \tag{15}$$

By (14), we can write

$$D_q w(x) = \sum_{j=1}^{\nu} e_{j,\nu}(x) \sum_{k=0}^{n-1} p_k^{(j)}(t) D_q^k u(t) = \sum_{k=0}^{n-1} f_{k,\nu}^{(1)}(x) D_q^k u(t),$$

where

$$f_{k,\nu}^{(1)}(x) = \sum_{j=1}^{\nu} p_k^{(j)}(x^\nu) e_{j,\nu}(x), \quad k = 0, 1, \ldots, n-1. \tag{16}$$

Furthermore,

$$D_q^2 w(x) = \sum_{k=0}^{n-1} D_q \left(f_{k,\nu}^{(1)}(x) D_q^k u(t) \right) = \sum_{k=0}^{n-1} D_q f_{k,\nu}^{(1)}(x) D_q^k u(t) + \sum_{k=0}^{n-1} f_{k,\nu}^{(1)}(qx) D_q \left(D_q^k u(t) \right).$$

As before, the second sum in the above term can be transformed to

$$\begin{aligned} \sum_{i=0}^{n-1} f_{i,\nu}^{(1)}(qx) D_q \left(D_q^i u(t) \right) &= \sum_{i=0}^{n-1} f_{i,\nu}^{(1)}(qx) \sum_{j=1}^{\nu} e_{j,\nu}(x) D_q^j \left(D_q^i u(t) \right) \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) D_q^{i+j} u(t) \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) \sum_{k=0}^{n-1} p_k^{(i+j)}(t) D_q^k u(t). \end{aligned}$$

Hence,

$$D_q^2 w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(2)}(x) D_q^k u(t),$$

where

$$f_{k,\nu}^{(2)}(x) = D_q f_{k,\nu}^{(1)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) p_k^{(i+j)}(x^{\nu}), \quad k = 0, 1, \ldots, n-1.$$

By induction, we obtain the representations

$$D_q^l w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x) D_q^k u(t), \quad l = 0, 1, 2, \ldots, n \tag{17}$$

where $f_{k,\nu}^{(0)}(x) = \delta_{k0}, f_{k,\nu}^{(1)}(x)$ is given in (16) and

$$f_{k,\nu}^{(l)}(x) = D_q f_{k,\nu}^{(l-1)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(l-1)}(qx) e_{j,\nu}(x) p_k^{(i+j)}(x^{\nu}). \tag{18}$$

Taking the first n of the identities (17), we can determine

$$D_q^k u(t) = \sum_{l=0}^{n-1} b_{l,\nu}^{(k)}(x) D_q^l w(x), \quad k = 0, 1, \ldots, n-1,$$

where $b_{l,\nu}^{(k)}(x)$ are rational functions. Substituting this in identity (17), we get

$$D_q^n w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x) \sum_{l=0}^{n-1} b_{l,\nu}^{(k)}(x) D_q^l w(x) = \sum_{l=0}^{n-1} c_{l,\nu}(x) D_q^l w(x),$$

where

$$c_{l,\nu}(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x) b_{l,\nu}^{(k)}(x).$$

By multiplying with the common denominator of $\{c_{l,\nu}(x), l = 0, 1, \ldots, n-1\}$, we obtain

$$\sum_{l=0}^{n} \tilde{c}_{l,\nu}(x) D_q^l w(x) = 0.$$

□

Example 3.4. In Example 2.2, it was proved that

$$u(x) = (x; q)_\lambda \Rightarrow (q-1)(x-1) D_q u(x) - (q^\lambda - 1) u(x) = 0.$$

Using our algorithm we get for $w(x) = u(x^2) = (x^2; q)_\lambda$ the q-holonomic equation

$$(q-1)(x-1)(x+1)(x^2 q - 1) D_q w(x) - x(q^\lambda - 1)(x^2 q^{\lambda+1} - q - 1 + x^2 q) f(x) = 0$$

and similar, but more complicated, equations for $(x^\nu; q)_\lambda$ for higher $\nu \in \mathbb{N}$.

