```
> restart;
> read `qsum17.mpl`:
```

Package "q-Hypergeometric Summation", Maple V-17

Copyright 1998-2013, Harald Boeing & Wolfram Koepf, University of Kassel **(1)**

```
> Digits:=15:
> _qsum_local_specialsolution:= false:
```

Shift on one parameter on the right-hand side of the equation

```
> qMixRec1:=proc(F,q,k,Sn,alpha,shift0)
            local zeit,pp,qq,rr,Rat,evalS,z,lo,hi,sigma,sigmasol,
  Poly,K,j,J,\
                      PQR,f,rec,S,n;
     zeit:= time();
       lo:=1; hi:=5;
       S:=op(0,Sn); n:=op(1,Sn);
            sigmasol:= NULL;
            for J from 1 to hi while (sigmasol = NULL) do
          Poly:=qsimpcomb(F - add(sigma[j]*subs({n=n-j,alpha=alpha*
  q^shift0}, F), j=0..J));
                      Rat:= `power/subs`({q^k=K,q^n=N,q^(-n)=1/N},
  qratio(Poly,k));
                      if has(Rat,{k,qpochhammer}) then
                                ERROR(`Algorithm not applicable.`);
                      fi;
                      if (J < lo) then next; fi;
          pp:=1: qq:=numer(Rat): rr:=denom(Rat):
                      PQR:= `qgosper/update`(pp,qq,rr,q,K);
                      f:= `qgosper/findf`(op(PQR),q,K,[seq(sigma
  [j],j=0..J)],'sigmasol');
            od;
            if (sigmasol = NULL) then
                  ERROR(cat(`Found no q-derivative rule of order
  smaller than `,J,`.`));
            fi;
            rec:= subs(sigmasol, add(sigma[j]*S(n-j,alpha*
  q^shift0), j=0..J-1));
      rec:=combine(map(factor, subs({N=q^n,K=q^k},rec)),power);
            if (_qsum_profile) then
                      printf(`CPU-time: %.1f seconds`, time()-zeit);
            fi;
            RETURN(S(n,alpha)=rec);
  end:
```

Shift on two parameters in the equation or on one parameter on the left-hand side of the equation

```
> qMixRec2:=proc(F,q,k,Sn,alpha,shift0,beta,shift1,shift2,r)
            local zeit,pp,qq,rr,Rat,evalS,z,lo,hi,sigma,sigmasol,
  Poly,K,j,J,\
                      PQR,f,rec,S,n;
     zeit:= time();
       lo:=1; hi:=5;
       S:=op(0,Sn); n:=op(1,Sn);
            sigmasol:= NULL;
            for J from 1 to hi while (sigmasol = NULL) do
          Poly:=qsimpcomb(subs(beta=beta*q^shift1,F) - add(sigma[j]
  *subs({n=n-j,alpha=alpha*q^shift0,beta=beta*q^(shift2+r*j)}, F),
```

```
j=0..J));
                    Rat:= `power/subs`({q^k=K,q^n=N,q^(-n)=1/N},
qratio(Poly,k));
                    if has(Rat,{k,qpochhammer}) then
                              ERROR(`Algorithm not applicable.`);
                    fi;
                    if (J < lo) then next; fi;
        pp:=1: qq:=numer(Rat): rr:=denom(Rat):
                    PQR:= `qgosper/update`(pp,qq,rr,q,K);
                    f:= `qgosper/findf`(op(PQR),q,K,[seq(sigma
[j],j=0..J)],'sigmasol');
          od;
          if (sigmasol = NULL) then
                ERROR(cat(`Found no q-derivative rule of order
smaller than `,J,`.`));
          fi;
          rec:= subs(sigmasol, add(sigma[j]*S(n-j,alpha*q^shift0,
beta*q^(shift2+r*j)), j=0..J-1));
      rec:=combine(map(factor, subs({N=q^n,K=q^k},rec)),power);
          if (_qsum_profile) then
                    printf(`CPU-time: %.1f seconds`, time()-zeit);
          fi;
          RETURN(S(n,alpha,beta*q^shift1)=rec);
end:
```

