On a bivariate kind of Bernoulli polynomials

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Abstract

In this paper, we introduce a bivariate kind of Bernoulli polynomials and study their basic properties. We also compute the Fourier expansion of these polynomials and obtain some new series involving Bernoulli numbers.

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1 Introduction

The Appell polynomials $A_n(x)$ defined by

$$f(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1}$$

where f is a formal power series in t, have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [2, 15]. Two special cases of Appell polynomials are Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ that are, respectively, generated by choosing $f(t) = \frac{t}{\mathrm{e}^t-1}$ and $f(t) = \frac{2}{\mathrm{e}^t+1}$ in (1). Also, Bernoulli numbers $B_n := B_n(0)$ and Euler numbers $E_n := 2^n E_n(\frac{1}{2})$ are of considerable importance in number theory, special functions, combinatorics and numerical analysis.

Bernoulli numbers are given by

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi),$$

or by the recurrence relation

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \ \text{ for } n \geq 1 \ \text{ and } \ B_0 = 1.$$

They are directly related to various combinatorial numbers such as Stirling, Cauchy and harmonic numbers. For example, except B_1 we have

$$B_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{m+1} S_2(n,m), \tag{2}$$

where

$$S_2(n,m) = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n,$$

denotes the second kind of Stirling numbers [5, 7] with $S_2(n,m) = 0$ for $n < m$.

There are some algorithms for computing Bernoulli numbers. One of them is Euler's formula

$$B_{2n} = \frac{(-1)^{n-1} 2n}{2^{2n}(2^{2n}-1)} T_n,$$

where $\{T_n\}$, known as Tangent numbers, are generated by

$$\tan t = \sum_{n=1}^{\infty} T_n \frac{t^{2n-1}}{(2n-1)!}.$$

In 2001, Akiyama and Tanigawa [1] (see also [13]) found an algorithm for computing $A_{n,0} := (-1)^n B_n$ without computing Tangent numbers as

$$A_{n+1,m} = (m+1)(A_{n,m} - A_{n,m+1}),$$

where $A_{0,m} = \frac{1}{m+1}$.

Later on, a modified version of the above-mentioned algorithm was proposed by Chen [4] for computing $C_{n,0} := B_n$ as

$$C_{n+1,m} = mC_{n,m} - (m+1)C_{n,m+1}$$

where $C_{0,m} = \frac{1}{m+1}$.

Bernoulli numbers have found various extensions such as poly-Bernoulli numbers, which are somehow connected to multiple zeta values. For recent extensions of poly-Bernoulli numbers see e.g. [3, 6, 8, 9, 14]. In [12], the author has defined a new family of poly-Bernoulli numbers in terms of Gaussian hypergeometric functions and obtained its basic properties. He has also presented an algorithm for computing Bernoulli numbers and polynomials and showed that poly-Bernoulli numbers are related to the certain regular values of the Euler-Zagiers multiple zeta function at non-positive integers of depth $p \geq 1$, i.e.

$$\zeta(s_1, s_2, \ldots, s_p) = \sum_{0<n_1<n_2<\cdots<n_p} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_p^{s_p}},$$

where $s_1, s_2, \ldots, s_p$ are positive integers with $s_p > 1$.

Another combinatorial aspect of Bernoulli numbers is that they have several symmetry properties with Cauchy numbers. The first kind of Cauchy numbers is defined by [5, 11]

$$C_n = \int_0^1 t(t-1)\cdots(t-n+1)\ \mathrm{d}t = n!\int_0^1 \binom{t}{n}\ \mathrm{d}t,$$

having the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!},$$

and the second kind is defined by

$$\hat{C}_n = \int_{-1}^0 t(t-1)\cdots(t-n+1)\ \mathrm{d}t = n!\int_{-1}^0 \binom{t}{n}\ \mathrm{d}t.$$

Both C_n and $\hat{C}_n$ can be explicitly written as

$$C_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m S_1(n,m)}{m+1} \quad \text{and} \quad \hat{C}_n = (-1)^n \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1},$$

such that $S_1(n,m)$ are the first kind of Stirling numbers given by

$$(t)_n = t(t+1)\cdots(t+n-1) = \sum_{m=0}^{n} S_1(n,m)t^m,$$

where $S_1(n,m) = 0$ for $n < m$.

This paper is organized as follows: In the next section, we introduce an extension of Bernoulli polynomials and present several basic properties of them in section 3. We also compute the Fourier expansion of the extended polynomials in section 4 and obtain some new series involving Bernoulli numbers.

2 A Bivariate Kind of Bernoulli Polynomials

If $p, q \in \mathbb{R}$, it is known that the Taylor expansion of the two functions $\mathrm{e}^{pt}\cos qt$ and $\mathrm{e}^{pt}\sin qt$ are respectively as follows [10]

$$\mathrm{e}^{pt}\cos qt = \sum_{k=0}^{\infty} C_k(p,q)\frac{t^k}{k!}, \tag{3}$$

and

$$e^{pt}\sin qt = \sum_{k=0}^{\infty} S_k(p,q)\frac{t^k}{k!}, \tag{4}$$

where

$$C_k(p,q) = \sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}p^{k-2j}q^{2j}, \tag{5}$$

and

$$S_k(p,q) = \sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1}p^{k-2j-1}q^{2j+1}. \tag{6}$$

By referring to relations (3)-(6), we can introduce two kinds of bivariate Bernoulli polynomials as

$$\frac{te^{pt}}{e^t-1}\cos qt = \sum_{n=0}^{\infty} B_n^{(c)}(p,q)\frac{t^n}{n!} \quad (|t|<2\pi), \tag{7}$$

and

$$\frac{te^{pt}}{e^t-1}\sin qt = \sum_{n=0}^{\infty} B_n^{(s)}(p,q)\frac{t^n}{n!} \quad (|t|<2\pi). \tag{8}$$

