On computing two special cases of Gauss hypergeometric function

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Abstract. In this paper, a computational technique is given to re-obtain the explicit forms of two cases of the Gauss hypergeometric function ${}_2F_1(a,b,c;x)$ for $b=a+1/2$ and $c=1/2,3/2$. Some special identities related to the two aforesaid cases are also introduced. Finally a special Maple package, called FormalPowerSeries (FPS), is used to automatically compute some results given in the paper.

Keywords. Gauss hypergeometric function, Symmetric orthogonal polynomials, Euler identity, Legendre's duplication formula, Kummer formula.

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1. Introduction

Let us begin with the Gauss hypergeometric differential equation [2, 8]:

$$x(1-x)\,y''(x)+(\gamma-(\alpha+\beta+1)x)\,y'(x)-\alpha\beta\,y(x)=0\,, \tag{1}$$

where α,β and γ are constant parameters. Since the indicial equation of (1), i.e.

$$r^2-(1-\gamma)\,r=0\,, \tag{2}$$

has two roots $r_1=0$ and $r_2=1-\gamma$, by using the Frobenius method the series solution of the Gauss equation for $r_1=0$ can be expressed as

$$y_1(x)=1+\frac{\alpha\beta}{\gamma}\frac{x}{1!}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!}+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{x^3}{3!}+\dots, \tag{3}$$

where $\gamma\neq 0,-1,-2,-3,\dots$ and the series converges for $-1<x<1$. This Taylor series expansion at $x=0$ is called *Gauss hypergeometric series* and its sum, denoted by ${}_2F_1\left(\begin{matrix}\alpha & \beta\\ & \gamma\end{matrix}\middle|\, x\right)$ or ${}_2F_1(\alpha,\beta,\gamma;x)$, is called Gauss hypergeometric function. So, we have

$$y_1(x) = {}_2F_1\left(\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!} \quad \text{where} \quad (\alpha)_n = \prod_{i=0}^{n-1} (\alpha + i). \tag{4}$$

A solution basis of differential equation (1) is given by [2]

$$y_1 = {}_2F_1\left(\begin{matrix}\alpha & \beta \\ & \gamma\end{matrix}\middle| x\right) \quad \text{and} \quad y_2 = x^{1-\gamma}\, {}_2F_1\left(\begin{matrix}\alpha+1-\gamma, & \beta+1-\gamma \\ & 2-\gamma\end{matrix}\middle| x\right), \tag{5}$$

in which $\gamma, \alpha-\beta$ and $\gamma-\alpha-\beta$ are all non-integers so that the general solution of (1) is given as linear combination $y = A\, y_1 + B\, y_2$ where A, B are two constants.
The importance of the Gauss hypergeometric function is that many elementary and special functions of mathematical physics can directly be expressed in terms of it, see e.g. [3, 5, 6, 9, 13]. Some examples are, respectively:

$${}_2F_1(-p,1,1;x) = (1-x)^p, \tag{6}$$

$${}_2F_1(1,1,2;-x) = \frac{\ln(1+x)}{x}, \tag{7}$$

$${}_2F_1(a,1-b,a+1;x) = ax^{-a} B_x(a,b) = ax^{-a} \int_0^x t^{a-1} (1-t)^{b-1}\, dt, \tag{8}$$

$${}_2F_1\left(\frac{1}{2},\frac{1}{2},\frac{3}{2};x\right) = \frac{\arcsin\sqrt{x}}{\sqrt{x}}, \tag{9}$$

$${}_2F_1\left(\frac{1}{2},1,\frac{3}{2};-x\right) = \frac{\arctan\sqrt{x}}{\sqrt{x}}, \tag{10}$$

classical orthogonal polynomials like the Jacobi polynomials

$$\binom{n+\alpha}{n}\, {}_2F_1\left(\begin{matrix}-n, & n+\alpha+\beta+1 \\ & \alpha+1\end{matrix}\middle| \frac{1-x}{2}\right) = P_n^{(\alpha,\beta)}(x), \tag{11}$$

