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Bounded nonvanishing functions and Bateman functions. (English summary)

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An outstanding, difficult problem is that of finding sharp bounds for the coefficients of functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ which are analytic, bounded by 1, and nonvanishing in the unit disc. Trivially, all $|a_n|$ are bounded by 1. The Krzyż conjecture is that $|a_n| \le 2/e$ for all $n > 0$. This has been proved for only $n = 1, 2, 3$, and 4. In an earlier paper, S. Scheinberg, L. Zalcman and the reviewer [J. Analyse Math. **31** (1977), 169–190; MR **58** #11358] considered this problem from a number of different points of view and since then a number of papers have appeared considering different aspects of the problem. The authors consider the problem from the point of view of subordination, observing that any $f(z)$ in the class is subordinate to $\exp\{-t(1+z)/(1-z)\} = \sum_{n=1}^{\infty} F_n(t) z^n$, and studying bounds on the $F_n(t)$. (Notice that it is not even immediately evident that all of the $F_n(t)$ satisfy the Krzyż conjecture.) These $F_n(t)$ turn out to be functions studied by Bateman via an integral representation. Results of Bateman and in this paper give a difference equation, a differential equation, and residue representations. Examples of results obtained are that $|F_n(t)|$ is bounded above by $(1-e^{-2t})^{1/2}$, $2n/t$, and $(4t/(2n-t))^{1/2}$. It is then shown that $|F_n(t)| \le 2/e$ for all t and all $n > 21138$. The paper makes use of methods more usually used in the study of special functions and contains a great many results about the Bateman functions $F_n(t)$.

Reviewed by *James A. Hummel*