Two Finite Classes of Special Functions Using Fourier Transforms of Two Symmetric Sequences of Finite Orthogonal Polynomials

Mohammad Masjed-Jamei[a] **Wolfram Koepf**[b,*]

[a] *Department of Mathematics, K.N.Toosi University of Technology, P.O.Box 16315-1618, Tehran, Iran,*
E-mail: mmjamei@kntu.ac.ir , mmjamei@yahoo.com

[b,*] *Institute of Mathematics, University of Kassel, Heinrich-Plett-Str. 40, D-34132 Kassel, Germany,*
E-mail: koepf@mathematik.uni-kassel.de , Corresponding author[*]

Abstract. Some orthogonal polynomial systems are mapped onto each other by the Fourier transform. Based on the results of the paper [A basic class of symmetric orthogonal polynomials using the extended Sturm–Liouville theorem for symmetric functions, *J. Math. Anal. Appl.*, 325 (2007), 753-775], in this paper we introduce two finite classes of orthogonal functions, which are Fourier transforms of two symmetric sequences of finite orthogonal polynomials, and then obtain their orthogonality relations using Parseval's identity.

Keywords. finite symmetric orthogonal polynomials, Fourier transform, orthogonality relation, Parseval identity, hypergeometric functions

MSC(2010): 33C45

1. Introduction. Let us begin with the differential equation [9]:

$$x^2(px^2+q)\Phi_n''(x)+x(rx^2+s)\Phi_n'(x)-\big(n(r+(n-1)p)x^2+(1-(-1)^n)s/2\big)\Phi_n(x)=0\,, \tag{1}$$

where p,q,r,s are real parameters and n is a nonnegative integer.
According to [9], one of the basis solutions of this equation is a specific class of symmetric orthogonal polynomials (with four free parameters) in the explicit form

$$\Phi_n(x)=S_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\,x\right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{\big(2i+(-1)^{n+1}+2[n/2]\big)p+r}{\big(2i+(-1)^{n+1}+2\big)q+s}\right)x^{n-2k}\,, \tag{2}$$

and the hypergeometric representation [9]

$$\bar{S}_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\,x\right)=x^n\,{}_2F_1\left(\begin{matrix}-[n/2],\quad (q-s)/2q-[(n+1)/2]\\ -(r+(2n-3)p)/2p\end{matrix}\,\middle|\,-\frac{q}{px^2}\right)\,, \tag{3}$$

such that $\bar{S}_n(p,q,r,s;x)$ is the monic type of polynomials (2) satisfying the three-term recurrence relation:

$$\bar{S}_{n+1}(x)=x\,\bar{S}_n(x)+C_n\begin{pmatrix} r & s\\ p & q\end{pmatrix}\bar{S}_{n-1}(x)\quad;\quad \bar{S}_0(x)=1\,,\ \bar{S}_1(x)=x\,,\ n\in\mathbf{N}\,, \tag{4}$$

in which

$$C_n\begin{pmatrix} r & s\\ p & q\end{pmatrix}=\frac{pq\,n^2+\big((r-2p)q-(-1)^n ps\big)n+(r-2p)s(1-(-1)^n)/2}{(2pn+r-p)(2pn+r-3p)}, \qquad (4.1)$$

and

$$\bar{S}_n(x)=\bar{S}_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right)=\prod_{i=0}^{[n/2]-1}\frac{\big(2i+(-1)^{n+1}+2\big)\,q+s}{\big(2i+(-1)^{n+1}+2[n/2]\big)p+r}\,S_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right). \qquad (4.2)$$

Since (4) is explicitly known, the norm square value of the polynomials can be obtained by using Favard's theorem [1]. In other words, the generic form of the orthogonality relation of the polynomials (3) is given by [9]

$$\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right)\bar{S}_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right)\bar{S}_m\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right)dx=\left((-1)^n\prod_{i=1}^{n}C_i\begin{pmatrix} r & s\\ p & q\end{pmatrix}\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right)dx\right)\delta_{n,m}, \qquad (5)$$

in which $\delta_{n,m}=\begin{cases}0 & (n\neq m)\\ 1 & (n=m)\end{cases}$, the weight function is defined by [9]

$$W\left(\begin{matrix} r & s\\ p & q\end{matrix}\,\middle|\, x\right)=\exp\left(\int\frac{(r-2p)x^2+s}{x(px^2+q)}dx\right), \qquad (6)$$

and finally α takes the standard values 1 and ∞. In this sense, note according to [9], the function $(px^2+q)W(p,q,r,s;x)$ must vanish at $x=\alpha$ for (5) to be a valid orthogonality property.