further

$$x^n\, {}_2F_1\left(\begin{matrix}-[n/2], & (q-s)/2q - [(n+1)/2] \\ & -(r+(2n-3)p)/2p\end{matrix}\middle| -\frac{q}{px^2}\right) = \bar{S}_n\left(\begin{matrix}r & s \\ p & q\end{matrix}\middle| x\right), \tag{12}$$

where $\bar{S}_n(p,q,r,s;x)$ is the monic form of a basic class of symmetric orthogonal polynomials, see [12], and

$$x^{(n+\frac{1-(-1)^n}{2}(\theta-1))}\, {}_2F_1\left(\begin{matrix}-[n/2], & \dfrac{2-\theta+(-1)^n(\theta-1)}{2} - \dfrac{s}{2q} - [(n+1)/2] \\ & (4-\theta-2n+(-1)^n(\theta-1))/2 - r/2p\end{matrix}\middle| -\frac{q}{px^2}\right) = S_n^{(\theta)}\left(\begin{matrix}r & s \\ p & q\end{matrix}\middle| x\right) \tag{13}$$

where $S_n^{(\theta)}(p,q,r,s;x)$ is a basic class of symmetric orthogonal functions [11] that generalizes the polynomials (12) for $\theta = 1$ and finally

$$J_n^{(p,q)}(x;a,b,c,d) = (-1)^n((ab+cd)+i(ad-bc))^n(n+1-2p)_n \times$$
$$\sum_{k=0}^{n}\binom{n}{k}\left(\frac{a^2+c^2}{(ab+cd)+i(ad-bc)}\right)^k {}_2F_1\left(\left.\begin{matrix} k-n \quad p-n-iq/2 \\ 2p-2n \end{matrix}\right| \frac{2(ad-bc)}{(ad-bc)-i(ab+cd)}\right)x^k, \tag{14}$$

which is a class of real polynomials orthogonal with respect to the generalized T student distribution weight function $((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan\dfrac{ax+b}{cx+d})$ on $(-\infty,\infty)$, see [10].

There are also special formulas of the Gauss hypergeometric function including constant parameters. For instance, the Kummer and Gauss formulas [8] are two well-known examples:

$${}_2F_1(a,b,-a+b+1;-1) = \frac{\Gamma(\frac{1}{2}b+1)\Gamma(b-a+1)}{\Gamma(b+1)\Gamma(\frac{1}{2}b-a+1)} \quad \text{(Kummer formula)}, \tag{15}$$

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{(Gauss formula)}. \tag{16}$$

It should be added that at a number of very special points, the Gauss hypergeometric function can assume specific values. For example in [14, 15] it is shown that

$${}_2F_1(\frac{1}{6},\frac{1}{3},\frac{1}{2};\frac{25}{27}) = \frac{3}{4}\sqrt{3}, \tag{17}$$

$${}_2F_1(\frac{1}{6},\frac{1}{2},\frac{2}{3};\frac{125}{128}) = \frac{4}{3}\sqrt[6]{2}, \tag{18}$$

$${}_2F_1(\frac{1}{12},\frac{5}{12},\frac{1}{2};\frac{1323}{1331}) = \frac{3}{4}\sqrt[4]{11}. \tag{19}$$

In this reviewing article, we present a computational method for re-obtaining two other special cases of Gauss hypergeometric function, i.e.

${}_2F_1\left(\left.\begin{matrix} a, \quad a+1/2 \\ 1/2 \end{matrix}\right| x\right)$ and ${}_2F_1\left(\left.\begin{matrix} a, \quad a+1/2 \\ 3/2 \end{matrix}\right| x\right)$ [1, p.556, Re. 15.1.9 and Re. 15.1.10]

in which $a \in \mathbf{R}$ in the convergence region $-1 < x < 1$, and then introduce some special identities related to these two special cases. We also implement the Formal Power Series (FPS) algorithm, which was designed and constructed by W. Koepf in 1992 [7], in Maple on our computational results in the next section.