Big q-Jacobi

The big q-Jacobi polynomials

```
> bqj:=(n,x,alpha,beta,gamma,q)->1/(qpochhammer(alpha*beta*q^
  (n+1),q,n)/(qpochhammer(alpha*q, q, n)*qpochhammer(gamma*q, q,
  n)))*add(qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],[alpha*q,
  gamma*q],q,q,j),j=0..n);
```

$$bqj := (n, x, \alpha, \beta, \gamma, q) \rightarrow \frac{1}{qpochhammer(\alpha\,\beta\,q^{n+1}, q, n)} \big(qpochhammer(\alpha\,q, q, n)\; qpochhammer(\gamma\,q, q, n)\; add\big(qphihyperterm([q^{-n}, \alpha\,\beta\,q^{n+1}, x], [\alpha\,q, \gamma\,q], q, q, j), j = 0..n\big)\big) \tag{1.1}$$

The summand of the above sum is

```
> Fbqj:=1/(qpochhammer(alpha*beta*q^(n+1),q,n)/(qpochhammer
  (alpha*q, q, n)*qpochhammer(gamma*q, q, n)))*(qphihyperterm([q^
  (-n),alpha*beta*q^(n+1),x],[alpha*q,gamma*q],q,q,k))
```

$$Fbqj := \big(qpochhammer(\alpha\,q, q, n)\; qpochhammer(\gamma\,q, q, n)\; qpochhammer(q^{-n}, q, k)\; qpochhammer(\alpha\,\beta\,q^{n+1}, q, k)\; qpochhammer(x, q, k)\; q^{k}\big) \Big/ \big(qpochhammer(\alpha\,\beta\,q^{n+1}, q, n)\; qpochhammer(\alpha\,q, q, k)\; qpochhammer(\gamma\,q, q, k)\; qpochhammer(q, q, k)\big) \tag{1.2}$$

Proposition 3

```
> eq4a1:=qMixRec1(Fbqj,q,k,P(n),alpha,1):
> eq4a2:=subs(_C1=1,eq4a1):
> eq4a:=lhs(eq4a2)=combine(map(qsimpcomb,rhs(eq4a2)),power)
```

$$eq4a := P(n,\alpha) = P(n,\alpha q) + \frac{(q^n-1)\,(\beta q^n-1)\,(\gamma q^n-1)\,q\,\alpha\,P(n-1,\alpha q)}{(\alpha\beta q^{1+2n}-1)\,(\alpha\beta q^{2n}-1)} \quad \textbf{(1.3)}$$

```
> eq4b1:=qMixRec1(Fbqj,q,k,P(n),beta,1):
> eq4b2:=subs(_C1=1,eq4b1):
> eq4b:=lhs(eq4b2)=combine(map(qsimpcomb,rhs(eq4b2)),power)
```

$$eq4b := P(n,\beta) = P(n,q\beta) - \frac{(\alpha q^n-1)\,\beta\,\alpha\,P(n-1,q\beta)\,(q^n-1)\,(\gamma q^n-1)\,q^{n+1}}{(\alpha\beta q^{2n}-1)\,(\alpha\beta q^{1+2n}-1)} \quad \textbf{(1.4)}$$

```
> eq4c1:=qMixRec2(Fbqj,q,k,P(n),alpha,1,beta,1,0,1):
> eq4c2:=subs(_C1=1,eq4c1):
> eq4c:=lhs(eq4c2)=combine(map(qsimpcomb,rhs(eq4c2)),power)
```

$$eq4c := P(n,\alpha,q\beta) = P(n,\alpha q,\beta) + \frac{\alpha q\,(q^n-1)\,(\gamma q^n-1)\,P(n-1,\alpha q,q\beta)}{\alpha\beta q^{1+2n}-1} \quad \textbf{(1.5)}$$

Theorem 4 (We give some specific values to the parameters and do some simulations)

```
> alpha:=0.5;beta:=0.9;gam:=-5;q:=0.9;n:=7;
```

$$\alpha := 0.5$$
$$\beta := 0.9$$
$$gam := -5$$
$$q := 0.9$$
$$n := 7 \quad \textbf{(1.6)}$$

```
> hyp1:= 0 < alpha*q and alpha*q<1 and 0<= beta*q and beta*q<1
  and gam <0;
```

$$hyp1 := true \quad \textbf{(1.7)}$$

x_{n,i}

```
> xnbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha,beta,gam,q)),x)]);
```

$xnbqj := [-4.49882900089888, -4.01859703862839, -3.46179600904223, -2.78095406854346, -2.01244790190428, -1.21320789744502, -0.435754697993304]$ **(1.8)**

y_{n,i}

```
> ynbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha*q,beta,gam,q)),x)])
  ;
```