In general, four classes of symmetric orthogonal polynomials can be extracted from the differential equation (1). Two of them are infinitely orthogonal (namely the generalized ultraspherical polynomials and generalized Hermite polynomials) and two other ones, which are less known [9], are finitely orthogonal. See Table 1 in this regard.

Table 1: Four special subclasses of $S_n(p,q,r,s;x)$

Definition	Weight function	Interval and Kind
$S_n\left(\begin{matrix} -2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right)$	$W\left(\begin{matrix} -2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\, x\right)=x^{2a}(1-x^2)^b$	$[-1,1]$, Infinite
$S_n\left(\begin{matrix} -2, & 2a\\ 0, & 1\end{matrix}\,\middle|\, x\right)$	$W\left(\begin{matrix} -2, & 2a\\ 0, & 1\end{matrix}\,\middle|\, x\right)=x^{2a}\exp(-x^2)$	$(-\infty,\infty)$, Infinite
$S_n\left(\begin{matrix} -2a-2b+2, & -2a\\ 1, & 1\end{matrix}\,\middle|\, x\right)$	$W\left(\begin{matrix} -2a-2b+2, & -2a\\ 1, & 1\end{matrix}\,\middle|\, x\right)=\frac{x^{-2a}}{(1+x^2)^b}$	$(-\infty,\infty)$, Finite
$S_n\left(\begin{matrix} -2a+2, & 2\\ 1, & 0\end{matrix}\,\middle|\, x\right)$	$W\left(\begin{matrix} -2a+2, & 2\\ 1, & 0\end{matrix}\,\middle|\, x\right)=x^{-2a}\exp(-1/x^2)$	$(-\infty,\infty)$, Finite

Note that all four weight functions in Table 1 must be even and positive, i.e. the condition $(-1)^{2a}=1$ must be satisfied for any four cases.

In this paper, the general properties of the two finite sequences of symmetric orthogonal polynomials in Table 1 are required. So, we restate them here in summary.

1.1. Finite orthogonal polynomials with weight $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$

If $(p,q,r,s)=(1,1,-2a-2b+2,-2a)$ is substituted in (1), then the equation

$$x^2(x^2+1)\Phi_n''(x)-2x((a+b-1)x^2+a)\Phi_n'(x)+\big(n(2a+2b-(n+1))x^2+(1-(-1)^n)a\big)\Phi_n(x)=0, \tag{7}$$

appears that has the explicit polynomial solution

$$\Phi_n(x)=S_n\left(\begin{matrix}-2a-2b+2, & -2a\\ 1, & 1\end{matrix}\,\middle|\, x\right)=\sum_{k=0}^{[n/2]}\binom{[\frac{n}{2}]}{k}\left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2a-2b}{2i+(-1)^{n+1}+2-2a}\right)x^{n-2k}, \tag{7.1}$$

and its monic form is equivalent to the hypergeometric representation

$$A_n^{(a,b)}(x)=\bar{S}_n\left(\begin{matrix}-2a-2b+2, & -2a\\ 1, & 1\end{matrix}\,\middle|\, x\right)=x^n\,{}_2F_1\left(\begin{matrix}-[n/2], & a+1/2-[(n+1)/2]\\ & a+b-n+1/2\end{matrix}\,\middle|\, -\frac{1}{x^2}\right). \tag{8}$$

${}_2F_1(.)$ is a special case of the generalized hypergeometric function [2,6] of order (p,q) as

$${}_pF_q\left(\begin{matrix}a_1 & a_2 & \dots & a_p\\ b_1 & b_2 & \dots & b_q\end{matrix}\,\middle|\, x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\dots(a_p)_k}{(b_1)_k(b_2)_k\dots(b_q)_k}\frac{x^k}{k!}, \tag{8.1}$$

in which $(r)_k=\prod_{i=0}^{k-1}(r+i)$ denotes the Pochhammer symbol.

The finite orthogonality relation corresponding to polynomials (8) is given by [9]

$$\int_{-\infty}^{\infty}\frac{x^{-2a}}{(1+x^2)^b}A_n^{(a,b)}(x)\,A_m^{(a,b)}(x)\,dx=(-1)^n\prod_{j=1}^{n}C_j\begin{pmatrix}-2a-2b+2, & -2a\\ 1, & 1\end{pmatrix}\frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)}\,\delta_{n,m}, \tag{9}$$

where

$$C_j\begin{pmatrix}-2a-2b+2, & -2a\\ 1, & 1\end{pmatrix}=\frac{(j-(1-(-1)^j)a)(j-(1-(-1)^j)a-2b)}{(2j-2a-2b+1)(2j-2a-2b-1)} \tag{9.1}$$

and

$$\int_{-\infty}^{\infty}\frac{x^{-2a}}{(1+x^2)^b}dx=\int_0^{\infty}\frac{t^{-a-1/2}}{(1+t)^b}dt=B(-a+\frac{1}{2};b+a-\frac{1}{2})=\frac{\Gamma(-a+1/2)\Gamma(b+a-1/2)}{\Gamma(b)}. \tag{9.2}$$

