On a structure formula for classical $q$-orthogonal polynomials. (English summary)


Summary: “The classical orthogonal polynomials are given as the polynomial solutions $P_n(x)$ of the differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where $\sigma(x)$ turns out to be a polynomial of at most second degree and $\tau(x)$ is a polynomial of first degree. In a similar way, the classical discrete orthogonal polynomials are the polynomial solutions of the difference equation

$$\sigma(x)\Delta \nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0,$$

where $\Delta y(x) = y(x+1) - y(x)$ and $\nabla y(x) = y(x) - y(x-1)$ denote the forward and backward difference operators, respectively. Finally, the classical $q$-orthogonal polynomials of the Hahn tableau are the polynomial solutions of the $q$-difference equation

$$\sigma(x)D_q D_{1/q} y(x) + \tau(x)D_q y(x) + \lambda_{q,n} y(x) = 0,$$

where

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1,$$

denotes the $q$-difference operator. We show by a purely algebraic deduction—without using the orthogonality of the families considered—that a structure formula of the type

$$\sigma(x) D_{1/q} P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x),$$

$n \in \mathbb{N} := \{1, 2, 3, \ldots \}$, is valid. Moreover, our approach not only proves this assertion, but generates the form of this structure formula. A similar argument applies to the discrete and
continuous cases and yields
\[ \sigma(x)\nabla P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \in \mathbb{N}, \]
and
\[ \sigma(x)P'_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x), \quad n \in \mathbb{N}. \]
Whereas the latter formulas are well known, their previous deduction used the orthogonality property. Hence our approach is also of interest in these cases.”

Reviewed by H. S. P. Shrivastava

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