For instance, we have

$$\begin{aligned}
&B_0^{(c)}(p,q) = 1,\\
&B_1^{(c)}(p,q) = p-\frac{1}{2},\\
&B_2^{(c)}(p,q) = p^2-p-q^2+\frac{1}{6},\\
&B_3^{(c)}(p,q) = p^3-\frac{3}{2}p^2+(\frac{1}{2}-3q^2)p+\frac{3}{2}q^2,\\
&B_4^{(c)}(p,q) = p^4-2p^3+(1-6q^2)p^2+6q^2p+q^4-q^2-\frac{1}{30},\\
&B_5^{(c)}(p,q) = p^5-\frac{5}{2}p^4+(\frac{5}{3}-10q^2)p^3+15q^2p^2+(5q^4-5q^2-\frac{1}{6})p-\frac{5}{2}q^4,\\
&B_6^{(c)}(p,q) = p^6-3p^5+(\frac{5}{2}-15q^2)p^4+30q^2p^3+(15q^4-15q^2-\frac{1}{2})p^2-15q^4p\\
&\qquad\qquad -q^6+\frac{5}{2}q^4+\frac{1}{2}q^2+\frac{1}{42},
\end{aligned}$$

and

$$\begin{aligned}
&B_0^{(s)}(p,q)=0,\\
&B_1^{(s)}(p,q)=q,\\
&B_2^{(s)}(p,q)=2qp-q,\\
&B_3^{(s)}(p,q)=3qp^2-3qp-q^3+\frac{1}{2}q,\\
&B_4^{(s)}(p,q)=4qp^3-6qp^2+(2q-4q^3)p+2q^3,\\
&B_5^{(s)}(p,q)=5qp^4-10qp^3+(5q-10q^3)p^2+10q^3p+q^5-\frac{5}{3}q^3-\frac{1}{6}q,\\
&B_6^{(s)}(p,q)=6qp^5-15qp^4+(10q-20q^3)p^3+30q^3p^2+(6q^5-10q^3-q)p-3q^5.
\end{aligned}$$

3 Some Basic Properties of the Polynomials $B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$.

Proposition 1. *$B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$ can be represented in terms of Bernoulli numbers as follows*

$$B_n^{(c)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}B_kC_{n-k}(p,q), \tag{9}$$

and

$$B_n^{(s)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}B_kS_{n-k}(p,q). \tag{10}$$

Proof. By noting the general identity

$$\left(\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}b_k\frac{t^k}{k!}\right)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}a_jb_{k-j}\right)\frac{t^k}{k!},$$

we have

$$\begin{aligned}
\sum_{k=0}^{\infty}B_k^{(c)}(p,q)\frac{t^k}{k!}=\frac{t}{\mathrm{e}^t-1}\left(\mathrm{e}^{pt}\cos qt\right)&=\left(\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}C_k(p,q)\frac{t^k}{k!}\right)\\
&=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}B_jC_{k-j}(p,q)\right)\frac{t^k}{k!},
\end{aligned}$$

which proves (9). The proof of (10) is similar. □

Proposition 2. *For every* $n \in \mathbb{Z}^+$ *we have*

$$B_n^{(c)}(1-p,q) = (-1)^n B_n^{(c)}(p,q), \tag{11}$$

and

$$B_n^{(s)}(1-p,q) = (-1)^{n+1} B_n^{(s)}(p,q). \tag{12}$$

Proof. Applying the generating function (7) gives

$$\sum_{n=0}^{\infty} B_n^{(c)}(1-p,q)\frac{t^n}{n!} = \frac{t\mathrm{e}^{(1-p)t}}{\mathrm{e}^t-1}\cos qt,$$

as well as

$$\sum_{n=0}^{\infty}(-1)^n B_n^{(c)}(p,q)\frac{t^n}{n!} = \frac{-t\mathrm{e}^{-pt}}{\mathrm{e}^{-t}-1}\cos(-qt) = \frac{t\mathrm{e}^{(1-p)t}}{\mathrm{e}^t-1}\cos qt.$$

Similarly, property (12) can be proved. □

Corollary 1. *Relations* (11) *and* (12) *imply that*

$$B_{2n+1}^{(c)}(\frac{1}{2},q) = 0,$$

and

$$B_{2n}^{(s)}(\frac{1}{2},q) = 0.$$

Proposition 3. *For every* $n \in \mathbb{N}$, *the following identities hold*

$$B_n^{(c)}(1+p,q) - B_n^{(c)}(p,q) = nC_{n-1}(p,q), \tag{13}$$

and

$$B_n^{(s)}(1+p,q) - B_n^{(s)}(p,q) = nS_{n-1}(p,q). \tag{14}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(c)}(1+p,q)\frac{t^n}{n!} &= \frac{t\mathrm{e}^{pt}(\mathrm{e}^t-1+1)}{\mathrm{e}^t-1}\cos qt = t\mathrm{e}^{pt}\cos qt + \frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\cos qt\\ &= \sum_{n=0}^{\infty} C_n(p,q)\frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty} B_n^{(c)}(p,q)\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty} nC_{n-1}(p,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n^{(c)}(p,q)\frac{t^n}{n!},\end{aligned}$$

which proves (13). The proof of (14) is similar. □

Corollary 2. *Relations* (13) *and* (14) *first imply that*

$$B^{(c)}_{2n+1}(1,q) - B^{(c)}_{2n+1}(0,q) = (2n+1)(-1)^n q^{2n},$$

and

$$B^{(s)}_{2n}(1,q) - B^{(s)}_{2n}(0,q) = 2n(-1)^{n+1} q^{2n-1}.$$

Hence, combining proposition 2 respectively yields

$$B^{(c)}_{2n+1}(1,q) = -B^{(c)}_{2n+1}(0,q) = \frac{2n+1}{2}(-1)^n q^{2n},$$

and

$$B^{(s)}_{2n}(1,q) = -B^{(s)}_{2n}(0,q) = n(-1)^{n+1} q^{2n-1}.$$

Corollary 3. *For every $n \in \mathbb{N}$ and $m \in \mathbb{Z}^+$ we have*

$$\sum_{p=0}^{m} C_{n-1}(p,q) = \frac{B^{(c)}_n(1+m,q) - B^{(c)}_n(0,q)}{n},$$

and

$$\sum_{p=0}^{m} S_{n-1}(p,q) = \frac{B^{(s)}_n(1+m,q) - B^{(s)}_n(0,q)}{n}.$$

We recall that $C_{n-1}(p,0) = p^{n-1}$ and therefore

$$\sum_{p=1}^{m} p^{n-1} = \frac{B_n(m+1) - B_n}{n}.$$

Proposition 4. *For every $n \in \mathbb{Z}^+$ the following identities hold*

$$B^{(c)}_n(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} B^{(c)}_k(p,q) r^{n-k}, \tag{15}$$

and

$$B^{(s)}_n(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} B^{(s)}_k(p,q) r^{n-k}. \tag{16}$$