According to [9], relation (9) is valid only if $m,n=0,1,\dots,N\le a+b-1/2$ where $N=\max\{m,n\}$; $a<1/2$; $(-1)^{2a}=1$ and $b>0$. Moreover, $B(\lambda_1,\lambda_2)$ in (9.2) is in general the Beta integral [2,3] having various definitions such as

$$
\begin{aligned}
B(\lambda_1;\lambda_2) &= \int_0^1 x^{\lambda_1-1}(1-x)^{\lambda_2-1}\,dx = \int_{-1}^1 x^{2\lambda_1-1}(1-x^2)^{\lambda_2-1}\,dx = \int_0^\infty \frac{x^{\lambda_1-1}}{(1+x)^{\lambda_1+\lambda_2}}\,dx \\
&= 2\int_0^{\pi/2} \sin^{(2\lambda_1-1)} x\, \cos^{(2\lambda_2-1)} x\,dx = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} = B(\lambda_2;\lambda_1),
\end{aligned}
\tag{10}
$$

in which

$$
\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}\,dx \qquad \operatorname{Re}(z) > 0, \tag{10.1}
$$

denotes the well-known Gamma function satisfying the fundamental recurrence relation $\Gamma(z+1) = z\Gamma(z)$.

1.2. Finite orthogonal polynomials with weight $x^{-2a}e^{-1/x^2}$ on $(-\infty,\infty)$

Similarly, if $(p,q,r,s) = (1,0,-2a+2,2)$ is considered in (1), then the equation

$$
x^4\,\Phi''_n(x) + 2x((1-a)x^2+1)\Phi'_n(x) - \left(n(n+1-2a)x^2 + 1 - (-1)^n\right)\Phi_n(x) = 0 \tag{11}
$$

has the polynomial solution

$$
\Phi_n(x) = S_n\left(\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\middle| x\right) = \sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)} \frac{2i+2[n/2]+(-1)^{n+1}+2-2a}{2}\right)x^{n-2k}, \tag{11.1}
$$

whose monic type is equivalent to the hypergeometric form

$$
B_n^{(a)}(x) = \bar{S}_n\left(\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\middle| x\right) = x^n\,{}_1F_1\left(\begin{matrix}-[n/2]\\ a+(-1)^n/2\end{matrix}\middle|\frac{1}{x^2}\right). \tag{12}
$$

Also, the corresponding orthogonality relation takes the form

$$
\int_{-\infty}^{\infty} x^{-2a}e^{-\frac{1}{x^2}} B_n^{(a)}(x)B_m^{(a)}(x)\,dx = \left((-1)^n\prod_{j=1}^{n} C_j\begin{pmatrix}-2a+2 & 2\\ 1 & 0\end{pmatrix}\right)\Gamma(a-\frac{1}{2})\,\delta_{n,m}, \tag{13}
$$

with

$$
C_j\begin{pmatrix}-2a+2 & 2\\ 1 & 0\end{pmatrix} = \frac{-2(-1)^j(j-a)-2a}{(2j-2a+1)(2j-2a-1)}, \tag{13.1}
$$

where $m,n = 0,1,\ldots,N \le a-1/2$ where $N = \max\{m,n\}$ and $(-1)^{2a} = 1$ [9].

It is known that some orthogonal polynomial systems are mapped onto each other by the Fourier transform or other integral transforms such as the Mellin and Hankel transforms. Some illustrative examples in this regard are found e.g. in [5,7,8]. In this paper, we apply this viewpoint to introduce two finite classes of orthogonal functions by using Fourier transform and Parseval's identity.

2. Fourier transform of polynomials $A_n^{(a,b)}(x)$ and $B_n^{(a)}(x)$ and their orthogonality relations

The Fourier transform of a function, say $g(x)$, is defined as [4]

$$\mathbf{F}(s)=\mathbf{F}(g(x))=\int_{-\infty}^{\infty} e^{-isx}g(x)\,dx\,, \tag{14}$$

and for the inverse transform one has the formula

$$g(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx}\mathbf{F}(s)\,ds\,. \tag{15}$$

For $g,h\in L^2(\mathbb{R})$, the Parseval identity related to Fourier theory is given by [4]

$$\int_{-\infty}^{\infty} g(x)\,h(x)\,dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathbf{F}(g(x))\,\overline{\mathbf{F}(h(x))}\,ds\,. \tag{16}$$

