SOME SUMMATION THEOREMS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. Essentially whenever a generalized hypergeometric series can be summed in terms of gamma functions, the result will be important as only a few such summation theorems are available in the literature. In this paper, we apply two identities of generalized hypergeometric series in order to extend some classical summation theorems of hypergeometric functions such as Gauss, Kummer, Dixon, Watson, Whipple, Pfaff-Saalschutz and Dougall formulas and also obtain some new summation theorems in the sequel.

1. Introduction

Let $\mathbb{R}$ and $\mathbb{C}$ denote the sets of real and complex numbers and z be a complex variable. For real or complex parameters a and b, the generalized binomial coefficient

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \begin{pmatrix} a \\ a-b \end{pmatrix} \qquad (a, b \in \mathbb{C}),$$

in which

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx,$$

denotes the well-known gamma function for $Re(z) > 0$, can be reduced to the particular case

$$\begin{pmatrix} a \\ n \end{pmatrix} = \frac{(-1)^n (-a)_n}{n!},$$

where $(a)_b$ denotes the Pochhammer symbol [5] given by

$$(a)_b = \frac{\Gamma(a+b)}{\Gamma(a)} = \begin{cases} 1 & (b = 0,\, a \in \mathbb{C}\backslash\{0\}), \\ a(a+1)...(a+n-1) & (b \in \mathbb{N},\, a \in \mathbb{C}). \end{cases} \tag{1.1}$$

Based upon Pochhammer's symbol (1.1), the generalized hypergeometric functions [13]

$${}_pF_q\left(\begin{matrix} a_1, & ... & a_p \\ b_1, & ... & b_q \end{matrix} \,\middle|\, z \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k ... (a_p)_k}{(b_1)_k ... (b_q)_k} \frac{z^k}{k!}, \tag{1.2}$$

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are indeed a Taylor series expansion for a function, say f, as $\sum_{k=0}^{\infty} c_k^* z^k$ with $c_k^* = f^{(k)}(0)/k!$ for which the ratio of successive terms can be written as

$$\frac{c_{k+1}^*}{c_k^*} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}.$$

According to the ratio test [1, 2], the series (1.2) is convergent for any $p \le q+1$. In fact, it converges in $|z| < 1$ for $p = q+1$, converges everywhere for $p < q+1$ and converges nowhere ($z \ne 0$) for $p > q+1$. Moreover, for $p = q+1$ it absolutely converges for $|z| = 1$ if the condition

$$A^* = \mathrm{Re}\left(\sum_{j=1}^{q} b_j - \sum_{j=1}^{q+1} a_j\right) > 0,$$

holds and is conditionally convergent for $|z| = 1$ and $z \ne 1$ if $-1 < A^* \le 0$ and is finally divergent for $|z| = 1$ and $z \ne 1$ if $A^* \le -1$.

There are two important cases of the series (1.2) arising in many physics problems [6, 10]. The first case (convergent in $|z| \le 1$) is the Gauss hypergeometric function

$$y = {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| \, z\right) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

with the integral representation

$$(1.3)\quad {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| \, z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt,$$
$$(\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ |\arg(1-z)| < \pi).$$

Replacing $z = 1$ in (1.3) directly leads to the well-known Gauss identity [5]

$$(1.4)\quad {}_2F_1\left(\begin{array}{c} a,\ b \\ c \end{array} \middle| \, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \qquad \mathrm{Re}(c-a-b) > 0\,.$$

The second case, which converges everywhere, is the Kummer confluent hypergeometric function

$$y = {}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| \, z\right) = \sum_{k=0}^{\infty} \frac{(b)_k}{(c)_k} \frac{z^k}{k!},$$

with the integral representation

$${}_1F_1\left(\begin{array}{c} b \\ c \end{array} \middle| \, z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}e^{zt}dt,$$
$$(\mathrm{Re}\, c > \mathrm{Re}\, b > 0;\ |\arg(1-z)| < \pi).$$

Essentially whenever a generalized hypergeometric series can be summed in terms of gamma functions, the result will be important as only a few such summation theorems are available in the literature, see e.g. [3, 4, 7, 8, 9, 11, 14]. In this sense, the classical summation theorems such as Kummer and Gauss for ${}_2F_1$, Dixon, Watson, Whipple and Pfaff-Saalschutz for ${}_3F_2$, Whipple for ${}_4F_3$, Dougall for ${}_5F_4$ and Dougall for ${}_7F_6$ are well known [5, 12]. In this paper, we apply two identities of generalized hypergeometric functions in order to obtain some new summation theorems and extend the above-mentioned classical theorems. For this purpose, we should first recall the classical theorems as follows.

* **Kummer theorem** [5, p. 108]:

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ 1+a-b \end{matrix} \,\middle|\, -1\right) = \frac{\Gamma(1+a-b)\Gamma(1+(a/2))}{\Gamma(1-b+(a/2))\Gamma(1+a)}. \tag{1.5} $$

* **Second Gauss theorem** [5, p. 108]:

$$ {}_2F_1\left(\begin{matrix} a,\ b \\ (a+b+1)/2 \end{matrix} \,\middle|\, \frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma((a+b+1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)}. \tag{1.6} $$

* **Bailey theorem** [5, p. 108]:

$$ {}_2F_1\left(\begin{matrix} a,\ 1-a \\ b \end{matrix} \,\middle|\, \frac{1}{2}\right) = \frac{\Gamma(b/2)\Gamma((b+1)/2)}{\Gamma((a+b)/2)\Gamma((b-a+1)/2)}. \tag{1.7} $$

* **Dixon theorem** [5, p. 108]:

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ 1+a-b,\ 1+a-c \end{matrix} \,\middle|\, 1\right) = \frac{\Gamma(1+a/2)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1-b-c+a/2)}{\Gamma(1+a)\Gamma(1-b+a/2)\Gamma(1-c+a/2)\Gamma(1+a-b-c)}. \tag{1.8} $$

* **Watson theorem** [5, p. 108]:

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ c \\ (a+b+1)/2,\ 2c \end{matrix} \,\middle|\, 1\right) = \frac{\sqrt{\pi}\,\Gamma(1+c/2)\Gamma((a+b+1)/2)\Gamma(c-(a+b-1)/2)}{\Gamma((a+1)/2)\Gamma((b+1)/2)\Gamma(c-(a-1)/2)\Gamma(c-(b-1)/2)}. \tag{1.9} $$

* **Whipple theorem** [5, p. 108]:

$$ \begin{aligned} &{}_3F_2\left(\begin{matrix} a,\ 1-a,\ c \\ c,\ 2b-c+1 \end{matrix} \,\middle|\, 1\right) \\ &\quad = \frac{\pi\, 2^{1-2b}\Gamma(c)\Gamma(2b-c+1)}{\Gamma((a+c)/2)\Gamma(b+(a-c+1)/2)\Gamma((1-a+c)/2)\Gamma(b+1-(a+c)/2)}. \end{aligned} \tag{1.10} $$

* **Pfaff-Saalschutz theorem** [5, p. 108]:

$$ {}_3F_2\left(\begin{matrix} a,\ b,\ -n \\ c,\ 1+a+b-c-n \end{matrix} \,\middle|\, 1\right) = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{1.11} $$

* **Second Whipple theorem** [5, p. 108]:

$$ {}_4F_3\left(\begin{matrix} a,\ 1+a/2,\ b,\ c \\ a/2,\ a-b+1,\ a-c+1 \end{matrix} \,\middle|\, -1\right) = \frac{\Gamma(a-b+1)\Gamma(a-c+1)}{\Gamma(a+1)\Gamma(a-b-c+1)}. \tag{1.12} $$

* **Dougall theorem** [5, p. 108]:

$$ \begin{aligned} &{}_5F_4\left(\begin{matrix} a,\ 1+a/2,\ c,\ d,\ e \\ a/2,\ a-c+1,\ a-d+1,\ a-e+1 \end{matrix} \,\middle|\, 1\right) \\ &\quad = \frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+1)}{\Gamma(a+1)\Gamma(a-d-e+1)\Gamma(a-c-e+1)\Gamma(a-c-d+1)}. \end{aligned} \tag{1.13} $$

*** Second Dougall theorem** [5, p. 108]:

(1.14)
$$ {}_7F_6\left(\begin{array}{c} a,\ 1+a/2,\ b,\ c,\ d,\ 1+2a-b-c+n,\ -n \\ a/2,\ a-b+1,\ a-c+1,\ a-d+1,\ b+c+d-a-n,\ a+1+n \end{array}\middle|\, 1\right) $$
$$ = \frac{(a+1)_n(a-b-c+1)_n(a-b-d+1)_n(a-c-d+1)_n}{(a+1-b)_n(a+1-c)_n(a+1-d)_n(a+1-b-c-d)_n}\,. $$