Example 3.5. In Example 2.5, for the q-sine function, we got

$$u(x) = \sin_q(x) \Rightarrow (1-q)^2 D_q^2 u(x) + u(x) = 0.$$

Now, for $w(x) = u(x^2)$, we have

$$D_q w(x) = f_{0,2}^{(1)}(x) u(t) + f_{1,2}^{(1)}(x) D_q u(t),$$

with

$$f_{0,2}^{(1)}(x) = \frac{qx^3}{1-q}, \qquad f_{1,2}^{(1)}(x) = (1+q)x$$

and

$$D_q^2 w(x) = f_{0,2}^{(2)}(x) u(t) + f_{1,2}^{(2)}(x) D_q u(t),$$

with

$$f_{0,2}^{(2)}(x) = \frac{(qx)^2(-2-q-q^2+q^3x^4)}{(1-q)^2} \qquad f_{1,2}^{(2)}(x) = \frac{(1+q)(1-q+q^2(1+q^2)x^4)}{1-q}.$$

By eliminating $D_q u(t)$, we get

$$D_q^2 w(x) = c_{0,2}(x) w(x) + c_{1,2}(x) D_q w(x),$$

wherefrom we get for the function $w(x) = u(x^2)$ the following equation

$$xD_q^2 w(x) - \left(1 + q^2 \frac{1+q^2}{1-q} x^4\right) D_q w(x) + qx^3 \left(\frac{1-q^4}{(1-q)^3} + \frac{q^2}{(1-q)^2} x^4\right) w(x) = 0.$$

4. Sharpness of the algorithms

In the previous section we proved that the sum, product and composition with powers of q-holonomic functions are q-holonomic too. In this section we show that the given bounds for the orders are sharp in all algorithms considered.

Example 4.1. The functions $u(x) = x^2$ and $v(x) = x^3$ are q-holonomic of first order. According to Theorem 3.2, the function $h(x) = u(x) + v(x)$ is q-holonomic of order at most two. However, all polynomials are q-holonomic functions of first order, and we find that $h(x)$ satisfies the equation

$$x(1+x) D_q h(x) - ([2]_q + [3]_q x) h(x) = 0.$$

This example shows that the order of the sum of some q-holonomic functions can be strictly less than the sum of their orders. This applies if the two functions $u(x)$ and $v(x)$ are linearly dependent over $\mathbb{K}(q)(x)$.

However, we will prove that for every algorithm given in the previous section there are functions for which the maximal order is attained.

LEMMA 4.1. *The functions $E_q(x^\mu)$ ($\mu = 1, 2, \ldots, n$) are linearly independent over $\mathbb{K}(q)(x)$.*

Proof. Let us consider a linear combination

$$r_1 E_q(x) + r_2 E_q(x^2) + \cdots + r_n E_q(x^n) = 0,$$

where $r_\mu = r_\mu(x)$ ($\mu = 1, 2, \ldots, n$) are rational functions and suppose that $r_\nu \not\equiv 0$. Then,

$$r_\nu E_q(x^\mu) = -\sum_{\substack{\mu=0,\\ \mu\neq\nu}}^{n} r_\mu E_q(x^\mu),$$

i.e.

$$\sum_{\substack{\mu=0,\\ \mu\neq\nu}}^{n} \frac{r_\mu}{r_\nu} \frac{E_q(x^\mu)}{E_q(x^\nu)} = -1. \tag{19}$$

Since

$$A(m) = \lim_{x\to\infty} \frac{\sum_{n=0}^{m} \frac{q^{\binom{n}{2}}}{(q;q)_n}(x^\mu)^n}{\sum_{n=0}^{m} \frac{q^{\binom{n}{2}}}{(q;q)_n}(x^\nu)^n} = \lim_{x\to\infty} x^{m(\mu-\nu)} = \begin{cases} +\infty, & \mu > \nu, \\ 0, & \mu < \nu, \end{cases}$$

we have

$$\lim_{x\to\infty} \frac{E_q(x^\mu)}{E_q(x^\nu)} = \lim_{m\to\infty} A(m) = \begin{cases} +\infty, & \mu > \nu, \\ 0, & \mu < \nu. \end{cases}$$

This is a contradiction with (19). Hence, it follows that $r_\mu \equiv 0$ for all $\mu = 1, 2, \ldots, n$, i.e. $E_q(x^\mu)$ ($\mu = 1, 2, \ldots, n$) are linearly independent over $\mathbb{K}(q)[x]$. □

LEMMA 4.2. *The function*

$$F_n(x) = \sum_{\mu=1}^{n} E_q(x^\mu) \tag{20}$$

is q-holonomic of order n.