By noting (9) and (16) we now define the following specific functions

$$\begin{cases} g(x)=x^{-2\alpha}(1+x^2)^{-\beta}A_n^{(c,d)}(x) \quad \text{s.t. } (-1)^{2\alpha}=1, \\ h(x)=x^{-2l}(1+x^2)^{-u}A_m^{(v,w)}(x) \quad \text{s.t. } (-1)^{2l}=1, \end{cases} \tag{17}$$

in terms of the monic polynomials (8) to which we will apply the Fourier transform. Clearly for both above functions the Fourier transform exists. For instance, for the function $g(x)$ defined in (17) we get

$$\begin{aligned}\mathbf{F}(g(x))&=\int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta}x^{-2\alpha}A_n^{(c,d)}(x)\,dx \\ &=\int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta}x^{-2\alpha+n}\left(\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k(c+1/2-[(n+1)/2])_k}{(c+d-n+1/2)_k\,k!}\frac{(-1)^k}{x^{2k}}\right)dx \\ &=\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k(c+1/2-[(n+1)/2])_k(-1)^k}{(c+d-n+1/2)_k\,k!}\left(\int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta}x^{-2\alpha+n-2k}\,dx\right).\end{aligned} \tag{18}$$

Now, it remains to evaluate the definite integral:

$$I_{n,k}(s;\alpha,\beta)=\int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta}x^{-2\alpha+n-2k}\,dx\,. \tag{19}$$

There are two ways to compute the above integral. First, by noting that $(-1)^{2\alpha}=1$ in (19), we can directly compute $I_{n,k}(s;\alpha,\beta)$ for $n=2m$ as follows

$$\begin{aligned}I_{2m,k}(s;\alpha,\beta)&=\int_{-\infty}^{\infty}\left(\sum_{j=0}^{\infty}\frac{(-isx)^j}{j!}\right)x^{-2\alpha+2m-2k}(1+x^2)^{-\beta}\,dx=\sum_{j=0}^{\infty}\frac{(-1)^j i^j s^j}{j!}\left(\int_{-\infty}^{\infty}x^{j-2\alpha+2m-2k}(1+x^2)^{-\beta}\,dx\right)= \\ &\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}\left(2\int_0^{\infty}x^{2r-2\alpha+2m-2k}(1+x^2)^{-\beta}\,dx\right)=\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}B(r-\alpha+m-k+\frac{1}{2};\beta-r+\alpha-m+k-\frac{1}{2}),\end{aligned} \tag{20}$$

where we have used the third kind of beta integral in (10).

To simplify the last summation of (20) in terms of a hypergeometric function, one can for example apply Koepf's algorithm [6, Chapter 2] as follows

```
> read "hsum13.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 13

Copyright 1998-2009, Wolfram Koepf, University of Kassel

```
> summand:=convert((-1)^r*s^(2*r)/(2*r)!*
  Beta(r-alpha+m-k+1/2,beta-r+alpha-m+k-1/2),GAMMA);
```

$$summand := \frac{(-1)^r s^{(2\,r)}\, \Gamma\left(r-\alpha+m-k+\frac{1}{2}\right) \Gamma\left(\beta-r+\alpha-m+k-\frac{1}{2}\right)}{\Gamma(2\,r+1)\,\Gamma(\beta)}$$