In order to derive the first identity and only for simplicity, we will use the following symbol for representing finite sums of hypergeometric series
$$ {}_p\overset{(m)}{F}_q\left(\begin{array}{ccc} a_1, & \dots & a_p \\ b_1, & \dots & b_q \end{array}\middle|\, z\right) = \sum_{k=0}^{m} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k}\,\frac{z^k}{k!}\,. $$

For instance, we have
$$ {}_p\overset{(-1)}{F}_q(z) = 0,\ {}_p\overset{(0)}{F}_q(z) = 1 \ \text{ and }\ {}_p\overset{(1)}{F}_q(z) = 1 + \frac{a_1 \dots a_p}{b_1 \dots b_q} z\,. $$

2. First hypergeometric identity

Let m, n be two natural numbers so that $n \le m$. By referring to relation (1.1), since
$$ \frac{(n)_k}{(m)_k} = \frac{\Gamma(k+n)\Gamma(m)}{\Gamma(k+m)\Gamma(n)} = \frac{\Gamma(m)}{\Gamma(n)}\,\frac{1}{(k+n)(k+n+1)\dots(k+m-1)}, \tag{2.1} $$
substituting (2.1) in a special case of (1.2) yields

(2.2)
$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \dots\ a_{p-1}, & n \\ b_1, & \dots\ b_{q-1}, & m \end{array}\middle|\, z\right) = \frac{\Gamma(m)}{\Gamma(n)} \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_{p-1})_k}{(b_1)_k \dots (b_{q-1})_k}\,\frac{(k+1)(k+2)\dots(k+n-1)\,z^k}{(k+m-1)!} $$
$$ = \frac{\Gamma(m)}{\Gamma(n)} \sum_{j=m-1}^{\infty} \frac{(a_1)_{j-m+1} \dots (a_{p-1})_{j-m+1}}{(b_1)_{j-m+1} \dots (b_{q-1})_{j-m+1}}\,\frac{(j+2-m)(j+3-m)\dots(j-m+n)\,z^{j-m+1}}{j!}. $$

Relation (2.2) shows that we encounter with a complicated computational problem that cannot be easily evaluated. However, some particular cases such as $n = 1$ and $n = 2$ can be directly computed. We leave other cases as open problems.

The case $n = 1$ leads to a known result in the literature [12], because

(2.3)
$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \dots\ a_{p-1}, & 1 \\ b_1, & \dots\ b_{q-1}, & m \end{array}\middle|\, z\right) = \Gamma(m) \sum_{j=m-1}^{\infty} \frac{(a_1)_{j-m+1} \dots (a_{p-1})_{j-m+1}}{(b_1)_{j-m+1} \dots (b_{q-1})_{j-m+1}}\,\frac{z^{j-m+1}}{j!} $$
$$ = \Gamma(m)\left(\sum_{j=0}^{\infty} \frac{(a_1)_{j-m+1} \dots (a_{p-1})_{j-m+1}}{(b_1)_{j-m+1} \dots (b_{q-1})_{j-m+1}}\,\frac{z^{j-m+1}}{j!} - \sum_{j=0}^{m-2} \frac{(a_1)_{j-m+1} \dots (a_{p-1})_{j-m+1}}{(b_1)_{j-m+1} \dots (b_{q-1})_{j-m+1}}\,\frac{z^{j-m+1}}{j!}\right), $$

and since
$$ (a)_{j-m+1} = \frac{\Gamma(a-m+1)}{\Gamma(a)}(a-m+1)_j, $$

relation (2.3) is simplified as

$$(2.4)\quad {}_pF_q\left(\begin{array}{ccc} a_1, & ...\, a_{p-1}, & 1 \\ b_1, & ...\, b_{q-1}, & m \end{array}\,\middle|\, z\right) = \frac{\Gamma(b_1)...\,\Gamma(b_{q-1})}{\Gamma(a_1)...\,\Gamma(a_{p-1})}\frac{\Gamma(a_1-m+1)...\,\Gamma(a_{p-1}-m+1)}{\Gamma(b_1-m+1)...\,\Gamma(b_{q-1}-m+1)}\frac{(m-1)!}{z^{m-1}}$$
$$\times\left({}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+1,\ ...,\ a_{p-1}-m+1 \\ b_1-m+1,\ ...,\ b_{q-1}-m+1 \end{array}\,\middle|\, z\right) - {}_{p-1}F_{q-1}^{(m-2)}\left(\begin{array}{c} a_1-m+1,\ ...,\ a_{p-1}-m+1 \\ b_1-m+1,\ ...,\ b_{q-1}-m+1 \end{array}\,\middle|\, z\right)\right).$$

But the interesting point is that using relation (2.4), we can obtain various special cases that extend all classical summation theorems as follows.

Special case 1. When $p=3, q=2$ and $x=1$, relation (2.4) is simplified as

$$(2.5)\quad {}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ c,\ m \end{array}\,\middle|\, 1\right) = \frac{\Gamma(m)\Gamma(c)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(c-m+1)}$$
$$\times\left(\frac{\Gamma(c-m+1)\Gamma(c-a-b+m-1)}{\Gamma(c-a)\Gamma(c-b)} - {}_2F_1^{(m-2)}\left(\begin{array}{c} a-m+1,\ b-m+1 \\ c-m+1 \end{array}\,\middle|\, 1\right)\right).$$

For $m=1$, relation (2.5) exactly gives formula (1.4) while for $m=2,3$ we have

$${}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ c,\ 2 \end{array}\,\middle|\, 1\right) = \frac{c-1}{(a-1)(b-1)}\left(\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} - 1\right),$$

and

$${}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ c,\ 3 \end{array}\,\middle|\, 1\right) = \frac{2(c-2)_2}{(a-2)_2(b-2)_2}\times\left(\frac{\Gamma(c-2)\Gamma(c-a-b+2)}{\Gamma(c-a)\Gamma(c-b)} - \frac{ab+c-2a-2b+2}{c-2}\right).$$

These two formulas are given in [12].

Special case 2. When $p=3, q=2$ and $x=-1$, by noting the Kummer theorem (1.5), relation (2.4) is simplified as

(2.6)
$${}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ a-b+m,\ m \end{array}\,\middle|\, -1\right) = (-1)^{m-1}\frac{\Gamma(m)\Gamma(a-b+m)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(a-b+1)}$$
$$\times\left(\frac{\Gamma(a-b+1)\Gamma(1+(a-m+1)/2)}{\Gamma(2+a-m)\Gamma(m-b+(a-m+1)/2)} - {}_2F_1^{(m-2)}\left(\begin{array}{c} a-m+1,\ b-m+1 \\ a-b+1 \end{array}\,\middle|\, -1\right)\right).$$

For $m=1$, relation (2.6) exactly gives the Kummer formula while for $m=2,3$ we have

$${}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ a-b+2,\ 2 \end{array}\,\middle|\, -1\right) = \frac{a-b+1}{(a-1)(b-1)}\left(1 - \frac{\Gamma(a-b+1)\Gamma(1+(a-1)/2)}{\Gamma(a)\Gamma(-b+2+(a-1)/2)}\right),$$

and

$$ {}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ a-b+3,\ 3 \end{array}\middle| -1\right) = \frac{2(a-b+1)_2}{(a-2)_2(b-2)_2}\left(\frac{\Gamma(a-b+1)\Gamma(a/2)}{\Gamma(a-1)\Gamma(-b+2+a/2)} - \frac{3a+b-ab-3}{a-b+1}\right). $$

Special case 3. When $p=3, q=2$ and $x=1/2$, by noting the second kind of Gauss formula (1.6), relation (2.4) is simplified as

$$ (2.7)\quad {}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ (a+b+1)/2,\ m \end{array}\middle| \frac{1}{2}\right) = (2)^{m-1} $$

$$ \times \frac{\Gamma(m)\Gamma((a+b+1)/2)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(-m+1+(a+b+1)/2)} $$

$$ \times\left(\frac{\sqrt{\pi}\,\Gamma(-m+1+(a+b+1)/2)}{\Gamma(1+(a-m)/2)\,\Gamma(1+(b-m)/2)} - {}_2F_1^{(m-2)}\left(\begin{array}{c} a-m+1,\ b-m+1 \\ -m+1+(a+b+1)/2 \end{array}\middle| \frac{1}{2}\right)\right). $$