$ynbqj := [-4.49936232891947, -4.02912634732680, -3.50432761230100, -2.86499849776070, -2.13427562311069, -1.36028997075449, -0.586329561486357]$ **(1.9)**

z_{n,i}

```
> znbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha,beta*q,gam,q)),x)])
  ;
```

$znbqj := [-4.49720628821141, -4.00032174693591, -3.41235989525388, -2.71039084782294, -1.93794153048075, -1.14958921178580, -0.393391768986038]$ **(1.10)**

x_{n-1,i}

```
> xnm1bqj:= sort([solve(qsimpcomb(bqj(n-1,x,alpha,beta,gam,q)),x)
  ]);
```

$xnm1bqj := [-4.49495902236422, -3.96775598822957, -3.29652350897147, -2.47692968174774, -1.57581676731384, -0.664435543036401]$ **(1.11)** y_{n-1,i}

```
> ynm1bqj:= sort([solve(qsimpcomb(bqj(n-1,x,alpha*q,beta,gam,q)),
  x)]);
```

ynm1bqj := [-4.49693686309745, -3.99028107216936, -3.36292435480235, -2.59104838987980, -1.72704350788236, -0.828585039755976] **(1.12)**

z_{n-1,i}

```
> znm1bqj:= sort([solve(qsimpcomb(bqj(n-1,x,alpha,beta*q,gam,q)),
  x)]);
```

znm1bqj := [-4.48921357052139, -3.92981883768300, -3.22139926833791, -2.38626536472727, -1.49278405577270, -0.606613900100430] **(1.13)**

t_{n-1,i}

```
> tnm1bqj:= sort([solve(qsimpcomb(bqj(n-1,x,alpha*q,beta*q,gam,q)
  ),x)]);
```

tnm1bqj := [-4.49322849305760, -3.96077191851065, -3.29822493826811, -2.50855343818780, -1.64792083977703, -0.770211320655107] **(1.14)**

(a)

```
> seq(evalb(ynbqj[i]<xnbqj[i] and xnbqj[i]<ynm1bqj[i] and ynm1bqj
  [i]<ynbqj[i+1] and ynbqj[i+1]<xnbqj[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true*, *true*, *true* **(1.15)**

(b)

```
> seq(evalb(xnbqj[i]<znbqj[i] and znbqj[i]<znm1bqj[i] and znm1bqj
  [i]<xnbqj[i+1] and xnbqj[i+1]<znbqj[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true*, *true*, *true* **(1.16)**

(c)

```
> seq(evalb(ynbqj[i]<xnbqj[i] and xnbqj[i]<znbqj[i] and znbqj[i]
  <tnm1bqj[i] and tnm1bqj[i]<ynbqj[i+1] and ynbqj[i+1]<xnbqj[i+1]
  and xnbqj[i+1]<znbqj[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true*, *true*, *true* **(1.17)**

Corollary 5

(a)

```
> seq(evalb(xnbqj[i]<ynm1bqj[i] and ynm1bqj[i]<xnm1bqj[i] and
  xnm1bqj[i]<xnbqj[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true*, *true*, *true* **(1.18)**

(b)

```
> seq(evalb(xnbqj[i]<xnm1bqj[i] and xnm1bqj[i]<znm1bqj[i] and
  znm1bqj[i]<xnbqj[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true*, *true*, *true* **(1.19)**

```
> unassign('alpha,beta,q,n')
```

q-Hahn

The q-Hahn polynomials are defined by

```
> QH:=(n,x,alpha,beta,N,q)->1/(qpochhammer(alpha*beta*q^(n+1), q,
  n)/(qpochhammer(alpha*q, q, n)*qpochhammer(q^(-N), q, n)))*add
  (qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],[alpha*q,q^(-N)],
  q,q,j),j=0..n);
```

(2.1)

$$QH := (n, x, \alpha, \beta, N, q) \rightarrow \frac{1}{qpochhammer(\alpha\beta q^{n+1}, q, n)} (qpochhammer(\alpha q, q, n)\, qpochhammer(q^{-N}, q, n)\, add(qphihyperterm([q^{-n}, \alpha\beta q^{n+1}, x], [\alpha q, q^{-N}], q, q, j), j = 0..n)) \quad (2.1)$$

The summand of the above sum is