```
> Sumtohyper(summand,r);
```

$$\Gamma\left(\frac{1}{2}-\alpha+m-k\right)\Gamma\left(\beta-\frac{1}{2}+\alpha-m+k\right) \mathrm{Hypergeom}\left(\left[\frac{1}{2}-\alpha+m-k\right],\left[\frac{1}{2},-\beta-k+\frac{3}{2}-\alpha+m\right],\frac{s^2}{4}\right)/\Gamma(\beta)$$

Therefore, (20) becomes

$$I_{2m,k}(s;\alpha,\beta)=\frac{\Gamma(-\alpha+m-k+1/2)\Gamma(\beta+\alpha-m+k-1/2)}{\Gamma(\beta)}\,{}_1F_2\left(\left.\begin{matrix} -\alpha+m-k+1/2 \\ 1/2,\quad -\beta-\alpha+m-k+3/2\end{matrix}\right|\frac{s^2}{4}\right). \tag{21}$$

Similarly this computational method can be applied to $I_{2m+1,k}(s;\alpha,\beta)$. We note that $\int_{-\infty}^{\infty} x^{j-2\alpha+2m+1-2k}(1+x^2)^{-\beta}\,dx=0$ for any $j=0,2,4,\ldots$. After some computations we obtain

$$I_{2m+1,k}(s;\alpha,\beta)=(-is)\frac{\Gamma(-\alpha+m-k+3/2)\Gamma(\beta+\alpha-m+k-3/2)}{\Gamma(\beta)}\,{}_1F_2\left(\left.\begin{matrix} -\alpha+m-k+3/2 \\ 3/2,\quad -\beta-\alpha+m-k+5/2\end{matrix}\right|\frac{s^2}{4}\right). \tag{22}$$

Thus, by combining both relations (21) and (22) and using the identity

$$[\frac{n+1}{2}]-[\frac{n}{2}]=\frac{1-(-1)^n}{2}, \tag{23}$$

we finally have

$$I_{n,k}(s;\alpha,\beta)=\Gamma(-\alpha-k+\frac{1}{2}+[\frac{n+1}{2}])\Gamma(\beta+\alpha+k-\frac{1}{2}-[\frac{n+1}{2}])$$
$$\times\frac{(-is)^{\frac{1-(-1)^n}{2}}}{\Gamma(\beta)}\,{}_1F_2\left(\left.\begin{matrix} 1/2-\alpha-k+[(n+1)/2] \\ 1-\frac{(-1)^n}{2},\quad -\beta-\alpha-k+3/2+[(n+1)/2]\end{matrix}\right|\frac{s^2}{4}\right). \tag{24}$$

The second way to compute $I_{n,k}(s;\alpha,\beta)$ is that we respectively consider $n=2m$ and $n=2m+1$ and directly apply the cosine and sine Fourier transforms [4] to the function $(1+x^2)^{-\beta}x^{-2\alpha+n-2k}$. In other words, by noting that $(-1)^{2\alpha}=1$ we have

$$\begin{aligned}
I_{2m,k}(s;\alpha,\beta) &= \int_{-\infty}^{\infty}\cos(sx)\,(1+x^2)^{-\beta}\,x^{-2\alpha+2m-2k}\,dx - i\int_{-\infty}^{\infty}\sin(sx)\,(1+x^2)^{-\beta}\,x^{-2\alpha+2m-2k}\,dx \\
&= 2\int_{0}^{\infty}\cos(sx)\,(1+x^2)^{-\beta}\,x^{-2\alpha+2m-2k}\,dx \\
&= \frac{\Gamma(-\alpha+m-k+1/2)\,\Gamma(\beta+\alpha-m+k-1/2)}{\Gamma(\beta)}\,{}_1F_2\left(\left.\begin{matrix}-\alpha+m-k+1/2\\ 1/2,\quad -\beta-\alpha+m-k+3/2\end{matrix}\right|\frac{s^2}{4}\right).
\end{aligned} \tag{25}$$

Remark 1. Note that to obtain (25) one should use the Dominated convergence theorem (DCT) [3,10]. In other words, by defining the sequence

$$\Phi_n(x) = (1+x^2)^{-\beta}x^{-2\lambda}\sum_{k=0}^{n}\frac{(-1)^k(sx)^{2k}}{(2k)!} \quad \text{for} \quad (-1)^{\lambda}=1, \tag{25.1}$$

in (25) we have

$$\left|\Phi_n(x)\right| \le \cosh(sx)\,(1+x^2)^{-\beta}x^{-2\lambda} = \Phi(x). \tag{25.2}$$

This allows us explicitly compute $I_{2m,k}(s;\alpha,\beta)$ in (25) as we have done in (20).
By noting remark 1, we can similarly conclude that

$$\begin{aligned}
I_{2m+1,k}(s;\alpha,\beta) &= \int_{-\infty}^{\infty}\cos(sx)\,(1+x^2)^{-\beta}\,x^{-2\alpha+2m+1-2k}\,dx - i\int_{-\infty}^{\infty}\sin(sx)\,(1+x^2)^{-\beta}\,x^{-2\alpha+2m+1-2k}\,dx \\