For $m=1$, relation (2.7) exactly gives the second kind of Gauss formula while for $m=2,3$ we have

$$ {}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ (a+b+1)/2,\ 2 \end{array}\middle| \frac{1}{2}\right) = \frac{a+b-1}{(a-1)(b-1)}\left(\frac{\sqrt{\pi}\,\Gamma(-1+(a+b+1)/2)}{\Gamma(a/2)\,\Gamma(b/2)} - 1\right), $$

and

$$ {}_3F_2\left(\begin{array}{c} a,\ b,\ 1 \\ (a+b+1)/2,\ 3 \end{array}\middle| \frac{1}{2}\right) = \frac{2(a+b-1)(a+b-3)}{(a-2)_2(b-2)_2} $$

$$ \times\left(\frac{\sqrt{\pi}\,\Gamma((a+b-3)/2)}{\Gamma((a-1)/2)\,\Gamma((b-1)/2)} - \frac{ab-a-b+1}{a+b-3}\right). $$

Special case 4. When $p=3, q=2$ and $x=1/2$, by noting the Bailey theorem (1.7), relation (2.4) is simplified as

$$ (2.8)\quad {}_3F_2\left(\begin{array}{c} a,\ 2m-a-1,\ 1 \\ b,\ m \end{array}\middle| \frac{1}{2}\right) = (2)^{m-1}\frac{\Gamma(m)\Gamma(b)\Gamma(a-m+1)\Gamma(m-a)}{\Gamma(a)\Gamma(2m-a-1)\Gamma(b-m+1)} $$

$$ \times\left(\frac{\Gamma((b-m+1)/2)\Gamma((b-m+2)/2)}{\Gamma(-m+1+(a+b)/2)\,\Gamma((b-a+1)/2)} - {}_2F_1^{(m-2)}\left(\begin{array}{c} a-m+1,\ m-a \\ b-m+1 \end{array}\middle| \frac{1}{2}\right)\right). $$

For $m=1$, relation (2.8) exactly gives the Bailey formula while for $m=2,3$ we have

$$ {}_3F_2\left(\begin{array}{c} a,\ 3-a,\ 1 \\ b,\ 2 \end{array}\middle| \frac{1}{2}\right) = \frac{2(1-b)}{(1-a)_2}\left(\frac{\Gamma((b-1)/2)\Gamma(b/2)}{\Gamma(-1+(a+b)/2)\,\Gamma((b-a+1)/2)} - 1\right), $$

and

$$ {}_3F_2\left(\begin{array}{c} a,\ 5-a,\ 1 \\ b,\ 3 \end{array} \,\middle|\, \frac{1}{2} \right) = \frac{8(b-2)_2}{(a-4)_4} \times \left(\frac{\Gamma((b-1)/2)\Gamma((b-2)/2)}{\Gamma(-2+(a+b)/2)\,\Gamma((b-a+1)/2)} - \frac{5a-a^2+2b-10}{2(b-2)} \right). $$

Special case 5. When $p=4, q=3$ and $x=1$, by noting the Dixon theorem (1.8), relation (2.4) is simplified as

$$ (2.9)\quad {}_4F_3\left(\begin{array}{c} a,\ b,\ c,\ 1 \\ a-b+m,\ a-c+m,\ m \end{array} \,\middle|\, 1 \right) $$

$$ = \frac{\Gamma(m)\Gamma(a-b+m)\Gamma(a-c+m)\Gamma(a+1-m)\Gamma(b+1-m)\Gamma(c+1-m)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a-b+1)\Gamma(a-c+1)} $$

$$ \times \left(\frac{\Gamma((a+3-m)/2)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(-b-c+(a+3m-1)/2)}{\Gamma(a+2-m)\Gamma(-b+(a+m+1)/2)\Gamma(-c+(a+m+1)/2)\Gamma(a-b-c+m)} - {}_3F_2^{(m-2)}\left(\begin{array}{c} a-m+1,\ b-m+1,\ c-m+1 \\ a-b+1,\ a-c+1 \end{array} \,\middle|\, 1 \right) \right). $$

For $m=1$, relation (2.9) exactly gives the Dixon formula while for $m=2,3$ we have

$$ {}_4F_3\left(\begin{array}{c} a,\ b,\ c,\ 1 \\ a-b+2,\ a-c+2,\ 2 \end{array} \,\middle|\, 1 \right) = \frac{(a-b+1)(a-c+1)}{(a-1)(b-1)(c-1)} $$

$$ \times \left(\frac{\Gamma((a+1)/2)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(-b-c+(a+5)/2)}{\Gamma(a)\Gamma(-b+(a+3)/2)\Gamma(-c+(a+3)/2)\Gamma(a-b-c+2)} - 1 \right), $$

and

$$ {}_4F_3\left(\begin{array}{c} a,\ b,\ c,\ 1 \\ a-b+3,\ a-c+3,\ 3 \end{array} \,\middle|\, 1 \right) = \frac{2(a-b+1)_2(a-c+1)_2}{(a-2)_2(b-2)_2(c-2)_2} $$

$$ \times \left(\frac{\Gamma(a/2)\Gamma(a-b+1)\Gamma(a-c+1)\Gamma((a/2)-b-c+4)}{\Gamma(a-1)\Gamma((a/2)-b+2)\Gamma((a/2)-c+2)\Gamma(a-b-c+3)} - \frac{(a-2)(b-2)(c-2)}{(a-b+1)(a-c+1)} - 1 \right). $$

Special case 6. When $p=4, q=3$ and $x=1$, by noting the Watson theorem (1.9), relation (2.4) is simplified as

$$ (2.10)\quad {}_4F_3\left(\begin{array}{c} a,\ b,\ c,\ 1 \\ (a+b+1)/2,\ 2c+1-m,\ m \end{array} \,\middle|\, 1 \right) $$

$$ = \frac{\Gamma(m)\Gamma((a+b+1)/2)\Gamma(2c+1-m)\Gamma(a+1-m)\Gamma(b+1-m)\Gamma(c+1-m)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(-m+(a+b+3)/2)\Gamma(2c-2m+2)} $$

$$\times\left(\begin{array}{c}\frac{\sqrt{\pi}\,\Gamma(c-m+(3/2))\Gamma(-m+(a+b+3)/2)\Gamma(c-(a+b-1)/2)}{\Gamma(1+(a-m)/2)\Gamma(1+(b-m)/2)\Gamma(c+1-(a+m)/2)\Gamma(c+1-(b+m)/2)}\\ -\,{}_3F_2^{(m-2)}\left(\begin{array}{c}a-m+1,\ b-m+1,\ c-m+1\\ -m+1+(a+b+1)/2,\ 2c-2m+2\end{array}\,\middle|\,1\right)\end{array}\right).$$

For $m=1$, relation (2.10) exactly gives the Watson formula while for $m=2,3$ we have

$${}_4F_3\left(\begin{array}{c}a,\ b,\ c,\ 1\\ (a+b+1)/2,\ 2c-1,\ 2\end{array}\,\middle|\,1\right)=\frac{a+b-1}{(a-1)(b-1)}$$

$$\times\left(\frac{\sqrt{\pi}\,\Gamma(c-(1/2))\Gamma((a+b-1)/2)\Gamma(c-(a+b-1)/2)}{\Gamma(a/2)\Gamma(b/2)\Gamma(c-(a/2))\Gamma(c-(b/2))}-1\right),$$

and

$${}_4F_3\left(\begin{array}{c}a,\ b,\ c,\ 1\\ (a+b+1)/2,\ 2c-2,\ 3\end{array}\,\middle|\,1\right)=\frac{(2c-3)(a+b-1)(a+b-3)}{(a-2)_2(b-2)_2(c-1)}$$

$$\times\left(\frac{\sqrt{\pi}\,\Gamma(c-(3/2))\Gamma((a+b-3)/2)\Gamma(c-(a+b-1)/2)}{\Gamma((a-1)/2)\Gamma((b-1)/2)\Gamma(c-(a+1)/2)\Gamma(c-(b+1)/2)}-\frac{(a-2)(b-2)}{a+b-3}-1\right).$$