Proof. Apply (7) to obtain

$$\sum_{n=0}^{\infty} B^{(c)}_n(p+r,q)\frac{t^n}{n!} = \left(\frac{t\mathrm{e}^{pt}}{\mathrm{e}^t - 1}\cos qt\right)\mathrm{e}^{rt} = \left(\sum_{n=0}^{\infty} B^{(c)}_n(p,q)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} r^n \frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} B^{(c)}_k(p,q) r^{n-k}\right)\frac{t^n}{n!},$$

which proves (15). The result (16) can be similarly proved. □

Proposition 5. *We have*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(c)}(p,q) = (n+1)C_n(p,q), \tag{17}$$

and

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(s)}(p,q) = (n+1)S_n(p,q). \tag{18}$$

Proof. From (15), one can conclude that

$$B_{n+1}^{(c)}(p+1,q) - B_{n+1}^{(c)}(p,q) = \sum_{k=0}^{n} \binom{n+1}{k} B_k^{(c)}(p,q).$$

Hence, by referring to (13), the result (17) is derived. The proof of (18) can be done in a similar way. □

Corollary 4. *Relations* (17) *and* (18) *imply that*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(c)}(0,q) = (n+1)q^n \cos n\frac{\pi}{2} = \begin{cases} (-1)^m(2m+1)q^{2m} & n=2m \text{ even}, \\ 0 & n=2m+1 \text{ odd}, \end{cases}$$

and

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(s)}(0,q) = (n+1)q^n \sin n\frac{\pi}{2} = \begin{cases} 0 & n=2m \text{ even}, \\ (-1)^m(2m+2)q^{2m+1} & n=2m+1 \text{ odd}. \end{cases}$$

Proposition 6. *For every $n \in \mathbb{N}$, the following partial differential equations hold*

$$\frac{\partial}{\partial p} B_n^{(c)}(p,q) = nB_{n-1}^{(c)}(p,q), \tag{19}$$

$$\frac{\partial}{\partial q} B_n^{(c)}(p,q) = -nB_{n-1}^{(s)}(p,q), \tag{20}$$

$$\frac{\partial}{\partial p} B_n^{(s)}(p,q) = nB_{n-1}^{(s)}(p,q), \tag{21}$$

and

$$\frac{\partial}{\partial q} B_n^{(s)}(p,q) = nB_{n-1}^{(c)}(p,q). \tag{22}$$

Proof. Relation (7) yields

$$\sum_{n=1}^{\infty} \frac{\partial B_n^{(c)}(p,q)}{\partial p} \frac{t^n}{n!} = \frac{t^2 \mathrm{e}^{pt}}{\mathrm{e}^t - 1} \cos qt = \sum_{n=0}^{\infty} B_n^{(c)}(p,q) \frac{t^{n+1}}{n!}$$
$$= \sum_{n=1}^{\infty} B_{n-1}^{(c)}(p,q) \frac{t^n}{(n-1)!} = \sum_{n=1}^{\infty} n B_{n-1}^{(c)}(p,q) \frac{t^n}{n!},$$

proving (19). Other equations (20), (21) and (22) can be similarly derived. □

Corollary 5. *By combining the above results and proposition 2 and corollary 2, we obtain*

$$\int_0^1 B_{2n}^{(c)}(p,q)\ \mathrm{d}p = (-1)^n q^{2n},$$
$$\int_0^1 B_{2n+1}^{(c)}(p,q)\ \mathrm{d}p = 0,$$
$$\int_0^1 B_{2n}^{(s)}(p,q)\ \mathrm{d}p = 0,$$

and

$$\int_0^1 B_{2n+1}^{(s)}(p,q)\ \mathrm{d}p = (-1)^n q^{2n+1}.$$

Proposition 7. *If $B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$ are sorted in terms of the variable p, then they are polynomials of degree n and $n-1$ respectively, such that we have*

$$B_n^{(c)}(p,q) = p^n - \frac{n}{2} p^{n-1} + \cdots, \tag{23}$$

and

$$B_n^{(s)}(p,q) = nqp^{n-1} - \binom{n}{2} qp^{n-2} + \cdots. \tag{24}$$

Also, if they are sorted in terms of the variable q, then

$$B_n^{(c)}(p,q) = \begin{cases} (-1)^{\frac{n-1}{2}} n(p-\frac{1}{2})q^{n-1} + (-1)^{\frac{n+1}{2}} \binom{n}{3}(p^3 - \frac{3}{2}p^2 + \frac{1}{2}p)q^{n-3} + \cdots\ (n\ \textit{odd}), \\ (-1)^{\frac{n}{2}} q^n + (-1)^{\frac{n+2}{2}} \binom{n}{2}(p^2 - p + \frac{1}{6})q^{n-2} + \cdots\ (n\ \textit{even}), \end{cases} \tag{25}$$

and

$$B_n^{(s)}(p,q) = \begin{cases} (-1)^{\frac{n+2}{2}} n(p-\frac{1}{2})q^{n-1} + (-1)^{\frac{n}{2}} \binom{n}{3}(p^3 - \frac{3}{2}p^2 + \frac{1}{2}p)q^{n-3} + \cdots\ (n\ \textit{even}), \\ (-1)^{\frac{n-1}{2}} q^n + (-1)^{\frac{n+1}{2}} \binom{n}{2}(p^2 - p + \frac{1}{6})q^{n-2} + \cdots\ (n\ \textit{odd}). \end{cases} \tag{26}$$

Proof. We first prove (23) by induction. It is known from (17) that

$$B_0^{(c)}(p,q)=1,\ B_1^{(c)}(p,q)=p-\frac{1}{2}\quad\text{and}\quad B_2^{(c)}(p,q)=p^2-p-q^2+\frac{1}{6}.$$

Therefore (23) holds for $n=0,1,2$. Now assume that it is valid for $n-1$. By referring to (19), we have

$$\frac{\partial}{\partial p}B_n^{(c)}(p,q)=np^{n-1}-\frac{n(n-1)}{2}p^{n-2}+\cdots.$$

To complete the proof, it is enough to integrate the above equation with respect to the variable p to get the result (23). By referring to relation (22), the result (24) can be similarly derived.
To prove (25), suppose that it first holds for $0,1,\cdots,n-1$. If $n=2m$, then from (17) we have

$$B_{2m}^{(c)}(p,q)=-\frac{1}{2m+1}\sum_{k=0}^{2m-1}\binom{2m+1}{k}B_k^{(c)}(p,q)+\sum_{k=0}^{m}(-1)^k\binom{2m}{2k}p^{2m-2k}q^{2k}. \tag{27}$$