&= (-2i)\int_{0}^{\infty}\sin(sx)\,(1+x^2)^{-\beta}\,x^{-2\alpha+2m+1-2k}\,dx \\
&= (-is)\,\Gamma(-\alpha+m-k+\frac{3}{2})\,\Gamma(\beta+\alpha-m+k-\frac{3}{2})\,{}_1F_2\left(\left.\begin{matrix}-\alpha+m-k+3/2\\ 3/2,\quad -\beta-\alpha+m-k+5/2\end{matrix}\right|\frac{s^2}{4}\right).
\end{aligned} \tag{26}$$

Hence, the result (24) simplifies (18) as

$$\begin{aligned}
\mathbf{F}(g(x)) = {}&\frac{1}{\Gamma(\beta)}\Gamma(-\alpha+\frac{1}{2}+[\frac{n+1}{2}])\,\Gamma(\beta+\alpha-\frac{1}{2}-[\frac{n+1}{2}])\,(-is)^{\frac{1-(-1)^n}{2}}\times \\
&\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k\,(c+1/2-[(n+1)/2])_k\,(\beta+\alpha-1/2-[(n+1)/2])_k}{(c+d-n+1/2)_k\,(1/2+\alpha-[(n+1)/2])_k\,k!} \\
&\times{}_1F_2\left(\left.\begin{matrix}1/2-\alpha-k+[(n+1)/2]\\ 1-\dfrac{(-1)^n}{2},\quad -\beta-\alpha-k+3/2+[(n+1)/2]\end{matrix}\right|\frac{s^2}{4}\right).
\end{aligned} \tag{27}$$

By noting (27), if for simplicity we define

(28)

$$K_n(x;p_1,p_2,p_3,p_4)=x^{\frac{1-(-1)^n}{2}}\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k\,(p_3+1/2-[(n+1)/2])_k\,(p_1+p_2-1/2-[(n+1)/2])_k}{(p_3+p_4-n+1/2)_k\,(1/2+p_1-[(n+1)/2])_k\,k!}$$
$$\times{}_1F_2\left(\begin{matrix}1/2-p_1-k+[(n+1)/2]\\ 1-\frac{(-1)^n}{2},\quad -p_1-p_2-k+3/2+[(n+1)/2]\end{matrix}\middle|\,\frac{x^2}{4}\right),$$

then it is clear by (27) that

$$\mathbf{F}(g(x))=(-i)^{\frac{1-(-1)^n}{2}}\frac{1}{\Gamma(\beta)}\Gamma(-\alpha+\frac{1}{2}+[\frac{n+1}{2}])\Gamma(\beta+\alpha-\frac{1}{2}-[\frac{n+1}{2}])\,K_n(s;\alpha,\beta,c,d)\,. \tag{29}$$

Now, by substituting (29) in Parseval's identity (16) and noting (17) we get

$$2\pi\int_{-\infty}^{\infty}x^{-2(\alpha+l)}(1+x^2)^{-(\beta+u)}\,A_n^{(c,d)}(x)\,A_m^{(v,w)}(x)dx=i^{\frac{(-1)^n-(-1)^m}{2}}\,\Gamma(-\alpha+\frac{1}{2}+[\frac{n+1}{2}])$$
$$\times\frac{1}{\Gamma(\beta)\Gamma(u)}\,\Gamma(\beta+\alpha-\frac{1}{2}-[\frac{n+1}{2}])\Gamma(-l+\frac{1}{2}+[\frac{m+1}{2}])\Gamma(u+l-\frac{1}{2}-[\frac{m+1}{2}]) \tag{30}$$
$$\times\int_{-\infty}^{\infty}K_n(s;\alpha,\beta,c,d)\,K_m(s;l,u,v,w)\,ds\,.$$

On the other hand, if on the left hand side of (30) we set

$$c=v=\alpha+l \qquad \text{and} \qquad d=w=\beta+u\,, \tag{31}$$

then according to orthogonality relation (9), the following theorem will finally be derived.

Theorem 1. *The special function* $K_n(x;p_1,p_2,p_3,p_4)$ *defined in (28) satisfies a finite orthogonality relation as*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}K_n\big(x;\alpha,\beta,p,q\big)K_m\big(x;p-\alpha,q-\beta,p,q\big)dx=\prod_{j=1}^{n}\frac{\big(-j+(1-(-1)^j)p\big)\big(j-(1-(-1)^j)p-2q\big)}{\big(2j-2p-2q+1\big)\big(2j-2p-2q-1\big)}\times$$
$$\frac{\Gamma(\beta)\Gamma(q-\beta)\Gamma(p+q-1/2)\Gamma(-p+1/2)\;\delta_{n,m}}{\Gamma(q)\Gamma\left(-\alpha+\frac{1}{2}+[\frac{n+1}{2}]\right)\Gamma\left(\alpha+\beta-\frac{1}{2}-[\frac{n+1}{2}]\right)\Gamma\left(\alpha-p+\frac{1}{2}+[\frac{n+1}{2}]\right)\Gamma\left(p+q-\alpha-\beta-\frac{1}{2}-[\frac{n+1}{2}]\right)}, \tag{32}$$

where $m,n=0,1,...,N=\max\{m,n\}\le p+q-1/2$, $p<1/2$, $(-1)^{2p}=1$, $q>\beta>0$, $0<\alpha<1/2$ *and* $\alpha+\beta>1/2$.