Special case 7. When $p=4, q=3$ and $x=1$, by noting the Whipple theorem (1.10), relation (2.4) is simplified as

$$(2.11)\quad {}_4F_3\left(\begin{array}{c}a,\ 2m-1-a,\ b,\ 1\\ c,\ 2b-c+1,\ m\end{array}\,\middle|\,1\right)$$

$$=\frac{\Gamma(m)\Gamma(c)\Gamma(2b-c+1)\Gamma(a+1-m)\Gamma(m-a)\Gamma(b+1-m)}{\Gamma(a)\Gamma(2m-1-a)\Gamma(b)\Gamma(c+1-m)\Gamma(2b-c-m+2)}$$

$$\times\left(\begin{array}{c}\frac{\pi\,2^{2m-2b-1}\Gamma(c-m+1)\Gamma(2b-c+2-m)}{\Gamma(-m+1+(a+c)/2)\Gamma(-m+1+b+(a-c+1)/2)\Gamma((1-a+c)/2)\Gamma(b+1-(a+c)/2)}\\ -\,{}_3F_2^{(m-2)}\left(\begin{array}{c}a-m+1,\ m-a,\ \ b-m+1\\ c-m+1,\ 2b-c-m+2\end{array}\,\middle|\,1\right)\end{array}\right).$$

For $m=1$, relation (2.11) exactly gives the Whipple formula while for $m=2,3$ we have

$${}_4F_3\left(\begin{array}{c}a,\ 3-a,\ b,\ 1\\ c,\ 2b-c+1,\ 2\end{array}\,\middle|\,1\right)=\frac{(c-1)(c-2b)}{(a-2)_2(b-1)}\times$$
$$\left(\frac{\pi\,2^{3-2b}\Gamma(c-1)\Gamma(2b-c)}{\Gamma(-1+(a+c)/2)\Gamma(b+(a-c-1)/2)\Gamma((1-a+c)/2)\Gamma(b+1-(a+c)/2)}-1\right),$$

and

$$ {}_4F_3\left(\begin{array}{c} a,\ 5-a,\ b,\ 1 \\ c,\ 2b-c+1,\ 3 \end{array}\,\middle|\, 1\right) = \frac{2(c-2)_2(2b-c-1)_2}{(a-4)_4(b-2)_2} $$
$$ \times\left(\frac{\pi\, 2^{5-2b}\Gamma(c-2)\Gamma(2b-c-1)}{\Gamma(-2+(a+c)/2)\Gamma(b+(a-c-3)/2)\Gamma((1-a+c)/2)\Gamma(b+1-(a+c)/2)} - \frac{(a-2)(3-a)(b-2)}{(c-2)(2b-c-1)} - 1\right). $$

Special case 8. When $p=4, q=3$ and $x=1$, by noting the Pfaff-Saalschutz theorem (1.11), relation (2.4) is simplified as

(2.12)
$$ {}_4F_3\left(\begin{array}{c} a,\ b,\ -n+m-1,\ 1 \\ c,\ 1+a+b-c-n,\ m \end{array}\,\middle|\, 1\right) = \frac{(m-1)!(1-c)_{m-1}(c-a-b+n)_{m-1}}{(1-a)_{m-1}(1-b)_{m-1}(n+2-m)_{m-1}} $$
$$ \times\left(\frac{(c-a)_n(c-b)_n}{(c+1-m)_n(c-a-b+m-1)_n} - {}^{(m-2)}_{\ \ 3}F_2\left(\begin{array}{c} a-m+1,\ b-m+1,\ -n \\ c-m+1,\ 2+a+b-c-m-n \end{array}\,\middle|\, 1\right)\right). $$

For $m=1$, relation (2.12) exactly gives the Pfaff-Saalschutz formula while for $m=2,3$ we have

$$ {}_4F_3\left(\begin{array}{c} a,\ b,\ -n+1,\ 1 \\ c,\ 1+a+b-c-n,\ 2 \end{array}\,\middle|\, 1\right) = \frac{(1-c)(c-a-b+n)}{n\,(1-a)(1-b)} \times\left(\frac{(c-a)_n(c-b)_n}{(c-1)_n(c-a-b+1)_n} - 1\right), $$

and

$$ {}_4F_3\left(\begin{array}{c} a,\ b,\ -n+2,\ 1 \\ c,\ 1+a+b-c-n,\ 3 \end{array}\,\middle|\, 1\right) = \frac{2(1-c)_2(c-a-b+n)_2}{n(1-a)_2(1-b)_2(n-1)_2} $$
$$ \times\left(\frac{(c-a)_n(c-b)_n}{(c-2)_n(c-a-b+2)_n} + \frac{n(a-2)(b-2)}{(c-2)(a+b-c-n-1)} - 1\right). $$

Special case 9. When $p=5, q=4$ and $x=-1$, by noting the second theorem of Whipple (1.12), relation (2.4) is simplified as

$$ (2.13)\quad {}_5F_4\left(\begin{array}{c} a,\ (a+m+1)/2,\ b,\ c,\ 1 \\ (a+m-1)/2, a-b+m,\ a-c+m,\ m \end{array}\,\middle|\, -1\right) = (-1)^{m-1}\times $$

$$ \frac{\Gamma(m)\Gamma((a+m-1)/2)\Gamma(a-b+m)\Gamma(a-c+m)\Gamma(a-m+1)\Gamma((a-m+3)/2)\Gamma(b-m+1)\Gamma(c-m+1)}{\Gamma(a)\Gamma((a+m+1)/2)\Gamma(b)\Gamma(c)\Gamma((a-m+1)/2)\Gamma(a-b+1)\Gamma(a-c+1)} $$

$$ \times\left(\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(2-m+a)\Gamma(m+a-b-c)} - {}^{(m-2)}_{\ \ 4}F_3\left(\begin{array}{c} a-m+1,\ (a-m+3)/2,\ b-m+1,\ c-m+1 \\ (a-m+1)/2,\ a-b+1,\ a-c+1 \end{array}\,\middle|\, -1\right)\right). $$

For $m=1$, relation (2.13) exactly gives the Whipple formula while for $m=2,3$ we have

$$
{}_5F_4\left(\begin{matrix} a,\ (a+3)/2,\ \ b,\ c,\ 1 \\ (a+1)/2, a-b+2,\ a-c+2,\ 2 \end{matrix} \,\middle|\, -1\right)
$$

$$
=\frac{4(a-b+1)(a-c+1)}{(a^2-1)(a-1)(b-1)(c-1)}\left(1-\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(a)\Gamma(2+a-b-c)}\right),
$$

and

$$
{}_5F_4\left(\begin{matrix} a,\ (a+4)/2,\ \ b,\ c,\ 1 \\ (a+2)/2, a-b+3,\ a-c+3,\ 3 \end{matrix} \,\middle|\, -1\right)=\frac{2(a-b+1)_2(a-c+1)_2}{(a+2)(a-1)(b-2)_2(c-2)_2}
$$

$$
\times\left(\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(a-1)\Gamma(3+a-b-c)}+\frac{a(b-2)(c-2)}{(a-b+1)(a-c+1)}-1\right).
$$

Special case 10. When $p=6, q=5$ and $x=1$, by noting the Dougall theorem (1.13), relation (2.4) is simplified as

$$
(2.14)\quad {}_6F_5\left(\begin{matrix} a,\ (a+m+1)/2,\ c,\ d,\ e,\ 1 \\ (a+m-1)/2, a-c+m,\ a-d+m,\ a-e+m,\ m \end{matrix} \,\middle|\, 1\right)
$$

$$
=\Gamma(m)\Gamma((a+m-1)/2)\Gamma(a-c+m)
$$

$$
\times\frac{\Gamma(a-d+m)\Gamma(a-e+m)\Gamma(a-m+1)\Gamma((a-m+3)/2)\Gamma(c-m+1)\Gamma(d-m+1)\Gamma(e-m+1)}{\Gamma(a)\Gamma((a+m+1)/2)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma((a-m+1)/2)\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)}
$$

$$
\times\left(\begin{matrix} \frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+2m-1)}{\Gamma(a+2-m)\Gamma(a-d-e+m)\Gamma(a-c-e+m)\Gamma(a-c-d+m)} \\ -\,{}_5^{(m-2)}F_4\left(\begin{matrix} a-m+1,\ (a-m+3)/2,\ c-m+1,\ d-m+1,\ e-m+1 \\ (a-m+1)/2,\ a-c+1,\ a-d+1,\ a-e+1 \end{matrix} \,\middle|\, 1\right)\end{matrix}\right).
$$