2. Computing ${}_2F_1(a,\, a+1/2, 1/2; x)$ and ${}_2F_1(a,\, a+1/2, 3/2; x)$

Due to Euler's identity [1, 2]

$$\exp(it) = \cos t + i\sin t \quad (i = \sqrt{-1}), \tag{20}$$

and Moivre's identity $(\exp(it))^p = \exp(i(pt))$ for $p \in \mathbf{R}$, one gets first

$$\cos pt + i \sin pt = (\cos t + i \sin t)^p = \sum_{k=0}^{\infty} \binom{p}{k} i^k \sin^k t \cos^{p-k} t, \tag{21}$$

which results in

$$\begin{cases} \cos pt = \sum_{k=0}^{\infty} (-1)^k \binom{p}{2k} \sin^{2k} t \cos^{p-2k} t, \\ \sin pt = \sum_{k=0}^{\infty} (-1)^k \binom{p}{2k+1} \sin^{2k+1} t \cos^{p-(2k+1)} t. \end{cases} \tag{22}$$

The two above formulas can be rewritten as

$$\begin{cases} \dfrac{\cos pt}{\cos^p t} = \sum_{k=0}^{\infty} \binom{p}{2k} (-\tan^2 t)^k, \\ \dfrac{1}{\cos^{p-1} t} \dfrac{\sin pt}{\sin t} = \sum_{k=0}^{\infty} \binom{p}{2k+1} (-\tan^2 t)^k. \end{cases} \tag{23}$$

On the other hand, the coefficients $C(p,2k)$ and $C(p,2k+1)$ in (23) can be written in terms of the Pochhammer symbol

$$(p)_k = p(p+1)...(p+k-1) = \frac{\Gamma(p+k)}{\Gamma(p)}, \tag{24}$$

in which $\Gamma(p) = \int_0^\infty x^{p-1} \exp(-x)\, dx$ denotes the Gamma function [1,2,8]. For this purpose, we should first note that

$$\binom{p}{2k} = \frac{p!}{(2k)!(p-2k)!} = \frac{\Gamma(p+1)}{\Gamma(2k+1)\Gamma(p+1-2k)}. \tag{25}$$

Now, according to Legendre's duplication formula [2]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma(z+\frac{1}{2}), \tag{26}$$

and the two identities

$$\Gamma(\beta+k) = \Gamma(\beta)(\beta)_k \quad \text{and} \quad \Gamma(\beta-k) = \frac{\Gamma(\beta)(-1)^k}{(1-\beta)_k}, \tag{27}$$

following from (24) the denominator term in the fraction (25) is simplified as

$$
\begin{aligned}
\Gamma(2k+1)\Gamma(p+1-2k) &= \frac{2^p}{\pi}\Gamma(k+1)\Gamma(k+\frac{1}{2})\Gamma(\frac{p+1}{2}-k)\Gamma(\frac{p+2}{2}-k) \\
&= \frac{2^p}{\pi}k!\,\Gamma(\frac{1}{2})\,(\frac{1}{2})_k\,\Gamma(\frac{p+1}{2})\Gamma(\frac{p+2}{2})\Big/(\frac{1-p}{2})_k(-\frac{p}{2})_k\,.
\end{aligned}
\tag{28}
$$

Therefore we have

$$
\binom{p}{2k} = \frac{\sqrt{\pi}\,2^{-p}\,\Gamma(p+1)}{\Gamma(\frac{p+1}{2})\Gamma(\frac{p+2}{2})}\frac{(\frac{1-p}{2})_k(-\frac{p}{2})_k}{(\frac{1}{2})_k\,k!} = \frac{(\frac{1-p}{2})_k(-\frac{p}{2})_k}{(\frac{1}{2})_k}\frac{1}{k!}. \tag{29}
$$

Note in (29) that the Legendre formula implies $\sqrt{\pi}\,2^{-p}\,\Gamma(p+1) = \Gamma(\frac{p+1}{2})\Gamma(\frac{p+2}{2})$.
Consequently, substituting (29) into the first relation of (23) yields:

$$
{}_2F_1\left(\left.\begin{matrix} -\frac{p}{2}, & \frac{1-p}{2} \\ & 1/2 \end{matrix}\right| -\tan^2 t\right) = \frac{\cos pt}{\cos^p t}. \tag{30}
$$