```
> Fqh:=1/(qpochhammer(alpha*beta*q^(n+1), q, n)/(qpochhammer
  (alpha*q, q, n)*qpochhammer(q^(-NN), q, n)))*(qphihyperterm([q^
  (-n),alpha*beta*q^(n+1),x],[alpha*q,q^(-NN)],q,q,k)):
```

Proposition 7

```
> eq5a1:=qMixRec1(Fqh,q,k,Q(n),alpha,1):
> eq5a2:=subs({_C1=1,NN=N},eq5a1):
> eq5a:=lhs(eq5a2)=combine(map(qsimpcomb,rhs(eq5a2)),power)
```

$$eq5a := Q(n, \alpha) = Q(n, \alpha q) + \frac{(q^n - 1)(\beta q^n - 1)(-q^{N+1} + q^n)\,\alpha\, Q(n-1, \alpha q)\, q^{-N}}{(\alpha\beta q^{1+2n} - 1)(\alpha\beta q^{2n} - 1)} \quad (2.2)$$

```
> eq5b1:=qMixRec1(Fqh,q,k,Q(n),beta,1):
> eq5b2:=subs({_C1=1,NN=N},eq5b1):
> eq5b:=lhs(eq5b2)=combine(map(qsimpcomb,rhs(eq5b2)),power)
```

$$eq5b := Q(n, \beta) = Q(n, q\beta) - \frac{(\alpha q^n - 1)(-q^{N+1} + q^n)\,\beta\alpha\, Q(n-1, q\beta)(q^n - 1)\, q^{n-N}}{(\alpha\beta q^{2n} - 1)(\alpha\beta q^{1+2n} - 1)} \quad (2.3)$$

```
> eq5c1:=qMixRec2(Fqh,q,k,Q(n),alpha,1,beta,1,0,1):
> eq5c2:=subs({_C1=1,NN=N},eq5c1):
> eq5c:=lhs(eq5c2)=combine(map(qsimpcomb,rhs(eq5c2)),power)
```

$$eq5c := Q(n, \alpha, q\beta) = Q(n, \alpha q, \beta) + \frac{\alpha (q^n - 1)(-q^{N+1} + q^n)\, Q(n-1, \alpha q, q\beta)\, q^{-N}}{\alpha\beta q^{1+2n} - 1} \quad (2.4)$$

```
> eq5d1:=subs(NN=N,qMixRec2(Fqh,q,k,Q(n),beta,0,alpha,0,1,1))
```

$$eq5d1 := Q(n, \beta, \alpha) = \left((\alpha\beta q^{1+2n} - 1)(\beta\alpha^2 q^{2n+N+3} - \alpha\beta q^{n+N+2} - \alpha q^{n+N+2} + \alpha q^{N+2} - \alpha q^{n+1} + q^n)\, Q(n, \beta, \alpha q)\, q^{-n}\right) \Big/ \left((\alpha q^{n+1} - 1)(\alpha\beta q^{n+1} - 1)(\alpha\beta q^{n+N+2} - 1)\right) + \frac{(\alpha q^2 - x)(-q^{N+1} + q^n)(\beta q^n - 1)\,\alpha\, Q(n-1, \beta, \alpha q^2)(q^n - 1)\, q^{-n+1}}{(\alpha\beta q^{n+N+2} - 1)(\alpha\beta q^{n+1} - 1)(\alpha q^{n+1} - 1)} \quad (2.5)$$

Theorem 8

```
> alpha:=0.5;beta:=0.9;N:=10;q:=0.9;n:=7;
```

$$\alpha := 0.5$$
$$\beta := 0.9$$
$$N := 10$$
$$q := 0.9$$
$$n := 7 \quad (2.6)$$

```
> hyp1:= 0 < alpha*q and alpha*q<1 and 0<= beta*q and beta*q<1
  and n <=N;
```

$$hyp1 := true \tag{2.7}$$

```
> xnqh:= sort([solve(qsimpcomb(QH(n,x,alpha,beta,N,q)),x)]);
```

$xnqh$:= [1.16164046616506, 1.45283574304484, 1.75551856689881, 2.04673112296365, 2.31558061366060, 2.58076075377308, 2.86796717089659] **(2.8)**

```
> ynqh:= sort([solve(qsimpcomb(QH(n,x,alpha*q,beta,N,q)),x)]);
```

$ynqh$:= [1.18581651590948, 1.47978740323468, 1.77668569493920, 2.05768366318270, 2.31824352980350, 2.58095190335542, 2.86796978989324] **(2.9)**