Hence, the coefficient of q^{2m} in the right hand side of (27) is equal to

$$(-1)^m\binom{2m}{2m}p^{2m-2m}=(-1)^m,$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2m+1}\left(\binom{2m+1}{2m-1}(-1)^{m-1}(2m-1)(p-\frac{1}{2})+\binom{2m+1}{2m-2}(-1)^{m-1}\right)$$
$$+(-1)^{m-1}\binom{2m}{2m-2}p^2=(-1)^{m+1}\binom{2m}{2}(p^2-p+\frac{1}{6}).$$

So, (25) is true for $n=2m$. In the second case, taking $n=2m+1$ in (17) gives

$$B_{2m+1}^{(c)}(p,q)=-\frac{1}{2m+2}\sum_{k=0}^{2m}\binom{2m+2}{k}B_k^{(c)}(p,q)+\sum_{k=0}^{m}(-1)^k\binom{2m+1}{2k}p^{2m+1-2k}q^{2k}. \tag{28}$$

Hence, the coefficient of q^{2m} in the right hand side of (28) is equal to

$$\frac{-1}{2m+2}\binom{2m+2}{2m}(-1)^m+(-1)^m\binom{2m+1}{2m}p=(-1)^m(2m+1)(p-\frac{1}{2}),$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2m+2}\left(\binom{2m+2}{2m}(-1)^{m+1}\binom{2m}{2}(p^2-p+\frac{1}{6})+\binom{2m+2}{2m-1}(-1)^{m-1}(2m-1)(p-\frac{1}{2})\right.$$
$$\left.+\binom{2m+2}{2m-2}(-1)^{m-1}\right)+(-1)^{m-1}\binom{2m+1}{2m-2}p^3=(-1)^{m+1}\binom{2m+1}{3}(p^3-\frac{3}{2}p^2+\frac{1}{2}p),$$

which completes the proof of (25). By combining (22) and (25), we can also obtain the result (26). □

Proposition 8. *The following identities hold*

$$B_n^{(c)}(p,q)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}B_{n-2k}^{(c)}(p,0)q^{2k}, \tag{29}$$

and

$$B_n^{(s)}(p,q)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}B_{n-2k-1}^{(c)}(p,0)q^{2k+1}, \tag{30}$$

in which $B_{n-2k}^{(c)}(p,0)=B_{n-2k}(p)$ *and* $B_{n-2k-1}^{(c)}(p,0)=B_{n-2k-1}(p)$ *are usual Bernoulli polynomials.*

Proof. According to (20) and (22), first we have

$$\frac{\partial^{2k}}{\partial q^{2k}}B_n^{(c)}(p,q)=(-1)^k\frac{n!}{(n-2k)!}B_{n-2k}^{(c)}(p,q)\quad\text{for}\quad k=0,1,\cdots,[\frac{n}{2}],$$

and

$$\frac{\partial^{2k+1}}{\partial q^{2k+1}}B_n^{(c)}(p,q)=(-1)^{k+1}\frac{n!}{(n-2k-1)!}B_{n-2k-1}^{(s)}(p,q)\quad\text{for}\quad k=0,1,\cdots,[\frac{n-2}{2}],$$

because $B_n^{(c)}(p,q)$ is a polynomial of degree n for even n and of degree $n-1$ for odd n in terms of the variable q according to the proposition 7. The Taylor expansion of $B_n^{(c)}(p,q)$ gives

$$B_n^{(c)}(p,q+h)=\sum_{k=0}^{n}\frac{1}{k!}\frac{\partial^k}{\partial q^k}B_n^{(c)}(p,q)h^k,$$

in which $h\in\mathbb{R}$. Since $B_n^{(s)}(p,0)=0$ for every $n\in\mathbb{Z}^+$, by replacing $q=0$ and $h=q$, we obtain the relation (29). In a similar way, equality (30), can be derived. □

Proposition 9. *If* $m \in \mathbb{N}$ *and* $n \in \mathbb{Z}^+$ *, then we have*

$$B_n^{(c)}(mp,q) = m^{n-1}\sum_{k=0}^{m-1} B_n^{(c)}(p+\frac{k}{m},\frac{q}{m}), \tag{31}$$

and

$$B_n^{(s)}(mp,q) = m^{n-1}\sum_{k=0}^{m-1} B_n^{(s)}(p+\frac{k}{m},\frac{q}{m}). \tag{32}$$

Proof. To prove (31), it is enough to consider the relation

$$\sum_{n=0}^{\infty} B_n^{(c)}(p+\frac{k}{m},\frac{q}{m})\frac{t^n}{n!} = \frac{t\mathrm{e}^{(p+\frac{k}{m})t}}{\mathrm{e}^t-1}\cos(\frac{q}{m}t),$$

and then take a sum from both sides of the above equation to obtain

$$\begin{aligned}\sum_{k=0}^{m-1}\left(\sum_{n=0}^{\infty} B_n^{(c)}(p+\frac{k}{m},\frac{q}{m})\frac{t^n}{n!}\right) &= \frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\cos(\frac{q}{m}t)\sum_{k=0}^{m-1}\left(\mathrm{e}^{\frac{t}{m}}\right)^k \\ &= m\frac{\frac{t}{m}\mathrm{e}^{mp\frac{t}{m}}}{\mathrm{e}^{\frac{t}{m}}-1}\cos(q\frac{t}{m}) = \sum_{n=0}^{\infty} m^{1-n}B_n^{(c)}(mp,q)\frac{t^n}{n!}.\end{aligned}$$

In a similar way, equality (32) can be proved. □

For $m=2$, relations (31) and (32) respectively yield

$$B_{2n}^{(c)}(\frac{1}{2},q) = 2^{1-2n}B_{2n}^{(c)}(0,2q) - B_{2n}^{(c)}(0,q),$$

and

$$B_{2n+1}^{(s)}(\frac{1}{2},q) = 2^{-2n}B_{2n+1}^{(s)}(0,2q) - B_{2n+1}^{(s)}(0,q).$$

Proposition 10. *For every* $n \in \mathbb{N}$ *and* $q \in \mathbb{R}$ *, the two following propositions are valid:*

$\mathcal{P}_n$ *: The function* $p \mapsto (-1)^n B_{2n-1}^{(c)}(p,q)$ *is positive on* $(0,\frac{1}{2})$ *and negative on* $(\frac{1}{2},1)$ *. Moreover,* $p=\frac{1}{2}$ *is a unique simple root on* $(0,1)$ *, i.e. the aforesaid function has no zero in the intervals* $(0,\frac{1}{2})$ *and* $(\frac{1}{2},1)$ *.*

$\mathcal{Q}_n$ *: The function* $p \mapsto (-1)^n B_{2n}^{(c)}(p,q)$ *is strictly increasing on* $[0,\frac{1}{2}]$ *and strictly decreasing on* $[\frac{1}{2},1]$ *and always takes a positive value at* $p=\frac{1}{2}$ *.*

Proof. The proposition $\mathcal{P}_1$ is clear, because $-B_1^{(c)}(p,q) = -(p-\frac{1}{2}) = -p+\frac{1}{2}$. Now define $f(p) = (-1)^n B_{2n}^{(c)}(p,q)$ to get $f'(p) = 2n(-1)^n B_{2n-1}^{(c)}(p,q)$. By referring to $\mathcal{P}_n$, we see that f is strictly increasing on $[0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},1]$. Moreover, since $\int_0^1 f(p)\ \mathrm{d}p = q^{2n} \geq 0$ (by corollary 5) and $B_{2n}^{(c)}(1-p,q) = B_{2n}^{(c)}(p,q)$ (from proposition 2), one can conclude that $f(\frac{1}{2}) > 0$.