Proof. The function $E_q(x)$ satisfies the q-holonomic equation of first order (see Example 2.4)

$$(1 - q)(x + 1) D_q f(x) - f(x) = 0.$$

With respect to Theorem 3.4, for each $\mu \in \mathbb{N}$, the function $E_q(x^\mu)$ is q-holonomic of first order and one has

$$D_q^l(E_q(x^\mu)) = f_{0,\mu}^{(l)}(x) E_q(x^\mu), \quad l = 0, 1, \ldots, \tag{21}$$

where $f_{0,\mu}^{(l)}(x)$ are rational functions given as in (18).

According to Theorem 3.2, the function $F_n(x)$ is q-holonomic of order at most n. Therefore

$$D_q^l F_n(x) = \sum_{\mu=1}^{n} D_q^l(E_q(x^\mu)) = \sum_{\mu=1}^{n} f_{0,\mu}^{(l)}(x) E_q(x^\mu).$$

Let us suppose that the function $F_n(x)$ satisfies a q-holonomic equation of order m, i.e.

$$D_q^m F_n(x) + \sum_{i=0}^{m-1} A_i D_q^i F_n(x) = 0. \tag{22}$$

This equation can be represented in the form

$$\sum_{\mu=1}^{n} \left(f_{0,\mu}^{(m)}(x) + \sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) \right) E_q(x^\mu) = 0.$$

Since $E_q(x^\mu)$ ($\mu = 1, 2, \ldots, n$) are linearly independent over $\mathbb{K}(q)[x]$, it follows that

$$f_{0,\mu}^{(m)}(x) + \sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) = 0, \quad \mu = 1, 2, \ldots, n.$$

This can be written in the form of the system of equations

$$\sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) = -f_{0,\mu}^{(m)}(x), \quad \mu = 1, 2, \ldots, n$$

with unknown rational functions $A_i = A_i(x)$.

If $m < n$, then the system is overdetermined and has no solution. Hence it follows that $m = n$. □

Note that similar results as in Lemmas 4.1 and 4.2 hold for the small q-exponential function.

Using the functions (20) of Lemma 4.2, we get the following conclusions.

THEOREM 4.3. *For each $n \in \mathbb{N}$ there is a function F which is q-holonomic of order n, such that $H = D_q F$ is q-holonomic of order n.*

Proof. The function defined by (20) satisfies the statement. □

THEOREM 4.4. *For each $n, m \in \mathbb{N}$ there are functions U and V that are q-holonomic of order n and m respectively, such that $H = U + V$ is q-holonomic of order $n + m$.*

Proof. Consider the functions

$$U(x) = \sum_{\mu=1}^{n} E_q(x^\mu) \quad \text{and} \quad V(x) = \sum_{\mu=n+1}^{n+m} E_q(x^\mu). \tag{23}$$

According to Lemma 4.2, they are q-holonomic of order n and m respectively, and the function

$$H(x) = U(x) + V(x) = \sum_{\mu=1}^{n+m} E_q(x^{\mu})$$

is q-holonomic of order $n + m$. □

THEOREM 4.5. *For each $n, m \in \mathbb{N}$ there are functions U and V that are q-holonomic of order n and m respectively, such that $H = U \cdot V$ is q-holonomic of order $n \cdot m$.*

Proof. The statement is valid for the functions defined by (23), because in the function

$$H(x) = U(x) \cdot V(x) = \sum_{\mu=1}^{n} \sum_{\nu=n+1}^{n+m} E_q(x^{\mu}) E_q(x^{\nu})$$

there are nm linearly independent summands $E_q(x^{\mu})E_q(x^{\nu})$ $(\mu = 1, 2, \ldots, n;\ \nu = n+1, n+2, \ldots, n+m)$ over $\mathbb{K}(q)[x]$. The proof of their independence is again based on Lemma 4.1. □

THEOREM 4.6. *For each $n \in \mathbb{N}$ there is a function F which is q-holonomic of order n, such that $W(x) = F(x^{\nu})$ is q-holonomic of order n.*

Proof. Starting from the function $F_n(x)$ defined by (20), we can form

$$W(x) = F_n(x^{\nu}) = \sum_{\mu=1}^{n} E_q(x^{\mu\nu})$$

which is of the same type as $F_n(x)$. □

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