The mentioned approach can similarly be applied to the monic polynomials $B_n^{(a)}(x)$ in (12). For this purpose, we first define the following specific functions

$$u(x) = x^{-2a} e^{\frac{-1}{2x^2}} B_n^{(b)}(x) \quad \text{and} \quad v(x) = x^{-2c} e^{\frac{-1}{2x^2}} B_m^{(d)}(x) \quad \text{for} \quad (-1)^{2a} = (-1)^{2c} = 1 . \tag{33}$$

If we take the Fourier transform for e.g. $u(x)$, we get

$$\begin{aligned} \mathbf{F}(u(x)) &= \int_{-\infty}^{\infty} e^{-isx} x^{-2a} e^{\frac{-1}{2x^2}} B_n^{(b)}(x)\, dx = \int_{-\infty}^{\infty} e^{-isx} e^{\frac{-1}{2x^2}} x^{-2a+n} \left(\sum_{k=0}^{[n/2]} \frac{(-[n/2])_k}{(b+(-1)^n/2)_k} \frac{x^{-2k}}{k!} \right) dx \\ &= \sum_{k=0}^{[n/2]} \frac{(-[n/2])_k}{(b+(-1)^n/2)_k\, k!} \left(\int_{-\infty}^{\infty} e^{-isx} e^{\frac{-1}{2x^2}} x^{-2a+n-2k}\, dx \right). \end{aligned} \tag{34}$$

Again, we should here evaluate the definite integral:

$$R_{n,k}(s;a) = \int_{-\infty}^{\infty} e^{-isx} e^{\frac{-1}{2x^2}} x^{-2a+n-2k}\, dx . \tag{34.1}$$

To compute this integral we can repeat the method which we applied to $I_{n,k}(s;\alpha,\beta)$ in (19). Hence we have (35)

$$\begin{aligned} R_{2m,k}(s;a) &= \int_{-\infty}^{\infty} \left(\sum_{j=0}^{\infty} \frac{(-isx)^j}{j!} \right) e^{\frac{-1}{2x^2}} x^{-2a+2m-2k}\, dx = \sum_{j=0}^{\infty} \frac{(-1)^j i^j s^j}{j!} \left(\int_{-\infty}^{\infty} e^{\frac{-1}{2x^2}} x^{j-2a+2m-2k}\, dx \right) \\ &= \sum_{r=0}^{\infty} \frac{(-1)^r s^{2r}}{(2r)!} \left(2\int_0^{\infty} x^{2r-2a+2m-2k} e^{\frac{-1}{2x^2}}\, dx \right) = \sum_{r=0}^{\infty} \frac{(-1)^r s^{2r}}{(2r)!} 2^{-r+a-m+k-\frac{1}{2}} \Gamma(-r+a-m+k-\frac{1}{2}) \\ &= 2^{a-m+k-\frac{1}{2}} \Gamma(a-m+k-\frac{1}{2})\; {}_0F_2\left(\left. \begin{matrix} - \\ \frac{1}{2},\quad \frac{3}{2}-a+m-k \end{matrix} \right| \frac{s^2}{8} \right), \end{aligned}$$

as well as (36)

$$\begin{aligned} R_{2m+1,k}(s;a) &= \sum_{j=0}^{\infty} \frac{(-1)^j i^j s^j}{j!} \left(\int_{-\infty}^{\infty} e^{\frac{-1}{2x^2}} x^{j-2a+2m+1-2k}\, dx \right) = (-is) \sum_{r=0}^{\infty} \frac{(-1)^r s^{2r}}{(2r+1)!} \left(2\int_0^{\infty} x^{2r-2a+2m-2k+2} e^{\frac{-1}{2x^2}}\, dx \right) \\ &= (-is)\, 2^{a-m+k-\frac{3}{2}} \Gamma(a-m+k-\frac{3}{2})\; {}_0F_2\left(\left. \begin{matrix} - \\ \frac{3}{2},\quad \frac{5}{2}-a+m-k \end{matrix} \right| \frac{s^2}{8} \right). \end{aligned}$$

Therefore (37)

$$R_{n,k}(s;a) = 2^{a+k-\frac{1}{2}-[\frac{n+1}{2}]} \Gamma\left(a+k-\frac{1}{2}-[\frac{n+1}{2}] \right) (-is)^{\frac{1-(-1)^n}{2}}\; {}_0F_2\left(\left. \begin{matrix} - \\ 1-\frac{(-1)^n}{2},\quad -a-k+\frac{3}{2}+[\frac{n+1}{2}] \end{matrix} \right| \frac{s^2}{8} \right).$$

The result (37) simplifies (34) towards

$$\mathbf{F}(u(x)) = \Gamma(a-\frac{1}{2}-[\frac{n+1}{2}])\,2^{a-\frac{1}{2}-[\frac{n+1}{2}]}(-is)^{\frac{1-(-1)^n}{2}} \times$$

$$\sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (a-\frac{1}{2}-[\frac{n+1}{2}])_k}{(b+(-1)^n/2)_k} \frac{2^k}{k!}\, {}_0F_2\left(\begin{matrix} - \\ 1-\frac{(-1)^n}{2}, \quad -a-k+\frac{3}{2}+[\frac{n+1}{2}] \end{matrix} \middle| \frac{s^2}{8} \right). \tag{38}$$