If for simplicity $-p/2 = a$ and $-\tan^2 t = x$, identity (30) is finally transformed to

$$
{}_2F_1\left(\left.\begin{matrix} a, & a+1/2 \\ & 1/2 \end{matrix}\right| x\right) = (1-x)^{-a}\cos(2a\arctan\sqrt{-x})\,. \tag{31}
$$

But as we pointed out in (3), the convergence region of the Gauss hypergeometric series is $-1 < x < 1$ and since $\arctan\sqrt{-x}$ is just defined on the real line for $x \le 0$, formula (31) should only be considered for $-1 < x \le 0$. Therefore we also compute the explicit form of ${}_2F_1(a,\, a+1/2, 1/2; x)$ for positive $0 \le x < 1$. To reach this goal, we use two different ways. The first way is as follows:

Let $z = (a,b) = a + ib = \sqrt{a^2+b^2}\exp(i\arctan\frac{b}{a})$ and set $a = 1$ and $b = ix$ for $x \ge 0$. So

$$
1-x = \sqrt{1-x^2}\exp(i\arctan ix) \;\Rightarrow\; \arctan\sqrt{-x} = \arctan i\sqrt{x} = -\frac{i}{2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}. \tag{32}
$$

By substituting this result in (31) one gets for $x \ge 0$

$$
\begin{aligned}
{}_2F_1\left(\left.\begin{matrix} a, & a+1/2 \\ & 1/2 \end{matrix}\right| x\right) &= (1-x)^{-a}\cos(2a\arctan\sqrt{-x}) = (1-x)^{-a}\cos(-ia\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}) \\
&= (1-x)^{-a}\cosh(a\ln\frac{1-\sqrt{x}}{1+\sqrt{x}}) = \frac{1}{2}\left((1+\sqrt{x})^{-2a} + (1-\sqrt{x})^{-2a}\right),
\end{aligned}
\tag{33}
$$

which is valid for $0 \le x < 1$ and $a \in \mathbf{R}$. The second way is to use the Taylor expansion

$$\frac{(1+x)^p+(1-x)^p}{2}=\sum_{k=0}^{\infty}\binom{p}{2k}x^{2k}, \tag{34}$$

which subsequently yields

$$\frac{(1+\sqrt{x})^{-2a}+(1-\sqrt{x})^{-2a}}{2}=\sum_{k=0}^{\infty}\binom{-2a}{2k}x^k={}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 1/2\end{matrix}\middle|\, x\right) \quad (0\le x<1). \tag{35}$$

Combining (31) and (35) gives the following well-known identity:

$${}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 1/2\end{matrix}\middle|\, x\right)=\begin{cases}(1-x)^{-a}\cos(2a\arctan\sqrt{-x}) & \text{if} \quad -1<x\le 0,\\ \frac{1}{2}\left((1+\sqrt{x})^{-2a}+(1-\sqrt{x})^{-2a}\right) & \text{if} \quad 0\le x<1.\end{cases} \tag{36}$$

For instance, for $x=-1/3,1/2$ (36) yields

$$\begin{cases}{}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 1/2\end{matrix}\middle|\, -\frac{1}{3}\right)=(\frac{4}{3})^{-a}\cos(a\frac{\pi}{3}),\\ {}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 1/2\end{matrix}\middle|\, \frac{1}{2}\right)=2^{2a-1}\left((2+\sqrt{2})^{-2a}+(2-\sqrt{2})^{-2a}\right).\end{cases} \tag{37}$$

