
2003i:33010
Koepf, Wolfram (D-HTWK-NS); **Schmersau, Dieter** (D-FUB)

Recurrence equations and their classical orthogonal polynomial solutions. (English summary)

Orthogonal systems and applications.
Appl. Math. Comput. **128** (2002), no. 2-3, 303–327.
33C45 (34B30 42C05)

A classical theorem of S. Bochner [Math. Z. **29** (1929), 730–736; JFM 55.0260.01] characterizes the classical orthogonal polynomials of Bessel, Hermite, Laguerre and Legendre as the only orthogonal polynomials which are solutions of the second order Sturm-Liouville type equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n(x)y = 0, \tag{1}$$

where $\sigma(x)$ and $\tau(x)$ are quadratic and linear polynomials, respectively. Similarly, theorems of O. E. Lancaster [Amer. J. Math. **63** (1941), 185–207; MR **2**, 132g] and P. Lesky [Monatsh. Math. **66** (1962), 203–214; MR **25** #3288] characterize the Charlier, Meixner, Krawtchouk and Hahn polynomials as the only orthogonal polynomial solutions of the finite difference analog of (1). And the major work of W. Hahn [Math. Nachr. **2** (1949), 4–34; MR **11**, 29b] provided a characterization of a large class of orthogonal q-polynomials as the solutions of the q-difference analog of (1).

In the present paper, the authors consider the orthogonal polynomial solutions of (1) and describe a procedure for expressing the coefficients in the classical three-term recurrence relation

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$$

in terms of the coefficients in (1) and the leading coefficients of the polynomials. A similar procedure applies for the orthogonal solutions of the finite difference and q-difference analogs of (1). The results of the authors' procedures are summarized in a theorem which gives explicit formulas for these coefficients in terms of the coefficients in $\sigma(x), \tau(x)$, and λ_n and the leading coefficients of the orthogonal polynomials. The authors then consider a sort of inverse problem:

given a general three-term recurrence relation with polynomial coefficients

(2) $$q_n(x)P_{n+2}(x) + r_n(x)P_{n+1}(x) + s_n(x)P_n(x) = 0,$$

determine if there are classical orthogonal polynomial solutions. The authors then describe an algorithm that determines if (2) has such solutions and identifies them. Computer implementation of the algorithm is also discussed. The corresponding problems for the finite difference and q-difference analogs of (1) are also addressed and solved. A number of examples are provided.

Reviewed by *T. S. Chihara*