Finally define $g(p) = (-1)^{n+1}B_{2n+1}^{(c)}(p,q)$ to get $g'(p) = -(2n+1)(-1)^n B_{2n}^{(c)}(p,q)$. Since $B_{2n}^{(c)}(0,q) = B_{2n}^{(c)}(1,q)$, by noting $\mathcal{Q}_n$, only one of the following cases occurs:

i) $\alpha \in (0,\frac{1}{2})$ and $\beta \in (\frac{1}{2},1)$ exist such that

$g'(\alpha) = g'(\beta) = 0$ and $\forall p \in (\alpha,\beta),\ g'(p) < 0$ and $\forall p \in [0,\alpha) \cup (\beta,1],\ g'(p) > 0.$

ii) $g'(0) = g'(1) = 0$ and $\forall p \in (0,1),\ g'(p) < 0.$

iii) $\forall p \in [0,1],\ g'(p) < 0.$

In the first case i), by referring to corollary 2 we have

$$A = g(0) = (-1)^{n+1}B_{2n+1}^{(c)}(0,q) = \frac{2n+1}{2}q^{2n} \geq 0.$$

Therefore $g(1) = -A \leq 0$ and g takes the following table of variations

p	0		α		$\frac{1}{2}$		β		1
$g'(p)$		$+$	0		$-$		0	$+$	
$g(p)$	$A \geq 0$	$\nearrow$	$\frown$	$\searrow$	0	$\searrow$	$\smile$	$\nearrow$	$-A \leq 0$

As $g(\frac{1}{2}) = 0$ (by corollary 1) and $g'(\frac{1}{2}) > 0$, $p = \frac{1}{2}$ is a simple root of g. We can similarly observe that the two other cases also hold. So the proof of $\mathcal{P}_{n+1}$ is complete. $\square$

Proposition 11. *For every $n \in \mathbb{Z}^+$ and $q \in \mathbb{R}$ we have*

$$\sup_{p\in[0,1]} |B_{2n}^{(c)}(p,q)| = \max\{|B_{2n}^{(c)}(0,q)|,\ |B_{2n}^{(c)}(\frac{1}{2},q)|\}, \tag{33}$$

and

$$\sup_{p\in[0,1]} |B_{2n+1}^{(c)}(p,q)| \leq \frac{2n+1}{2}\max\{|B_{2n}^{(c)}(0,q)|,\ |B_{2n}^{(c)}(\frac{1}{2},q)|\}. \tag{34}$$

Proof. The result (33) is clear by referring to propositions 2 and 10. To prove (34), if $p \in [0,\frac{1}{2}]$ then we have

$$B_{2n+1}^{(c)}(p,q) = B_{2n+1}^{(c)}(p,q) - B_{2n+1}^{(c)}(\frac{1}{2},q) = (2n+1)\int_{\frac{1}{2}}^{p} B_{2n}^{(c)}(t,q)\ \mathrm{d}t.$$

Therefore

$$|B^{(c)}_{2n+1}(p,q)| \leq (2n+1)\int_p^{\frac{1}{2}} |B^{(c)}_{2n}(t,q)| \ \mathrm{d}t \leq (2n+1)(\frac{1}{2}-p) \sup_{t\in[p,\frac{1}{2}]} |B^{(c)}_{2n}(t,q)|$$
$$\leq (2n+1)(\frac{1}{2}-p)\max\{|B^{(c)}_{2n}(0,q)|,\ |B^{(c)}_{2n}(\frac{1}{2},q)|\},$$

which is equivalent to

$$\sup_{p\in[0,\frac{1}{2}]} |B^{(c)}_{2n+1}(p,q)| \leq \frac{2n+1}{2}\max\{|B^{(c)}_{2n}(0,q)|,\ |B^{(c)}_{2n}(\frac{1}{2},q)|\}.$$

On the other hand, $B^{(c)}_{2n+1}(1-p,q) = -B^{(c)}_{2n+1}(p,q)$ completes the proof of (34). $\square$

Proposition 12. *For every $n \in \mathbb{N}$ and $q > 0$, the two following propositions are valid:*

$\mathcal{P}_n$: *The function $p \mapsto (-1)^n B^{(s)}_{2n}(p,q)$ is positive on $[0,\frac{1}{2})$ and negative on $(\frac{1}{2},1]$. Moreover, $p = \frac{1}{2}$ is a unique simple root on $[0,1]$, i.e. the aforesaid function has no zero in the intervals $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$.*

$\mathcal{Q}_n$: *The function $p \mapsto (-1)^n B^{(s)}_{2n+1}(p,q)$ is strictly increasing on $[0,\frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2},1]$ and always takes a positive value at $p = \frac{1}{2}$.*

Proof. The proposition $\mathcal{P}_1$ is clear, because $-B^{(s)}_2(p,q) = -q(2p-1) = q(1-2p)$. Now define $f(p) = (-1)^n B^{(s)}_{2n+1}(p,q)$ to get $f'(p) = (2n+1)(-1)^n B^{(s)}_{2n}(p,q)$. By noting $\mathcal{P}_n$, we see that f is strictly increasing on $[0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},1]$. Moreover, since $\int_0^1 f(p)\ \mathrm{d}p = q^{2n+1} > 0$ (by corollary 5) and $B^{(s)}_{2n+1}(1-p,q) = B^{(s)}_{2n+1}(p,q)$ (from proposition 2), one can conclude that $f(\frac{1}{2}) > 0$.