```
> znqh:= sort([solve(qsimpcomb(QH(n,x,alpha,beta*q,N,q)),x)]);
```

$znqh$:= [1.15260110307267, 1.43937573066722, 1.74167834750979, 2.03719920451099, 2.31230506207323, 2.58040026198878, 2.86795880025802] **(2.10)**

```
> xnm1qh:= sort([solve(qsimpcomb(QH(n-1,x,alpha,beta,N,q)),x)]);
```

$xnm1qh$:= [1.25524223106918, 1.61081694836532, 1.96451819900839, 2.28820337489456, 2.57752019177601, 2.86789443962382] **(2.11)**

```
> ynm1qh:= sort([solve(qsimpcomb(QH(n-1,x,alpha*q,beta,N,q)),x)])
  ;
```

$ynm1qh$:= [1.28784969039447, 1.64343368420813, 1.98720048354850, 2.29754917303560, 2.57891266281773, 2.86793190270405] **(2.12)**

```
> znm1qh:= sort([solve(qsimpcomb(QH(n-1,x,alpha,beta*q,N,q)),x)])
  ;
```

$znm1qh$:= [1.24083776907828, 1.59090914050049, 1.94575649863061, 2.27741029686448, 2.57506981295959, 2.86778031023554] **(2.13)**

```
> tnm1qh:= sort([solve(qsimpcomb(QH(n-1,x,alpha*q,beta*q,N,q)),x)
  ]);
```

$tnm1qh$:= [1.27316329848469, 1.62456235780771, 1.97064709765738, 2.28904831563461, 2.57730155814938, 2.86787063077651] **(2.14)**

(a)

```
> seq(evalb(xnqh[i]<ynqh[i] and ynqh[i]<ynm1qh[i] and ynm1qh[i]
  <xnqh[i+1] and xnqh[i+1]<ynqh[i+1]),i=1..n-1 )
```

true, true, true, true, true, true **(2.15)**

(b)

```
> seq(evalb(znqh[i]<xnqh[i] and xnqh[i]<znm1qh[i] and znm1qh[i]
  <znqh[i+1] and znqh[i+1]<xnqh[i+1]),i=1..n-1 )
```

true, true, true, true, true, true **(2.16)**

(c)

```
> seq(evalb(znqh[i]<xnqh[i] and xnqh[i]<ynqh[i] and ynqh[i]
  <tnm1qh[i] and tnm1qh[i]<znqh[i+1] and znqh[i+1]<xnqh[i+1] and
  xnqh[i+1]<ynqh[i+1]),i=1..n-1 )
```

true, true, true, true, true, true **(2.17)**

Corollary 9

(a)

```
> seq(evalb(xnqh[i]<xnm1qh[i] and xnm1qh[i]<ynm1qh[i] and ynm1qh
  [i]<xnqh[i+1]),i=1..n-1 )
```

true, true, true, true, true, true **(2.18)**

(b)

```
> seq(evalb(xnqh[i]<znm1qh[i] and znm1qh[i]<xnm1qh[i] and xnm1qh
  [i]<xnqh[i+1]),i=1..n-1 )
```

true, true, true, true, true, true **(2.19)**

```
> unassign('alpha,beta,N,q,n')
```

Big q-Laguerre (see Remark 6-ii)

```
> BQL:=(n,x,alpha,beta,q)->1/(1/(qpochhammer(alpha*q, q, n)*
  qpochhammer(beta*q, q, n)))*add(qphihyperterm([q^(-n),x,0],
  [alpha*q,beta*q],q,q,j),j=0..n);
```

$$BQL := (n, x, \alpha, \beta, q) \rightarrow qpochhammer(\alpha\, q, q, n)\; qpochhammer(q\, \beta, q, n)\; add(qphihyperterm([q^{-n}, x, 0], [\alpha\, q, q\, \beta], q, q, j), j = 0..n) \quad (3.1)$$

```
> alpha:=0.4;beta:=-10;q:=0.9;n:=7;
```

$$\alpha := 0.4$$
$$\beta := -10$$
$$q := 0.9$$
$$n := 7 \quad (3.2)$$

```
> hyp1:= 0 < alpha*q and alpha*q<1 and  beta<0;
```

$$hyp1 := true \quad (3.3)$$