It is interesting to notice that (36) is also valid for the two boundary points $x=-1$ and $x=1$, because according to Gauss's formula (16) we have

$${}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 1/2\end{matrix}\middle|\, 1\right)=\frac{\sqrt{\pi}\,\Gamma(-2a)}{\Gamma(1/2-a)\Gamma(-a)}=2^{-2a-1}, \tag{38}$$

which can directly be proved for $a<0$ if one applies Legendre's duplication formula. Similarly, for $x=-1$, according to Kummer's formula (15) we have

$${}_2F_1\left(\begin{matrix}a, & a+1/2\\ & 1/2\end{matrix}\middle|\, -1\right)={}_2F_1\left(\begin{matrix}a+1/2, & a\\ & 1/2\end{matrix}\middle|\, -1\right)=\frac{\sqrt{\pi}\,\Gamma(a/2+1)}{\Gamma(a+1)\Gamma((1-a)/2)}=2^{-a}\cos(a\frac{\pi}{2}), \tag{39}$$

which again can be proved for $-1<a<1$ if one applies Legendre's duplication formula and furthermore the well-known identity $\Gamma(q)\Gamma(1-q)=\pi/\sin(q\pi)$ $(0<q<1)$.

Similarly, to compute ${}_2F_1(a,a+1/2,3/2;x)$, the coefficient $C(p,2k+1)$ can be rewritten as

$$\binom{p}{2k+1}=\frac{(\frac{2-p}{2})_k(\frac{1-p}{2})_k}{(\frac{3}{2})_k}\frac{p}{k!}, \tag{40}$$

which changes the second formula of (23) to

$$ {}_2F_1\left(\left.\begin{matrix} \frac{1-p}{2}, & 1-\frac{p}{2} \\ & 3/2 \end{matrix}\right| -\tan^2 t\right) = \frac{\sin(pt)}{p\sin t}\frac{1}{\cos^{p-1} t}. \tag{41} $$

Now if for simplicity $(1-p)/2 = a$ and $-\tan^2 t = x$, (41) becomes

$$ {}_2F_1\left(\left.\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\right| x\right) = (1-x)^{-a}\frac{\sin((2a-1)\arctan\sqrt{-x})}{(2a-1)\sin(\arctan\sqrt{-x})}, \tag{42} $$

which holds for $-1 < x \le 0$ because $\arctan\sqrt{-x}$ is defined on $x \le 0$.
Again, to compute the explicit form of ${}_2F_1(a, a+1/2, 3/2; x)$ for positive $0 \le x < 1$, there are two different ways. The first way is to substitute (32) in (42), i.e.

$$ \begin{aligned} {}_2F_1\left(\left.\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\right| x\right) &= (1-x)^{-a}\frac{\sin((2a-1)\arctan\sqrt{-x})}{(2a-1)\sin(\arctan\sqrt{-x})} = (1-x)^{-a}\frac{\sin(-(a-\frac{1}{2})i\ln\frac{1-\sqrt{x}}{1+\sqrt{x}})}{(2a-1)\sin(-\frac{i}{2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}})} \\ &= (1-x)^{-a}\frac{\sinh((a-\frac{1}{2})\ln\frac{1-\sqrt{x}}{1+\sqrt{x}})}{(2a-1)\sinh(\frac{1}{2}\ln\frac{1-\sqrt{x}}{1+\sqrt{x}})} = \frac{(1+\sqrt{x})^{1-2a}-(1-\sqrt{x})^{1-2a}}{2(1-2a)\sqrt{x}}. \end{aligned} \tag{43} $$

The second way is to apply the Taylor expansion

$$ \frac{(1+x)^p-(1-x)^p}{2x} = \sum_{k=0}^{\infty}\binom{p}{2k+1}x^{2k}, \tag{44} $$

which results in

$$ \frac{(1+\sqrt{x})^{1-2a}-(1-\sqrt{x})^{1-2a}}{2\sqrt{x}} = \sum_{k=0}^{\infty}\binom{1-2a}{2k+1}x^k = (1-2a)\ {}_2F_1\left(\left.\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\right| x\right). \tag{45} $$

Combining (42) and (45) gives the second well-known identity as:

$$ {}_2F_1\left(\left.\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\right| x\right) = \begin{cases} (1-x)^{-a}\dfrac{\sin((2a-1)\arctan\sqrt{-x})}{(2a-1)\sin(\arctan\sqrt{-x})} & \text{if} \quad -1 < x \le 0, \\ \dfrac{(1+\sqrt{x})^{1-2a}-(1-\sqrt{x})^{1-2a}}{2(1-2a)\sqrt{x}} & \text{if} \quad 0 \le x < 1. \end{cases} \tag{46} $$