Using (38) if we now define

$$J_n(x;q_1,q_2) = x^{\frac{1-(-1)^n}{2}} \sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (q_1-\frac{1}{2}-[\frac{n+1}{2}])_k}{(q_2+(-1)^n/2)_k} \frac{2^k}{k!}\, {}_0F_2\left(\begin{matrix} - \\ 1-\frac{(-1)^n}{2}, \quad -q_1-k+\frac{3}{2}+[\frac{n+1}{2}] \end{matrix} \middle| \frac{x^2}{8} \right), \tag{39}$$

then by referring to definitions (33) and applying Parseval’s identity we get

$$2\pi \int_{-\infty}^{\infty} x^{-2(a+c)} e^{\frac{-1}{x^2}} B_n^{(b)}(x)\, B_m^{(d)}(x)dx$$

$$= i^{\frac{(-1)^n-(-1)^m}{2}} \frac{\Gamma(a-\frac{1}{2}-[\frac{n+1}{2}])\Gamma(c-\frac{1}{2}-[\frac{m+1}{2}])}{2^{-a+\frac{1}{2}+[\frac{n+1}{2}]}\,2^{-c+\frac{1}{2}+[\frac{m+1}{2}]}} \int_{-\infty}^{\infty} J_n(s;a,b)\, J_m(s;c,d)\, ds. \tag{40}$$

Finally it is sufficient in (40) to assume that $b = d = a + c$ and then refer to the finite orthogonality relation (13) to reach the following theorem.

Theorem 2. *The special function* $J_n(x;q_1,q_2)$ *defined in (39) satisfies the orthogonality relation*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} J_n(x;a,b)\, J_m(x;b-a,b)\, dx = 2^{-b+1+2[\frac{n+1}{2}]} \prod_{j=1}^{n} \frac{2(-1)^j(j-b)+2b}{(2j-2b+1)(2j-2b-1)}$$

$$\times \frac{\Gamma(b-1/2)}{\Gamma(a-\frac{1}{2}-[\frac{n+1}{2}])\Gamma(b-a-\frac{1}{2}-[\frac{n+1}{2}])}\,\delta_{n,m}\,, \tag{41}$$

where $m,n = 0,1,...,N = \max\{m,n\} \le b-\frac{1}{2},\ (-1)^{2b} = 1$ *and* $\frac{1}{2} < a < b-\frac{1}{2}.$

References

[1] R. Askey, An integral of Ramanujan and orthogonal polynomials, *J. Indian Math. Soc.* **51** (1987) 27-36.

[2] W. N. Bailey, *Generalized Hypergeometric Series*. Cambridge Tracts 32, Cambridge University PFTV, 1935. Reprinted by Hafner Publishing Company, 1972.

[3] Digital Library of Mathematical Functions, DLMF available at: http://dlmf.nist.gov/

[4] A. Erdelyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Tables of Integral Transforms*. Vol. 2, McGraw-Hill, 1954.

[5] H. T. Koelink, On Jacobi and continuous Hahn polynomials, *Proc. Amer. Math. Soc.,* **124** (1996) 997-898.

[6] W. Koepf, *Hypergeometric Summation*. Braunschweig/Wiesbaden, Vieweg, 1988.

[7] W. Koepf, M. Masjed-Jamei, Two classes of special functions using Fourier transforms of some finite classes of classical orthogonal polynomials, *Proc. Amer. Math. Soc.,* **135** (2007) 3599-3606.

[8] T. H. Koornwinder, Special orthogonal polynomial systems mapped onto each other by the Fourier-Jacobi transform, *Polynômes Orthogonaux et Applications* (C. Brezinski, A. Draux, A. P. Magnus, P. Maroni and A. Ronveaux, Eds.), Lecture Notes Math. 1171, Springer, 1985, 174-183.

[9] M. Masjed-Jamei, A basic class of symmetric orthogonal polynomials using the extended Sturm–Liouville theorem for symmetric functions, *J. Math. Anal. Appl.*, 325 (2007), 753-775.

[10] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*. 4th ed., Cambridge University Press, Cambridge, 1927.