```
> xnbql:= sort([solve(qsimpcomb(BQL(n,x,alpha,beta,q)),x)]);
```

$$xnbql := [-8.95738562981191, -7.82127143778649, -6.53268029903154, -5.16799821230597, -3.80736438970594, -2.50781497878821, -1.29758977635687] \quad (3.4)$$

```
> ynbql:= sort([solve(qsimpcomb(BQL(n,x,alpha*q,beta,q)),x)]);
```

$$ynbql := [-8.97303849741946, -7.89281263606771, -6.67059477821005, -5.35826887252032, -4.03001703203827, -2.74024732668595, -1.51324300846638] \quad (3.5)$$

```
> znbql:= sort([solve(qsimpcomb(BQL(n,x,alpha,beta*q,q)),x)]);
```

$$znbql := [-8.05945252674578, -7.02949006030592, -5.86037935487747, -4.62318815841983, -3.39047388371623, -2.21369625627684, -1.11840089506636] \quad (3.6)$$

```
> xnm1bql:= sort([solve(qsimpcomb(BQL(n-1,x,alpha,beta,q)),x)]);
```

$$xnm1bql := [-8.87281030879282, -7.52826202049989, -6.02592703523841, -4.48861454062568, -3.00038049790744, -1.60309051025195] \quad (3.7)$$

```
> ynm1bql:= sort([solve(qsimpcomb(BQL(n-1,x,alpha*q,beta,q)),x)])
  ;
```

$$ynm1bql := [-8.91088535484520, -7.64523492205165, -6.21290953900536, -4.72172929324458, -3.25208677817942, -1.84136153465856] \quad (3.8)$$

```
> znm1bql:= sort([solve(qsimpcomb(BQL(n-1,x,alpha,beta*q,q)),x)])
  ;
```

$$znm1bql := [-7.98074235809208, -6.76078201519472, -5.39860947301396, -4.00589892691518, -2.65844673550939, -1.39401567325957] \quad (3.9)$$

```
> tnm1bql:= sort([solve(qsimpcomb(BQL(n-1,x,alpha*q,beta*q,q)),x)
  ]);
```

$$tnm1bql := [-8.01622841180800, -6.86842371008524, -5.56984404829435, -4.21901010129707, -2.88848452735919, -1.61198276494215] \quad (3.10)$$

(a)

```
> seq(evalb(ynbql[i]<xnbql[i] and xnbql[i]<ynm1bql[i] and ynm1bql
  [i]<ynbql[i+1] and ynbql[i+1]<xnbql[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true \quad (3.11)$$

(b)

```
> seq(evalb(xnbql[i]<znbql[i] and znbql[i]<znm1bql[i] and znm1bql
  [i]<xnbql[i+1] and xnbql[i+1]<znbql[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true \quad (3.12)$$

(c)

```
> seq(evalb(ynbql[i]<xnbql[i] and xnbql[i]<znbql[i] and znbql[i]
  <tnm1bql[i] and tnm1bql[i]<ynbql[i+1] and ynbql[i+1]<xnbql[i+1]
  and xnbql[i+1]<znbql[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true \quad (3.13)$$

(d)

```
> seq(evalb(xnbql[i]<ynm1bql[i] and ynm1bql[i]<xnm1bql[i] and
  xnm1bql[i]<xnbql[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true \quad (3.14)$$

(e)

```
> seq(evalb(xnbql[i]<xnm1bql[i] and xnm1bql[i]<znm1bql[i] and
  znm1bql[i]<xnbql[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true \quad (3.15)$$

```
> unassign('alpha,beta,q,n')
```

little q-jacobi

The little q-Jacobi polynomials are given by

```
> LQJ:=(n,x,alpha,beta,q)->1/((-1)^n*q^(-binomial(n,2))*
  qpochhammer(alpha*beta*q^(n+1), q, n)/qpochhammer(alpha*q, q,
  n))*add(qphihyperterm([q^(-n),alpha*beta*q^(n+1)],[alpha*q],q,
  q*x,j),j=0..n);
```

$$LQJ := (n, x, \alpha, \beta, q) \rightarrow (qpochhammer(\alpha q, q, n)\ add(qphihyperterm([q^{-n}, \alpha \beta q^{n+1}], [\alpha q], q, q x, j), j=0..n)) / ((-1)^n q^{-\mathrm{binomial}(n,2)}\ qpochhammer(\alpha \beta q^{n+1}, q, n)) \quad (4.1)$$

```
> Flqj:=1/((-1)^n*q^(-binomial(n,2))*qpochhammer(alpha*beta*q^
  (n+1), q, n)/qpochhammer(alpha*q, q, n))*(qphihyperterm([q^(-
  n),alpha*beta*q^(n+1)],[alpha*q],q,q*x,k)):
```