For instance, if $x = -1/3, 1/4$ then (46) is reduced to

$$\begin{cases} {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\middle| -\frac{1}{3}\right) = \dfrac{3^a}{2^{2a-1}(2a-1)} \sin((2a-1)\dfrac{\pi}{6}), \\ {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\middle| \frac{1}{4}\right) = \dfrac{1-3^{(1-2a)}}{2^{(1-2a)}(2a-1)}. \end{cases} \tag{47}$$

Moreover, for $a = 1/2$ there exists a limiting case in (46) which directly generates identity (10), because we have

$${}_2F_1\left(\begin{matrix} 1/2, & 1 \\ 3/2 & \end{matrix}\middle| x\right) = \lim_{a \to 1/2} \frac{(1-x)^{-a}}{\sin(\arctan\sqrt{-x})} \frac{\sin((2a-1)\arctan\sqrt{-x})}{(2a-1)} = \frac{\arctan\sqrt{-x}}{\sqrt{-x}}. \tag{48}$$

Note that the result (46) is also valid for the two boundary points $x = -1$ and $x = 1$, because according to (16) the identity

$${}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\middle| 1\right) = \frac{\Gamma(3/2)\Gamma(1-2a)}{\Gamma(3/2-a)\Gamma(1-a)} = \frac{2^{-2a}}{1-2a}, \tag{49}$$

can be proved for $a < 1/2$ if one applies Legendre's duplication formula, and also according to (15) identity

$${}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 3/2 \end{matrix}\middle| -1\right) = \frac{\Gamma(5/4+a/2)\Gamma(3/2)}{\Gamma(5/4-a/2)\Gamma(3/2+a)} = \frac{2^{(1/2-a)}}{2a-1} \sin((2a-1)\frac{\pi}{4}), \tag{50}$$

can be proved for $-3 < 2a < 5$ if Legendre's duplication formula and the relation $\Gamma(q)\Gamma(1-q) = \pi/\sin(q\pi)$ are used simultaneously.

2.1. Some identities related to ${}_2F_1(a, a+1/2, 1/2; x)$ and ${}_2F_1(a, a+1/2, 3/2; x)$

Since relations (30) and (41) have trigonometric forms, various identities can be derived for the two mentioned hypergeometric functions. Here we introduce three examples of these identities.

i) By referring to (30) and (41) one can first conclude

$$\begin{aligned} &{}_2F_1\left(\begin{matrix} \frac{1-p}{2}, & -\frac{p}{2} \\ & 1/2 \end{matrix}\middle| -\tan^2 t\right) {}_2F_1\left(\begin{matrix} \frac{1-p}{2}, & 1-\frac{p}{2} \\ & 3/2 \end{matrix}\middle| -\tan^2 t\right) = \frac{\sin(2p)t}{(2p)\sin t} \frac{1}{\cos^{2p-1} t} \\ &= {}_2F_1\left(\begin{matrix} 1/2-p, & 1-p \\ & 3/2 \end{matrix}\middle| -\tan^2 t\right). \end{aligned} \tag{51}$$

Clearly this relation is simplified as follows if one assumes $-p/2 = a$ and $-\tan^2 t = x$:

$$ {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 1/2 \end{matrix}\middle| x\right) {}_2F_1\left(\begin{matrix} a+1, & a+1/2 \\ & 3/2 \end{matrix}\middle| x\right) = {}_2F_1\left(\begin{matrix} 2a+1, & 2a+1/2 \\ & 3/2 \end{matrix}\middle| x\right). \tag{52} $$

ii) Since we have

$$ \left({}_2F_1\left(\begin{matrix} \frac{1-p}{2}, & -\frac{p}{2} \\ & 1/2 \end{matrix}\middle| -\tan^2 t\right)\right)^2 = \frac{\cos^2 pt}{\cos^{2p} t} = \frac{1}{2\cos^{2p} t} + \frac{1}{2}\, {}_2F_1\left(\begin{matrix} \frac{1-2p}{2}, & -p \\ & 1/2 \end{matrix}\middle| -\tan^2 t\right), \tag{53} $$