Finally define $g(p) = (-1)^{n+1} B^{(s)}_{2n+2}(p,q)$ to get $g'(p) = -(2n+2)(-1)^n B^{(s)}_{2n+1}(p,q)$. Since $B^{(s)}_{2n+1}(0,q) = B^{(s)}_{2n+1}(1,q)$, by noting $\mathcal{Q}_n$, only one of the three following cases occurs:

i) $\alpha \in (0,\frac{1}{2})$ and $\beta \in (\frac{1}{2},1)$ exist such that

$$g'(\alpha) = g'(\beta) = 0 \text{ and } \forall p \in (\alpha,\beta),\ g'(p) < 0 \text{ and } \forall p \in [0,\alpha) \cup (\beta,1],\ g'(p) > 0.$$

ii) $g'(0) = g'(1) = 0$ and $\forall p \in (0,1),\ g'(p) < 0.$

iii) $\forall p \in [0,1],\ g'(p) < 0.$

In the first case i), by referring to corollary 2, we have

$$A^* = g(0) = (-1)^{n+1} B^{(s)}_{2n+2}(0,q) = (n+1)q^{2n+1} > 0.$$

Therefore $g(1) = -A^* < 0$ and g takes the following table of variations
As $g(\frac{1}{2}) = 0$ (by corollary 1) and $g'(\frac{1}{2}) < 0$, then $p = \frac{1}{2}$ is a simple root of function g. Similarly, we can observe that the two other cases also hold. $\square$

p	0		α		$\frac{1}{2}$		β		1
$g'(p)$		$+$	0		$-$		0	$+$	
$g(p)$	$A^* > 0$	$\nearrow$	$\frown$	$\searrow$	0	$\searrow$	$\smile$	$\nearrow$	$-A^* < 0$

Corollary 6. *For every* $n \in \mathbb{N}$ *and* $q \in \mathbb{R}$ *we have*

$$\sup_{p\in[0,1]} |B^{(s)}_{2n+1}(p,q)| = \max\{|B^{(s)}_{2n+1}(0,q)|,\ |B^{(s)}_{2n+1}(\frac{1}{2},q)|\},$$

and

$$\sup_{p\in[0,1]} |B^{(s)}_{2n}(p,q)| \leq n \max\{|B^{(s)}_{2n-1}(0,q)|,\ |B^{(s)}_{2n-1}(\frac{1}{2},q)|\}.$$

Proposition 13. *Let* m *and* n *be two positive integers and*

$$I^{(c)} = \int_0^1 B^{(c)}_m(p,q) B^{(c)}_n(p,q)\ \mathrm{d}p.$$

If $m+n$ *is odd then* $I^{(c)} = 0$ *and if it is even then*

$$I^{(c)} = \sum_{k=0}^{m+n} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} B^{(c)}_{n-j}(0,q) B^{(c)}_{m-k+j}(0,q) \right),$$

where $A = \max\{0, k-m\}$ *and* $B = \min\{n,k\}$.

Proof. First, suppose that $m+n$ is odd. By using (11) we have

$$I^{(c)} = \int_0^1 B^{(c)}_m(1-p,q) B^{(c)}_n(1-p,q)\ \mathrm{d}p = (-1)^{m+n} \int_0^1 B^{(c)}_m(p,q) B^{(c)}_n(p,q)\ \mathrm{d}p = -I^{(c)}.$$

Now, assume that $m+n$ is even. Since $\deg_p\big(B^{(c)}_m B^{(c)}_n\big) = m+n$ (from proposition 7), by referring to (19) we obtain

$$\begin{aligned} B^{(c)}_m(p,q) B^{(c)}_n(p,q) &= \sum_{k=0}^{m+n} \left(\frac{\partial^k}{\partial p^k} \big(B^{(c)}_m(p,q) B^{(c)}_n(p,q) \big) \right)\Bigg|_{p=0} \frac{p^k}{k!} \\ &= \sum_{k=0}^{m+n} \left(\sum_{j=0}^{k} \binom{k}{j} \left(\frac{\partial^j}{\partial p^j} B^{(c)}_n(p,q) \frac{\partial^{k-j}}{\partial p^{k-j}} B^{(c)}_m(p,q) \right)\Bigg|_{p=0} \right) \frac{p^k}{k!} \\ &= \sum_{k=0}^{m+n} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} B^{(c)}_{n-j}(0,q) B^{(c)}_{m-k+j}(0,q) \right) \frac{p^k}{k!}, \end{aligned}$$

which leads to the second result. □

Corollary 7. *Let m and n be two positive integers and*

$$I^{(s)} = \int_0^1 B_m^{(s)}(p,q)B_n^{(s)}(p,q)\ \mathrm{d}p.$$

If $m+n$ is odd then $I^{(s)}=0$ and if $m+n$ is even then

$$I^{(s)} = \sum_{k=0}^{m+n-2} \frac{1}{(k+1)!}\left(\sum_{j=A}^{B}\binom{k}{j}\frac{n!m!}{(n-j)!(m-k+j)!}B_{n-j}^{(s)}(0,q)B_{m-k+j}^{(s)}(0,q)\right),$$

where $A=\max\{0,k-m\}$ and $B=\min\{n,k\}$.

4 Fourier expansions of the polynomials $B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$

The Fourier series of a periodic function f on $[0,L]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(\frac{2k\pi}{L}x) + b_k\sin(\frac{2k\pi}{L}x)\right),$$

where

$$a_0 = \frac{2}{L}\int_0^L f(x)\ \mathrm{d}x,$$

$$a_k = \frac{2}{L}\int_0^L f(x)\cos(\frac{2k\pi}{L}x)\ \mathrm{d}x,$$

and

$$b_k = \frac{2}{L}\int_0^L f(x)\sin(\frac{2k\pi}{L}x)\ \mathrm{d}x.$$

It can also be extended to complex coefficients so that, by considering a real-valued periodic function f on $[0,L]$, we have

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\frac{2\mathrm{i}k\pi}{L}x},$$

in which

$$c_k = \frac{1}{L}\int_0^L f(x)\mathrm{e}^{\frac{-2\mathrm{i}k\pi}{L}x}\ \mathrm{d}x.$$