For $m=1$, relation (2.14) exactly gives the Dougall formula while for $m=2,3$ we have

$$
{}_6F_5\left(\begin{matrix} a,\ (a+3)/2,\ c,\ d,\ e,\ 1 \\ (a+1)/2, a-c+2,\ a-d+2,\ a-e+2,\ 2 \end{matrix} \,\middle|\, 1\right)
$$

$$
=\frac{(a-c+1)(a-d+1)(a-e+1)}{(c-1)(d-1)(e-1)(a+1)}
$$

$$
\times\left(\frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+3)}{\Gamma(a)\Gamma(a-d-e+2)\Gamma(a-c-e+2)\Gamma(a-c-d+2)}-1\right),
$$

and

$$
{}_6F_5\left(\begin{matrix} a,\ (a+4)/2,\ c,\ d,\ e,\ 1 \\ (a+2)/2, a-c+3,\ a-d+3,\ a-e+3,\ 3 \end{matrix}\ \middle|\ 1\right)
$$

$$
= \frac{2\,(a-c+1)_2(a-d+1)_2(a-e+1)_2}{(a-1)(a+2)(c-2)_2(d-2)_2(e-2)_2}
$$

$$
\times \left(\frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+5)}{\Gamma(a-1)\Gamma(a-d-e+3)\Gamma(a-c-e+3)\Gamma(a-c-d+3)}\right.
$$

$$
\left. - \frac{a(c-2)(d-2)(e-2)}{(a-c+1)(a-d+1)(a-e+1)} - 1\right).
$$

Special case 11. When $p = 8, q = 7$ and $x = 1$, by noting the second theorem of Dougall (1.14), relation (2.4) is simplified as

(2.15)

$$
{}_8F_7\left(\begin{matrix} a,\ (a+m+1)/2,\ b, c,\ d,\ 2a-b-c-d+2m-1+n,\ m-n-1,\ 1 \\ (a+m-1)/2,\ a-b+m,\ a-c+m,\ a-d+m,\ b+c+d-a+1-m-n, a+n+1,\ m \end{matrix}\ \middle|\ 1\right)
$$

$$
= (-1)^{m-1}(m-1)! \times
$$

$$
\frac{((3-a-m)/2)_{m-1}(1-a+b-m)_{m-1}(1-a+c-m)_{m-1}(1-a+d-m)_{m-1}(m+n+a-b-c-d)_{m-1}(-a-n)_{m-1}}{((1-a-m)/2)_{m-1}(1-a)_{m-1}(1-b)_{m-1}(1-c)_{m-1}(1-d)_{m-1}(b+c+d-2a+2-2m-n)_{m-1}(n+2-m)_{m-1}}
$$

$$
\times \left(\begin{matrix} \frac{(a-m+2)_n(a-b-c+m)_n(a-b-d+m)_n(a-c-d+m)_n}{(a-b+1)_n(a-c+1)_n(a-d+1)_n(a-b-c-d+2m-1)_n} \\ - {}^{(m-2)}_{\ \ 7}F_6\left(\begin{matrix} a-m+1,\ (a-m+3)/2,\ b-m+1, c-m+1,\ d-m+1,\ 2a-b-c-d+m+n, -n \\ (a-m+1)/2,\ a-b+1,\ a-c+1,\ a-d+1,\ b+c+d-a+2-2m-n, a-m+n+2 \end{matrix}\ \middle|\ 1\right) \end{matrix}\right).
$$

For $m = 1$, relation (2.15) exactly gives the Dougall formula while for $m = 2, 3$ we have

$$
{}_8F_7\left(\begin{matrix} a,\ (a+3)/2,\ b, c,\ d,\ 2a-b-c-d+3+n,\ -n+1,\ 1 \\ (a+1)/2,\ a-b+2,\ a-c+2,\ a-d+2,\ b+c+d-a-1-n, a+n+1,\ 2 \end{matrix}\ \middle|\ 1\right)
$$

$$
= \frac{(-a+b-1)(-a+c-1)(-a+d-1)(n+2+a-b-c-d)(a+n)}{n\,(1+a)(1-b)(1-c)(1-d)(b+c+d-2a-2-n)}
$$

$$
\times \left(1 - \frac{(a)_n(a-b-c+2)_n(a-b-d+2)_n(a-c-d+2)_n}{(a-b+1)_n(a-c+1)_n(a-d+1)_n(a-b-c-d+3)_n}\right),
$$

and

$$ {}_8F_7\left(\begin{array}{c} a,\ (a+4)/2,\ b, c,\ d,\ 2a-b-c-d+5+n,\ -n+2,\ 1 \\ (a+2)/2,\ a-b+3,\ a-c+3,\ a-d+3,\ b+c+d-a-2-n, a+n+1,\ 3 \end{array}\middle|\ 1\right) $$

$$ = \frac{(a-2)(-a+b-2)_2(-a+c-2)_2(-a+d-2)_2(3+n+a-b-c-d)_2(-a-n)_2}{(a+2)(1-a)_2(1-b)_2(1-c)_2(1-d)_2(b+c+d-2a-4-n)_2(n-1)_2} $$

$$ \times\left(\begin{array}{l} \frac{(a-1)_n(a-b-c+3)_n(a-b-d+3)_n(a-c-d+3)_n}{(a-b+1)_n(a-c+1)_n(a-d+1)_n(a-b-c-d+5)_n} \\ +\frac{na(b-2)(c-2)(d-2)(2a-b-c-d+3+n)}{(a-b+1)(a-c+1)(a-d+1)(b+c+d-a-n-4)(n+a-1)}-1 \end{array}\right). $$

Remark 2.1. There are two further special cases which however do not belong to classical summation theorems. When $p=q=1$, relation (2.4) is simplified as

$$ {}_1F_1\left(\begin{array}{c} 1 \\ m \end{array}\middle|\ z\right) = \frac{(m-1)!}{z^{m-1}}\left(e^z-\sum_{j=0}^{m-2}\frac{z^j}{j!}\right), $$

and when $p=q+1=2$, it yields

$$ {}_2F_1\left(\begin{array}{c} a,\ 1 \\ m \end{array}\middle|\ z\right) = \frac{(m-1)!}{z^{m-1}}\frac{\Gamma(a-m+1)}{\Gamma(a)}\left((1-z)^{m-a-1}-\sum_{j=0}^{m-2}(a-m+1)_j\frac{z^j}{j!}\right). $$

Similarly, for the case $n=2$, relation (2.2) changes to

$$ {}_pF_q\left(\begin{array}{c} a_1,\ \ldots\ a_{p-1},\quad 2 \\ b_1,\ \ldots\ b_{q-1},\quad m \end{array}\middle|\ z\right) = \frac{\Gamma(m)}{z^{m-1}}\sum_{j=m-1}^{\infty}\frac{(a_1)_{j-m+1}\cdots(a_{p-1})_{j-m+1}}{(b_1)_{j-m+1}\cdots(b_{q-1})_{j-m+1}}\frac{(j+2-m)\,z^j}{j!} $$

$$ = \frac{\Gamma(m)}{z^{m-1}}\left(z\sum_{r=m-2}^{\infty}\frac{(a_1)_{j-m+2}\cdots(a_{p-1})_{j-m+2}}{(b_1)_{j-m+2}\cdots(b_{q-1})_{j-m+2}}\frac{z^r}{r!}+(2-m)\sum_{j=m-1}^{\infty}\frac{(a_1)_{j-m+1}\cdots(a_{p-1})_{j-m+1}}{(b_1)_{j-m+1}\cdots(b_{q-1})_{j-m+1}}\frac{z^j}{j!}\right) $$

$$ = \frac{(m-1)!}{z^{m-2}}\frac{\Gamma(b_1)\ldots\Gamma(b_{q-1})}{\Gamma(a_1)\ldots\Gamma(a_{p-1})}\frac{\Gamma(a_1-m+2)\ldots\Gamma(a_{p-1}-m+2)}{\Gamma(b_1-m+2)\ldots\Gamma(b_{q-1}-m+2)} $$

$$ \times\left({}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+2,\ \ldots,\ a_{p-1}-m+2 \\ b_1-m+2,\ \ldots,\ b_{q-1}-m+2 \end{array}\middle|\ z\right)-{}^{(m-3)}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+2,\ \ldots,\ a_{p-1}-m+2 \\ b_1-m+2,\ \ldots,\ b_{q-1}-m+2 \end{array}\middle|\ z\right)\right) $$

$$ +\ (2-m)\frac{(m-1)!}{z^{m-1}}\frac{\Gamma(b_1)\ldots\Gamma(b_{q-1})}{\Gamma(a_1)\ldots\Gamma(a_{p-1})}\frac{\Gamma(a_1-m+1)\ldots\Gamma(a_{p-1}-m+1)}{\Gamma(b_1-m+1)\ldots\Gamma(b_{q-1}-m+1)} $$

$$ \times\left({}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+1,\ \ldots,\ a_{p-1}-m+1 \\ b_1-m+1,\ \ldots,\ b_{q-1}-m+1 \end{array}\middle|\ z\right)-{}^{(m-2)}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+1,\ \ldots,\ a_{p-1}-m+1 \\ b_1-m+1,\ \ldots,\ b_{q-1}-m+1 \end{array}\middle|\ z\right)\right). $$