by taking $-\tan^2 t = x$ and $-p = a$ in (53) and noting the well-known identity [1,2]

$$ \cos^{2a}(\arctan\sqrt{-x}) = (1-x)^{-a} = {}_2F_1\left(\begin{matrix} a, & 1/2 \\ & 1/2 \end{matrix}\middle| x\right), \tag{54} $$

we finally obtain

$$ {}_2F_1\left(\begin{matrix} a, & 1/2 \\ & 1/2 \end{matrix}\middle| x\right) + {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 1/2 \end{matrix}\middle| x\right) = 2\ {}_2F_1^2\left(\begin{matrix} a/2, & (a+1)/2 \\ & 1/2 \end{matrix}\middle| x\right). \tag{55} $$

iii) Since

$$ \left({}_2F_1\left(\begin{matrix} \frac{1-p}{2}, & 1-\frac{p}{2} \\ & 3/2 \end{matrix}\middle| -\tan^2 t\right)\right)^2 = \frac{\sin^2 pt}{p^2 \sin^2 t \cos^{2p-2} t} $$
$$ = \frac{1}{2p^2 \tan^2 t\, \cos^{2p} t} - \frac{1}{2p^2 \tan^2 t}\, {}_2F_1\left(\begin{matrix} \frac{1-2p}{2}, & -p \\ & 1/2 \end{matrix}\middle| -\tan^2 t\right), \tag{56} $$

again taking $-\tan^2 t = x$ and $-p = a$ in (56) yields

$$ {}_2F_1\left(\begin{matrix} a, & 1/2 \\ & 1/2 \end{matrix}\middle| x\right) - {}_2F_1\left(\begin{matrix} a, & a+1/2 \\ & 1/2 \end{matrix}\middle| x\right) = (2a^2 x)\ {}_2F_1^2\left(\begin{matrix} a/2+1, & (a+1)/2 \\ & 3/2 \end{matrix}\middle| x\right). \tag{57} $$

2.2. Appendix: FPS Algorithm yields previous results

In 1992, W. Koepf introduced the FormalPowerSeries (FPS) algorithm [7] and implemented it in the Maple software. The algorithm is accessible via the Maple command "convert(…,FormalPowerSeries)", see also [4]. In this section, using this special package, we wish to directly compute some results given in the paper and point out that they can be determined completely automatically from right to left by the FPS algorithm, see also [9]. Similarly note that the algorithmic procedure given by Roach in [13], which is also partially implemented in Maple, yields the conversion from left to right.

2.3. Maple computations

```
> read "hsum13.mpl":
```

Package "Hypergeometric Summation", Maple V - Maple 13

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Formula (6) :

```
> fps:=convert((1-x)^p,FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{\mathrm{pochhammer}(-p, k)\, x^k}{k!}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \mathrm{Hypergeom}([-p], [\ ], x)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$(1-x)^p$$

Formula (9):

```
> fps:=convert(arcsin(sqrt(x))/sqrt(x),FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{(2\,k)!\, 4^{(-k)}\, x^k}{(k!)^2\,(2\,k+1)}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \mathrm{Hypergeom}\left(\left[\frac{1}{2}, \frac{1}{2}\right], \left[\frac{3}{2}\right], x\right)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$\frac{\arcsin(\sqrt{x})}{\sqrt{x}}$$

Formula (10):

```
> fps:=convert(arctan(sqrt(x))/sqrt(x),FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{(-1)^k\, x^k}{2\,k+1}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \mathrm{Hypergeom}\left(\left[\frac{1}{2}, 1\right], \left[\frac{3}{2}\right], -x\right)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$\frac{1}{2}\,\frac{\ln\left(-\frac{1+\sqrt{-x}}{-1+\sqrt{-x}}\right)}{\sqrt{-x}}$$

Formula (11):

```
> fps:=convert(JacobiP(n,alpha,beta,x),FPS,x=1);
```

$$fps := \sum_{k=0}^{\infty} \text{binomial}(n+\alpha, n)\,(-1)^k\, 2^{(-k)}\, \text{pochhammer}(-n, k)\, \text{pochhammer}(n+\alpha+\beta+1, k)\,(-1+x)^k/(\text{pochhammer}(1+\alpha, k)\, k!)$$