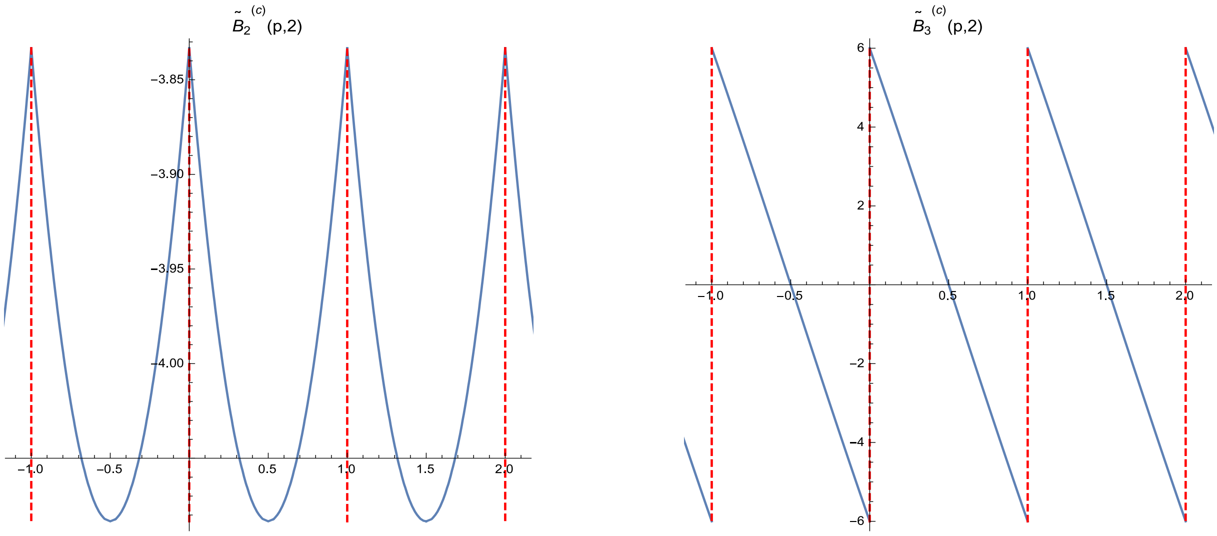

Figure 1: The graphs of $\tilde{B}_2(p,2)$ and $\tilde{B}_3(p,2)$

By periodically extending the restrictions of the introduced bivariate Bernoulli polynomials to $p \in [0,1)$, we would encounter with periodic piecewise continuous functions. In other words, for every real p and q we can define

$$\tilde{B}_n^{(c)}(p,q) = B_n^{(c)}(\{p\},q),$$
$$\tilde{B}_n^{(s)}(p,q) = B_n^{(s)}(\{p\},q),$$

where $\{p\} = p - [p]$ is the fractional part of the real p.
In figure 1, the graphs of the periodic functions $\tilde{B}_2(p,q)$ and $\tilde{B}_3(p,q)$ are displayed for $q = 2$.

Theorem 4.1. *Let $q \in \mathbb{R}$. Then for any $p \in (0,1)$ we have*

$$B_1^{(c)}(p,q) = p - \frac{1}{2} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kp)}{k}, \tag{35}$$

and for every $n \in \mathbb{N}$ we respectively have

$$B_{2n}^{(c)}(p,q) = (-1)^n q^{2n} + \sum_{k=1}^{\infty} a_{k,n}\cos(2\pi kp), \quad p \in [0,1], \tag{36}$$

where

$$a_{k,n} = 2(2n)!(-1)^{n+1}\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}},$$

and

$$B_{2n+1}^{(c)}(p,q) = \sum_{k=1}^{\infty} b_{k,n} \sin(2\pi kp), \quad p \in (0,1), \tag{37}$$

where

$$b_{k,n} = (-1)^{n+1}(2n+1)\left(\frac{q^{2n}}{\pi k} + 2(2n)! \sum_{j=1}^{n} \frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\right).$$

Proof. First, let us consider $\tilde{B}_1^{(c)}$. It is clear that

$$c_0(\tilde{B}_1^{(c)}) = \int_0^1 B_1^{(c)}(p,q)\ \mathrm{d}p = \int_0^1 (p - \frac{1}{2})\ \mathrm{d}p = 0,$$

and for $k \in \mathbb{Z}\backslash\{0\}$ we have

$$c_k(\tilde{B}_1^{(c)}) = \int_0^1 B_1^{(c)}(p,q)\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p = \int_0^1 (p - \frac{1}{2})\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p = \frac{-1}{2\mathrm{i}\pi k}. \tag{38}$$

Since $B_1^{(c)}(0,q) \neq B_1^{(c)}(1,q)$, according to Dirichlet's conditions we can conclude that for every $p \in \mathbb{R}\backslash\mathbb{Z}$ we have

$$\tilde{B}_1^{(c)}(p,q) = \sum_{k\in\mathbb{Z}} c_k(\tilde{B}_1^{(c)})\mathrm{e}^{2\mathrm{i}\pi kp} = \sum_{k\in\mathbb{Z}\backslash\{0\}} \frac{-1}{2\mathrm{i}\pi k}\mathrm{e}^{2\mathrm{i}\pi kp} = -\frac{1}{\pi}\sum_{k=1}^{\infty} \frac{\sin(2\pi kp)}{k},$$

where we use $c_{-k}(\tilde{B}_1^{(c)}) = -c_k(\tilde{B}_1^{(c)})$, which proves (35).

We now consider the case $\tilde{B}_{2n}^{(c)}$. According to corollary 5 we have

$$c_0(\tilde{B}_{2n}^{(c)}) = \int_0^1 B_{2n}^{(c)}(p,q)\ \mathrm{d}p = (-1)^n q^{2n},$$

and for $k \in \mathbb{Z}\backslash\{0\}$

$$\begin{aligned} c_k(\tilde{B}_{2n}^{(c)}) = \int_0^1 B_{2n}^{(c)}(p,q)\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p &= \frac{2n}{2\mathrm{i}\pi kp}\int_0^1 B_{2n-1}^{(c)}(p,q)\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p \\ &= \frac{n}{\mathrm{i}\pi k} c_k(\tilde{B}_{2n-1}^{(c)}), \end{aligned} \tag{39}$$

where we have used $B_{2n}^{(c)}(0,q) = B_{2n}^{(c)}(1,q)$ in proposition 2. Similarly, we can find that

$$c_0(\tilde{B}_{2n+1}^{(c)}) = 0 \quad \text{and} \quad c_k(\tilde{B}_{2n+1}^{(c)}) = \frac{2n+1}{2\mathrm{i}\pi k}\left((-1)^{n+1}q^{2n} + c_k(\tilde{B}_{2n}^{(c)})\right). \tag{40}$$

Now, for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}\backslash\{0\}$ we show that

$$c_k(\tilde{B}_{2n}^{(c)}) = (-1)^{n+1}(2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}}, \tag{41}$$

and

$$c_k(\tilde{B}_{2n+1}^{(c)}) = \frac{(-1)^{n+1}(2n+1)}{\mathrm{i}}\left(\frac{q^{2n}}{2\pi k} + (2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\right). \tag{42}$$