Therefore

(2.16)

$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \dots\, a_{p-1}, & 2 \\ b_1, & \dots\, b_{q-1}, & m \end{array} \middle| \, z\right) = \frac{(m-1)!}{z^{m-1}}\frac{\Gamma(b_1)\dots\Gamma(b_{q-1})}{\Gamma(a_1)\dots\Gamma(a_{p-1})}\frac{\Gamma(a_1-m+1)\dots\Gamma(a_{p-1}-m+1)}{\Gamma(b_1-m+1)\dots\Gamma(b_{q-1}-m+1)} $$

$$ \times\left(\begin{array}{l} \frac{(a_1-m+1)\dots(a_{p-1}-m+1)}{(b_1-m+1)\dots(b_{q-1}-m+1)}z \\ \times\left({}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+2,\ \dots,\ a_{p-1}-m+2 \\ b_1-m+2,\ \dots,\ b_{q-1}-m+2 \end{array}\middle| \, z\right) - {}_{p-1}^{(m-3)}F_{q-1}\left(\begin{array}{c} a_1-m+2,\ \dots,\ a_{p-1}-m+2 \\ b_1-m+2,\ \dots,\ b_{q-1}-m+2 \end{array}\middle| \, z\right)\right) \\ -(m-2)\left({}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-m+1,\ \dots,\ a_{p-1}-m+1 \\ b_1-m+1,\ \dots,\ b_{q-1}-m+1 \end{array}\middle| \, z\right) - {}_{p-1}^{(m-2)}F_{q-1}\left(\begin{array}{c} a_1-m+1,\ \dots,\ a_{p-1}-m+1 \\ b_1-m+1,\ \dots,\ b_{q-1}-m+1 \end{array}\middle| \, z\right)\right) \end{array}\right) $$

For instance, if $m=3$ relation (2.16) reads as

$$ {}_pF_q\left(\begin{array}{ccc} a_1, & \dots\, a_{p-1}, & 2 \\ b_1, & \dots\, b_{q-1}, & 3 \end{array} \middle| \, z\right) = \frac{2}{z^2}\frac{\Gamma(b_1)\dots\Gamma(b_{q-1})}{\Gamma(a_1)\dots\Gamma(a_{p-1})}\frac{\Gamma(a_1-2)\dots\Gamma(a_{p-1}-2)}{\Gamma(b_1-2)\dots\Gamma(b_{q-1}-2)} $$

$$ \times\left(\frac{(a_1-2)\dots(a_{p-1}-2)}{(b_1-2)\dots(b_{q-1}-2)}z\,{}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-1,\ \dots,\ a_{p-1}-1 \\ b_1-1,\ \dots,\ b_{q-1}-1 \end{array}\middle| \, z\right)\right. $$

$$ \left. -{}_{p-1}F_{q-1}\left(\begin{array}{c} a_1-2,\ \dots,\ a_{p-1}-2 \\ b_1-2,\ \dots,\ b_{q-1}-2 \end{array}\middle| \, z\right)+1\right). $$

Hence, for $p=q+1=3$ and $z=1$ we have

$$ {}_3F_2\left(\begin{array}{ccc} a, & b, & 2 \\ & c, & 3 \end{array} \middle| \, 1\right) = \frac{2}{(a-2)_2(b-2)_2} \times\left((c-2)_2+\frac{\Gamma(c)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)}(ab-a-b-c+3)\right). $$

3. Second hypergeometric identity

By noting relation (1.1), first it is not difficult to verify that

$$ (a+m)_k = \frac{(a)_{k+m}}{(a)_m}. \tag{3.1} $$

Now if the identity (3.1) is applied in a special case of (1.2), we obtain

$$ {}_pF_q\left(\begin{array}{ccc} a_1+m, & \dots\, a_{p-1}+m, & 1 \\ b_1+m, & \dots & b_q+m \end{array} \middle| \, z\right) = \sum_{k=0}^{\infty}\frac{(a_1+m)_k\dots(a_{p-1}+m)_k}{(b_1+m)_k\dots(b_q+m)_k}\,z^k $$

$$ = \frac{(b_1)_m\dots(b_q)_m}{(a_1)_m\dots(a_{p-1})_m}\sum_{k=0}^{\infty}\frac{(a_1)_{k+m}\dots(a_{p-1})_{k+m}}{(b_1)_{k+m}\dots(b_{q-1})_{k+m}}\,z^k $$

$$ = \frac{(b_1)_m\dots(b_q)_m}{(a_1)_m\dots(a_{p-1})_m}\sum_{j=m}^{\infty}\frac{(a_1)_j\dots(a_{p-1})_j}{(b_1)_j\dots(b_{q-1})_j}\,z^{j-m} $$

$$= \frac{(b_1)_m \cdots (b_q)_m}{(a_1)_m \cdots (a_{p-1})_m} z^{-m} \left(\sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_{p-1})_j}{(b_1)_j \cdots (b_{q-1})_j} z^j - \sum_{j=0}^{m-1} \frac{(a_1)_j \cdots (a_{p-1})_j}{(b_1)_j \cdots (b_{q-1})_j} z^j \right),$$

leading to the second identity

$$(3.2)\quad {}_pF_q\left(\begin{array}{ccc} a_1+m, & \ldots\ a_{p-1}+m, & 1 \\ b_1+m, & \ldots & b_q+m \end{array} \,\middle|\, z \right) = \frac{(b_1)_m \cdots (b_q)_m}{(a_1)_m \cdots (a_{p-1})_m} z^{-m}$$

$$\times \left({}_pF_q\left(\begin{array}{ccc} a_1, & \ldots\ a_{p-1}, & 1 \\ b_1, & \ldots & b_q \end{array} \,\middle|\, z \right) - {}_pF_q^{(m-1)}\left(\begin{array}{ccc} a_1, & \ldots\ a_{p-1}, & 1 \\ b_1, & \ldots & b_q \end{array} \,\middle|\, z \right) \right),$$

which is equivalent to

$$(3.3)\quad {}_pF_q\left(\begin{array}{ccc} a_1, & \ldots\ a_{p-1}, & 1 \\ b_1, & \ldots & b_q \end{array} \,\middle|\, z \right) = \frac{(b_1-m)_m \cdots (b_q-m)_m}{(a_1-m)_m \cdots (a_{p-1}-m)_m} z^{-m}$$

$$\times \left({}_pF_q\left(\begin{array}{ccc} a_1-m, & \ldots\ a_{p-1}-m, & 1 \\ b_1-m, & \ldots & b_q-m \end{array} \,\middle|\, z \right) \right.$$

$$\left. - {}_pF_q^{(m-1)}\left(\begin{array}{ccc} a_1-m, & \ldots\ a_{p-1}-m, & 1 \\ b_1-m, & \ldots & b_q-m \end{array} \,\middle|\, z \right) \right).$$

Once again, the interesting point is that by using relations (3.2) or (3.3) various special cases can be considered as follows.