```
> hyper:=convert(Sumtohyper(op(1,fps),k),binomial);
```

$$hyper := \text{Hypergeom}\left([-n, n+\alpha+\beta+1], [\alpha+1], -\frac{x}{2}+\frac{1}{2}\right)\text{binomial}(n+\alpha, \alpha)$$

Formula (30):

```
> fps:=convert(cos(p*arctan(x))/cos(arctan(x))^p,FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{(-1)^k\, \text{pochhammer}(-p, 2\,k)\, x^{(2\,k)}}{(2\,k)!}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \text{Hypergeom}\left(\left[-\frac{p}{2}, \frac{1}{2}-\frac{p}{2}\right], \left[\frac{1}{2}\right], -x^2\right)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$\frac{(-x^2)^{(1/4)}\,(1+x^2)^{\left(\frac{p}{2}\right)}\left(\left(\sqrt{\frac{1}{1+x^2}}+\sqrt{\frac{x^2}{1+x^2}}\,I\right)^p+\left(\sqrt{\frac{1}{1+x^2}}+\sqrt{\frac{x^2}{1+x^2}}\,I\right)^{(-p)}\right)}{2\,\sqrt{1+x^2}\left(\frac{1}{1+x^2}\right)^{(1/4)}\left(-\frac{x^2}{1+x^2}\right)^{(1/4)}}$$

Formula (31):

```
> fps:=convert((1-x)^(-a)*cos(2*a*arctan(sqrt(-x))),FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{\text{pochhammer}(2\,a, 2\,k)\, x^k}{(2\,k)!}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \text{Hypergeom}\left(\left[a+\frac{1}{2}, a\right], \left[\frac{1}{2}\right], x\right)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$\frac{x^{(1/4)}\,(1-x)^{(-a-1/2)}\cos\left(2\,a\arcsin\left(\sqrt{\frac{x}{-1+x}}\right)\right)}{\left(-\frac{1}{-1+x}\right)^{(1/4)}\left(-\frac{x}{-1+x}\right)^{(1/4)}}$$

Formula (35):

```
> fps:=convert(1/2*((1+sqrt(x))^(-2*a)+(1-sqrt(x))^(-2*a)),
FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{\mathrm{pochhammer}(2\,a,\,2\,k)\,x^k}{(2\,k)!}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \mathrm{Hypergeom}\left(\left[a+\frac{1}{2},\,a\right],\left[\frac{1}{2}\right],\,x\right)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$\frac{x^{(1/4)}\,(1-x)^{(-a-1/2)}\cos\left(2\,a\arcsin\left(\sqrt{\frac{x}{-1+x}}\right)\right)}{\left(-\frac{1}{-1+x}\right)^{(1/4)}\left(-\frac{x}{-1+x}\right)^{(1/4)}}$$

Formula (46):

```
> fps:=convert((1-x)^(-a)*sin((2*a-1)*arctan(sqrt(-x)))/
((2*a-1)*sin(arctan(sqrt(-x)))),FPS,x);
```

$$fps := \sum_{k=0}^{\infty} \frac{\mathrm{pochhammer}(2\,a,\,2\,k)\,x^k}{(2\,k+1)!}$$

```
> hyper:=Sumtohyper(op(1,fps),k);
```

$$hyper := \mathrm{Hypergeom}\left(\left[a+\frac{1}{2},\,a\right],\left[\frac{3}{2}\right],\,x\right)$$

```
> simplify(subs(Hypergeom=hypergeom,hyper));
```

$$\frac{(1-x)^{(-a)}\left(-\frac{x}{-1+x}\right)^{(1/4)}\sin\left((2\,a-1)\arcsin\left(\sqrt{\frac{x}{-1+x}}\right)\right)}{x^{(1/4)}\left(-\frac{1}{-1+x}\right)^{(1/4)}(2\,a-1)\sqrt{\frac{x}{-1+x}}}$$

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