Since $c_k(\tilde{B}_1^{(c)}) = -\frac{1}{2\mathrm{i}\pi k}$ by (38), from equation (39) we obtain

$$c_k(\tilde{B}_2^{(c)}) = \frac{1}{\mathrm{i}\pi k}(-\frac{1}{2\mathrm{i}\pi k}) = \frac{2}{(2\pi k)^2}.$$

Assume that (41) is true for n. Then using (40) gives

$$\begin{aligned} c_k(\tilde{B}_{2n+1}^{(c)}) &= \frac{2n+1}{2\mathrm{i}\pi k}\left((-1)^{n+1}q^{2n} + (-1)^{n+1}(2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}}\right) \\ &= \frac{(-1)^{n+1}(2n+1)}{\mathrm{i}}\left(\frac{q^{2n}}{2\pi k} + (2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\right). \end{aligned}$$

So, (42) is satisfied for n. Now let (42) be true for n. Then for $n+1$, relation (39) gives

$$\begin{aligned} c_k(\tilde{B}_{2n+2}^{(c)}) &= \frac{n+1}{\mathrm{i}\pi k}\frac{(-1)^{n+1}(2n+1)}{\mathrm{i}}\left(\frac{q^{2n}}{2\pi k} + (2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\right) \\ &= (-1)^{n+2}(2n+2)!\left(\frac{q^{2n}}{(2n)!(2\pi k)^2} + \sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+2}}\right) \\ &= (-1)^{n+2}(2n+2)!\left(\frac{q^{2n}}{(2n)!(2\pi k)^2} + \sum_{j=2}^{n+1}\frac{q^{2n-2j+2}}{(2n-2j+2)!(2\pi k)^{2j}}\right) \\ &= (-1)^{n+2}(2n+2)!\sum_{j=1}^{n+1}\frac{q^{2(n+1)-2j}}{(2(n+1)-2j)!(2\pi k)^{2j}}, \end{aligned}$$

which approves (41) for $n+1$. From (41) and (42), it is clear that

$$c_{-k}(\tilde{B}_{2n}^{(c)}) = c_k(\tilde{B}_{2n}^{(c)}) \quad \text{and} \quad c_{-k}(\tilde{B}_{2n+1}^{(c)}) = -c_k(\tilde{B}_{2n+1}^{(c)}).$$

As

$$B_{2n}^{(c)}(0,q) = B_{2n}^{(c)}(1,q) \quad \text{and} \quad B_{2n+1}^{(c)}(0,q) \neq B_{2n}^{(c)}(1,q),$$

we can directly obtain the identities (36) and (37) by Dirichlet's theorem. □

Theorem 4.2. *Let $q \in \mathbb{R}$. Then for every $p \in (0,1)$ we have*

$$B_2^{(s)}(p,q) = 2qp - q = -\frac{2q}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kp)}{k},$$

and for every $n \geq 2$ we respectively have

$$B_{2n-1}^{(s)}(p,q) = (-1)^{n-1}q^{2n-1} + \sum_{k=1}^{\infty} a'_{k,n}\cos(2\pi kp), \quad p \in [0,1], \tag{43}$$

where

$$a'_{k,n} = 2(-1)^n(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j}},$$

and

$$B_{2n}^{(s)}(p,q) = \sum_{k=1}^{\infty} b'_{k,n}\sin(2\pi kp), \quad p \in (0,1), \tag{44}$$

where

$$b'_{k,n} = 2n(-1)^n\left(\frac{q^{2n-1}}{\pi k} + 2(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j+1}}\right).$$

Proof. The proof of this theorem is similar to the previous one. However, note that for $k \in \mathbb{Z}\backslash\{0\}$ we have

$$c_k(\tilde{B}_{2n-1}^{(s)}) = (-1)^n(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j}},$$

and

$$c_k(\tilde{B}_{2n}^{(s)}) = \frac{2n(-1)^n}{\mathrm{i}}\left(\frac{q^{2n-1}}{2\pi k} + (2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j+1}}\right),$$

and from corollary 5

$$c_0(\tilde{B}_{2n-1}^{(s)}) = (-1)^{n-1}q^{2n-1}, \;\; c_0(\tilde{B}_{2n}^{(s)}) = 0.$$

□

By using theorems 4.1 and 4.2, one can now obtain the following series in terms of the introduced bivariate polynomials:

Corollary 8. *From relations* (36) *and* (37), *we have*

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}}\right)=\frac{(-1)^{n+1}}{2(2n)!}\left(B_{2n}^{(c)}(0,q)-(-1)^n q^{2n}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\left(\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}}\right)=\frac{(-1)^{n+1}}{2(2n)!}\left(B_{2n}^{(c)}(\frac{1}{2},q)-(-1)^n q^{2n}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\left(\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(4\pi k)^{2j}}\right)=\frac{(-1)^{n+1}}{2(2n)!}\left(B_{2n}^{(c)}(\frac{1}{4},q)-(-1)^n q^{2n}\right),$$

and

$$\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{q^{2n}}{(2k-1)\pi}+2(2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!\big((4k-2)\pi\big)^{2j+1}}\right)=\frac{(-1)^{n+1}}{2n+1}B_{2n+1}^{(c)}(\frac{1}{4},q).$$

Corollary 9. *From relations* (43) *and* (44), *we have*

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j}}\right)=\frac{(-1)^{n}}{2(2n-1)!}\left(B_{2n-1}^{(s)}(0,q)-(-1)^{n-1} q^{2n-1}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\left(\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j}}\right)=\frac{(-1)^{n}}{2(2n-1)!}\left(B_{2n-1}^{(s)}(\frac{1}{2},q)-(-1)^{n-1} q^{2n-1}\right),$$

$$\sum_{k=1}^{\infty}(-1)^k\left(\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(4\pi k)^{2j}}\right)=\frac{(-1)^{n}}{2(2n-1)!}\left(B_{2n-1}^{(s)}(\frac{1}{4},q)-(-1)^{n-1} q^{2n-1}\right),$$

and

$$\sum_{k=1}^{\infty}(-1)^{k+1}\left(\frac{q^{2n-1}}{(2k-1)\pi}+2(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!\big((4k-2)\pi\big)^{2j+1}}\right)=\frac{(-1)^{n}}{2n}B_{2n}^{(s)}(\frac{1}{4},q).$$

We finally point out not only the main approach, used in this paper, can be applied for other special numbers and polynomials [in preprint], but can also be used for explicit computation of some new power-trigonometric series [10].

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