Special case 12. When $p = 2, q = 1$ and $x = -1$, by noting the Kummer formula and relation (3.2) we get

$$(3.4)\quad {}_2F_1\left(\begin{array}{c} b+m,\ 1 \\ 2-b+m \end{array} \,\middle|\, -1 \right)$$

$$= (-1)^m \frac{(2-b)_m}{(b)_m} \left(\frac{\sqrt{\pi}}{2} \frac{\Gamma(2-b)}{\Gamma(-b+(3/2))} - {}_2F_1^{(m-1)}\left(\begin{array}{c} b,\ 1 \\ 2-b \end{array} \,\middle|\, -1 \right) \right).$$

For instance, if $m = 1, 2$, relation (3.4) is simplified as

$${}_2F_1\left(\begin{array}{c} b+1,\ 1 \\ 3-b \end{array} \,\middle|\, -1 \right) = -\frac{\sqrt{\pi}}{2} \frac{\Gamma(3-b)}{\Gamma((3/2)-b)} + \frac{2}{b} - 1,$$

and

$${}_2F_1\left(\begin{array}{c} b+2,\ 1 \\ 4-b \end{array} \,\middle|\, -1 \right) = \frac{\sqrt{\pi}}{2b(b+1)} \frac{\Gamma(4-b)}{\Gamma((3/2)-b)} - \frac{2(b-1)(b-3)}{b(b+1)}.$$

Special case 13. When $p = 2, q = 1$ and $x = 1/2$, by noting the second kind of Gauss formula and relation (3.2) we get

$$ {}_2F_1\left(\begin{matrix} a+m,\ 1 \\ (a/2)+\ m+1 \end{matrix} \,\middle|\, \frac{1}{2} \right) = \frac{(1+(a/2))_m}{(a)_m} 2^m \left(\frac{\sqrt{\pi}\,\Gamma(1+(a/2))}{\Gamma((a+1)/2)} - {}^{(m-1)}_{\ \ \ \ 2}F_1\left(\begin{matrix} a,\ 1 \\ (a/2)+1 \end{matrix} \,\middle|\, \frac{1}{2} \right) \right). \tag{3.5} $$

For instance, if $m = 1, 2$, relation (3.5) is simplified as

$$ {}_2F_1\left(\begin{matrix} a+1,\ 1 \\ (a/2)+\ 2 \end{matrix} \,\middle|\, \frac{1}{2} \right) = \frac{a+2}{a} \left(\sqrt{\pi}\frac{\Gamma(1+(a/2))}{\Gamma((a+1)/2)} - 1 \right), $$

and

$$ {}_2F_1\left(\begin{matrix} a+2,\ 1 \\ (a/2)+3 \end{matrix} \,\middle|\, \frac{1}{2} \right) = \sqrt{\pi}\frac{(a+2)(a+4)}{a(a+1)} \frac{\Gamma(1+(a/2))}{\Gamma((a+1)/2)} - 2(1+\frac{4}{a}). $$

Special case 14. When $p = 3, q = 2$ and $x = 1$, by noting the Dixon formula and relation (3.2) we get

$$ {}_3F_2\left(\begin{matrix} b+m,\ c+m,\ 1 \\ 2-b+m,\ 2-c+m \end{matrix} \,\middle|\, 1 \right) = \frac{(2-b)_m(2-c)_m}{(b)_m(c)_m} \tag{3.6} $$

$$ \times\left(\frac{\sqrt{\pi}}{2} \frac{\Gamma(2-b)\Gamma(2-c)\Gamma(-b-c+(3/2))}{\Gamma(-b+(3/2))\Gamma(-c+(3/2))\Gamma(2-b-c)} - {}^{(m-1)}_{\ \ \ \ 3}F_2\left(\begin{matrix} b,\ c,\ 1 \\ 2-b,\ 2-c \end{matrix} \,\middle|\, 1 \right) \right). $$

For instance, if $m = 1, 2$, relation (3.6) is simplified as

$$ {}_3F_2\left(\begin{matrix} b+1,\ c+1,\ 1 \\ 3-b,\ 3-c \end{matrix} \,\middle|\, 1 \right) = \frac{\sqrt{\pi}}{2bc} \frac{\Gamma(3-b)\Gamma(3-c)\Gamma((3/2)-b-c)}{\Gamma((3/2)-b)\Gamma((3/2)-c)\Gamma(2-b-c)} - \frac{(b-2)(c-2)}{bc}, $$

and

$$ {}_3F_2\left(\begin{matrix} b+2,\ c+2,\ 1 \\ 4-b,\ 4-c \end{matrix} \,\middle|\, 1 \right) = \frac{\sqrt{\pi}}{2(b)_2(c)_2} \frac{\Gamma(4-b)\Gamma(4-c)\Gamma((3/2)-b-c)}{\Gamma((3/2)-b)\Gamma((3/2)-c)\Gamma(2-b-c)} - \frac{2(bc-b-c-2)(b-3)(c-3)}{(b)_2(c)_2}. $$

Special case 15. When $p = 3, q = 2$ and $x = 1$, by noting the Watson formula and relation (3.2) we get

$$ {}_3F_2\left(\begin{matrix} b+m,\ c+m,\ 1 \\ (b/2)+m+1,\ 2c+\ m \end{matrix} \,\middle|\, 1 \right) = \frac{(1+(b/2))_m(2c)_m}{(b)_m(c)_m} \tag{3.7} $$

$$ \times\left(\frac{\sqrt{\pi}\,\Gamma(1+(c/2))\Gamma(1+(b/2))\Gamma(c-(b/2))}{\Gamma((b+1)/2)\Gamma(c)\Gamma(c-(b-1)/2)} - {}^{(m-1)}_{\ \ \ \ 3}F_2\left(\begin{matrix} b,\ c,\ 1 \\ (b/2)+1,\ 2c \end{matrix} \,\middle|\, 1 \right) \right). $$

For instance, if $m = 1, 2$, relation (3.7) is simplified as

$${}_3F_2\left(\begin{array}{c} b+1,\ c+1,\ 1 \\ (b/2)+2,\ 2c+1 \end{array}\,\middle|\, 1\right) = (1+\frac{2}{b})(\frac{\sqrt{\pi}\,\Gamma(1+(c/2))\Gamma(1+(b/2))\Gamma(c-(b/2))}{\Gamma((b+1)/2)\Gamma(c)\Gamma(c-(b-1)/2)}-1),$$

and

$${}_3F_2\left(\begin{array}{c} b+2,\ c+2,\ 1 \\ (b/2)+3,\ 2c+2 \end{array}\,\middle|\, 1\right)$$
$$= \frac{2\sqrt{\pi}c(2c+1)}{b(b+1)}\frac{\Gamma((c/2)+1)\Gamma((b/2)+3)\Gamma(c-(b/2))}{\Gamma((b+1)/2)\Gamma(c+2)\Gamma(c-(b-1)/2)} - \frac{(b+4)(2c+1)}{b(c+1)}.$$

Special case 16. When $p = 3, q = 2$ and $x = 1$, by noting the Whipple formula and relation (3.2) we get

$$(3.8)\quad {}_3F_2\left(\begin{array}{c} a+m,\ 1-a+m,\ 1 \\ c+m,\ 3-c+\ m \end{array}\,\middle|\, 1\right) = \frac{(c)_m(3-c)_m}{(a)_m(1-a)_m}$$
$$\times\left(\frac{\pi\,\Gamma(c)\Gamma(3-c)}{2\,\Gamma((a+c)/2)\Gamma((a-c+3)/2)\Gamma((1-a+c)/2)\Gamma(2-(a+c)/2)}\right.$$
$$\left. - \ {}_3^{(m-1)}F_2\left(\begin{array}{c} a,\ 1-a,\ 1 \\ c,\ 3-c \end{array}\,\middle|\, 1\right)\right).$$

For instance, if $m = 1, 2$, relation (3.8) is simplified as

$${}_3F_2\left(\begin{array}{c} a+1,\ 2-a,\ 1 \\ c+1,\ 4-c \end{array}\,\middle|\, 1\right)$$
$$= \frac{\pi}{2a(1-a)}\frac{\Gamma(c+1)\Gamma(4-c)}{\Gamma((a+c)/2)\Gamma((a-c+3)/2)\Gamma((1-a+c)/2)\Gamma(2-(a+c)/2)} - \frac{c(c-3)}{a(a-1)},$$

and

$${}_3F_2\left(\begin{array}{c} a+2,\ 3-a,\ 1 \\ c+2,\ 5-c \end{array}\,\middle|\, 1\right)$$
$$= \frac{\pi}{2(a-2)_4}\frac{\Gamma(c+2)\Gamma(5-c)}{\Gamma((a+c)/2)\Gamma((a-c+3)/2)\Gamma((1-a+c)/2)\Gamma(2-(a+c)/2)}$$
$$- \frac{(c+1)(4-c)(c(3-c)+a(1-a))}{(a-2)_4}.$$

Special case 17. When $p = 3, q = 2$ and $x = 1$, by noting the Pfaff-Saalschutz formula and relation (3.2) we get

$$(3.9)\quad {}_3F_2\left(\begin{array}{c} b+m,\ \ -n+m,\ 1 \\ c+m,\ 2+b-c-n+\ m \end{array}\,\middle|\, 1\right)$$
$$= \frac{(c)_m(2+b-c-n)_m}{(b)_m(-n)_m}\left(\frac{(c-1)(c-1-b+n)}{(c-1-b)(c-1+n)} - {}_3^{(m-1)}F_2\left(\begin{array}{c} b,\ -n,\ 1 \\ c,\ 2+b-c-n \end{array}\,\middle|\, 1\right)\right).$$

For instance, if $m = 1, 2$, relation (3.9) is simplified as

$$ {}_3F_2\left(\begin{array}{c} b+1,\ -n+1,\ 1 \\ c+1,\ 3+b-c-n \end{array} \middle| 1\right) = \frac{c(c-2-b+n)}{nb}\left(\frac{(c-1)(c-1-b+n)}{(c-1-b)(c-1+n)} - 1\right), $$

and

$$ {}_3F_2\left(\begin{array}{c} b+2,\ -n+2,\ 1 \\ c+2,\ 4+b-c-n \end{array} \middle| 1\right) $$

$$ = \frac{(c-1)c(c+1)(c-1-b+n)(c-2-b+n)(c-3-b+n)}{n(n-1)b(b+1)(c-1-b)(c-1+n)} $$

$$ - \frac{(c+1)(c-3-b+n)(nb+c(c-2-b+n))}{n(n-1)b(b+1)}. $$

Special case 18. When $p = 4, q = 3$ and $x = -1$, by noting the Whipple formula and relation (3.2) we get

(3.10)
$$ {}_4F_3\left(\begin{array}{c} m+(3/2),\ b+m,\ c+m,\ 1 \\ m+(1/2),\ 2-b+m,\ 2-c+m \end{array} \middle| -1\right) = \frac{(-1)^m}{2m+1}\frac{(2-b)_m(2-c)_m}{(b)_m(c)_m} $$

$$ \times\left(\frac{\Gamma(2-b)\Gamma(2-c)}{\Gamma(2-b-c)} - {}^{(m-1)}_{\ \ 4}F_3\left(\begin{array}{c} 3/2,\ b,\ c,\ 1 \\ 1/2,\ 2-b,\ 2-c \end{array} \middle| -1\right)\right). $$

For instance, if $m = 1, 2$, relation (3.10) is simplified as

$$ {}_4F_3\left(\begin{array}{c} 5/2,\ b+1,\ c+1,\ 1 \\ 3/2,\ 3-b,\ 3-c \end{array} \middle| -1\right) = \frac{1}{3bc}((b-2)(c-2) - \frac{\Gamma(3-b)\Gamma(3-c)}{\Gamma(2-b-c)}), $$

and

$$ {}_4F_3\left(\begin{array}{c} 7/2,\ b+2,\ c+2,\ 1 \\ 5/2,\ 4-b,\ 4-c \end{array} \middle| -1\right) = \frac{(2-b)_2(2-c)_2}{5\,(b)_2(c)_2} $$

$$ \times\left(\frac{\Gamma(2-b)\Gamma(2-c)}{\Gamma(2-b-c)} + 2\frac{bc+b+c-2}{(b-2)(c-2)}\right). $$

Special case 19. When $p = 5, q = 4$ and $x = 1$, by noting the Dougall formula and relation (3.2) we get

$$ {}_5F_4\left(\begin{matrix} m+(3/2),\ c+m,\ d+m,\ e+m,\ 1 \\ m+(1/2),\ 2-c+m,\ 2-d+m,\ 2-e+m \end{matrix} \,\middle|\, 1 \right) \tag{3.11} $$

$$ = \frac{1}{2m+1} \frac{(2-c)_m(2-d)_m(2-e)_m}{(c)_m(d)_m(e)_m} $$

$$ \times \left(\frac{\Gamma(2-c)\Gamma(2-d)\Gamma(2-e)\Gamma(2-c-d-e)}{\Gamma(2-d-e)\Gamma(2-c-e)\Gamma(2-c-d)} - {}_5^{(m-1)}F_4\left(\begin{matrix} 3/2,\ c,\ d,\ e,\ 1 \\ 1/2,\ 2-c,\ 2-d,\ 2-e \end{matrix} \,\middle|\, 1 \right) \right). $$

For instance, if $m = 1, 2$, relation (3.11) is simplified as

$$ {}_5F_4\left(\begin{matrix} 5/2,\ c+1,\ d+1,\ e+1,\ 1 \\ 3/2,\ 3-c,\ 3-d,\ 3-e \end{matrix} \,\middle|\, 1 \right) = \frac{(2-c)(2-d)(2-e)}{3\,cde} $$

$$ \times \left(\frac{\Gamma(2-c)\Gamma(2-d)\Gamma(2-e)\Gamma(2-c-d-e)}{\Gamma(2-d-e)\Gamma(2-c-e)\Gamma(2-c-d)} - 1 \right), $$

and

$$ {}_5F_4\left(\begin{matrix} 7/2,\ c+2,\ d+2,\ e+2,\ 1 \\ 5/2,\ 4-c,\ 4-d,\ 4-e \end{matrix} \,\middle|\, 1 \right) = \frac{(2-c)_2(2-d)_2(2-e)_2}{5\,(c)_2(d)_2(e)_2} $$

$$ \times \left(\frac{\Gamma(2-c)\Gamma(2-d)\Gamma(2-e)\Gamma(2-c-d-e)}{\Gamma(2-d-e)\Gamma(2-c-e)\Gamma(2-c-d)} - \frac{3\,cde}{(2-c)(2-d)(2-e)} - 1 \right). $$

Special case 20. When $p = 7, q = 6$ and $x = 1$, by noting the Dougall formula and relation (3.2) we get

(3.12)
$$ {}_7F_6\left(\begin{matrix} m+(3/2),\ b+m, c+m,\ d+m,\ 3-b-c+n+m,\ m-n,\ 1 \\ m+(1/2), 2-b+m, 2-c+m, 2-d+m,\ b+c+d-n-1+m, n+2+m \end{matrix} \,\middle|\, 1 \right) $$

$$ = \frac{1}{2m+1} \frac{(2-b)_m(2-c)_m(2-d)_m(b+c+d-n-1)_m(n+2)_m}{(b)_m(c)_m(d)_m(3-b-c+n)_m(-n)_m} $$

$$ \times \left(\frac{(2)_n(2-b-c)_n(2-b-d)_n(2-c-d)_n}{(2-b)_n(2-c)_n(2-d)_n(2-b-c-d)_n} \right. $$

$$ \left. - {}_7^{(m-1)}F_6\left(\begin{matrix} 3/2,\ b, c,\ d,\ 3-b-c+n,\ -n,\ 1 \\ 1/2, 2-b, 2-c, 2-d,\ b+c+d-n-1, n+2 \end{matrix} \,\middle|\, 1 \right) \right). $$

For instance, if $m = 1, 2$, relation (3.12) is simplified as

$$
{}_7F_6\left(\begin{array}{c} 5/2,\ b+1, c+1,\ d+1,\ 3-b-c+n+1,\ 1-n,\ 1 \\ 3/2,\ 3-b,\ 3-c,\ 3-d,\ b+c+d-n, n+3 \end{array}\middle|\ 1\right)
$$

$$
= \frac{(2-b)(2-c)(2-d)(b+c+d-n-1)(n+2)}{3\,nbcd(3-b-c+n)}
$$

$$
\times\left(1 - \frac{(2)_n(2-b-c)_n(2-b-d)_n(2-c-d)_n}{(2-b)_n(2-c)_n(2-d)_n(2-b-c-d)_n}\right),
$$

and

$$
{}_7F_6\left(\begin{array}{c} 7/2,\ b+2, c+2,\ d+2,\ 5-b-c+n,\ 2-n,\ 1 \\ 5/2,\ 4-b,\ 4-c,\ 4-d,\ b+c+d-n+1, n+4 \end{array}\middle|\ 1\right)
$$

$$
= \frac{(2-b)_2(2-c)_2(2-d)_2(b+c+d-n-1)_2(n+2)_2}{5\,(b)_2(c)_2(d)_2(3-b-c+n)_2(-n)_2}
$$

$$
\times\left(\frac{(2)_n(2-b-c)_n(2-b-d)_n(2-c-d)_n}{(2-b)_n(2-c)_n(2-d)_n(2-b-c-d)_n}\right.
$$

$$
\left. + \frac{3\,bcd\,n\,(3-b-c+n)}{(2-b)(2-c)(2-d)(b+c+d-n-1)(n+2)} - 1\right).
$$

Conclusion. In this paper, we applied two identities for generalized hypergeometric series in order to extend some classical summation theorems of hypergeometric functions such as Gauss, Kummer, Dixon, Watson, Whipple, Pfaff-Saalschutz and Dougall formulas and then obtained some new summation theorems using the second introduced hypergeometric identity.

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