Proposition 11

```
> eq6a1:=qMixRec1(Flqj,q,k,p(n),alpha,1):
> eq6a2:=subs(_C1=1,eq6a1):
> eq6a:=lhs(eq6a2)=combine(map(qsimpcomb,rhs(eq6a2)),power)
```

$$eq6a := p(n, \alpha) = p(n, \alpha q) + \frac{p(n-1, \alpha q)\ (q^n - 1)\ (\beta q^n - 1)\ \alpha q^n}{(\alpha \beta q^{1+2n} - 1)\ (\alpha \beta q^{2n} - 1)} \quad (4.2)$$

```
> eq6b1:=qMixRec1(Flqj,q,k,p(n),beta,1):
> eq6b2:=subs(_C1=1,eq6b1):
> eq6b:=lhs(eq6b2)=combine(map(qsimpcomb,rhs(eq6b2)),power)
```

$$eq6b := p(n, \beta) = p(n, q \beta) - \frac{p(n-1, q \beta)\ (q^n - 1)\ (\alpha q^n - 1)\ \alpha \beta q^{2n}}{(\alpha \beta q^{1+2n} - 1)\ (\alpha \beta q^{2n} - 1)} \quad (4.3)$$

```
> eq6c:=qMixRec1(Flqj,q,k,p(n),alpha,2)
```

$$eq6c := p(n,\alpha) = \frac{(\alpha q-1)\,(\alpha\beta q^{1+2n}-1)\,p(n,\alpha q^2)}{(\alpha q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)} + \frac{q\,\alpha\,(q^n-1)\,(\beta q^n-1)\,(\alpha\beta q^{2n+2}x+\alpha q^{n+1}-q^n-x)\,p(n-1,\alpha q^2)}{(\alpha q^{n+1}-1)\,(\alpha\beta q^{2n+2}-1)\,(\alpha\beta q^{n+1}-1)} \quad (4.4)$$

```
> eq6d:=qMixRec2(Flqj,q,k,p(n),alpha,0,beta,0,1,1)
```

$$eq6d := p(n,\alpha,\beta) = -\frac{(\alpha\beta q^{1+2n}-1)\,(\alpha\beta q^{1+2n}-\alpha\beta q^{n+1}-\beta q^{n+1}+1)\,p(n,\alpha,q\beta)}{(\beta q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)} + \frac{q^{2n}\alpha\beta\,(\beta q^2x-1)\,(q^n-1)\,(\alpha q^n-1)\,p(n-1,\alpha,\beta q^2)}{(\beta q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)} \quad (4.5)$$

```
> eq6e1:=qMixRec2(Flqj,q,k,p(n),alpha,1,beta,1,0,1):
> eq6e2:=subs(_C1=1,eq6e1):
> eq6e:=lhs(eq6e2)=combine(map(qsimpcomb,rhs(eq6e2)),power)
```

$$eq6e := p(n,\alpha,q\beta) = p(n,\alpha q,\beta) + \frac{\alpha q^n\,(q^n-1)\,p(n-1,\alpha q,q\beta)}{\alpha\beta q^{1+2n}-1} \quad (4.6)$$

Theorem 12

```
> alpha:=3;beta:=-2;q:=0.2;n:=4;
```

$$\alpha := 3$$
$$\beta := -2$$
$$q := 0.2$$
$$n := 4 \quad (4.7)$$

```
> hyp1:= 0 < alpha*q and alpha*q<1 and  beta*q<1 and beta<0;
```

$hyp1 := true$ **(4.8)**

```
> xnlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta,q)),x)]);
```

$xnlqj$:= [0.00278360733572197, 0.0392116735187675, 0.199995243235831, 0.999999998935250] **(4.9)**

```
> ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha*q,beta,q)),x)]);
```

$ynlqj$:= [0.00683442474645475, 0.0399637043523054, 0.199999960739226, 0.999999999998274] **(4.10)**

```
> Ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha*q^2,beta,q)),x)]);
```

$Ynlqj$:= [0.00776111880160327, 0.0399984988909236, 0.199999999683965, 1.00000000000001] **(4.11)**

```
> znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta*q,q)),x)]);
```

$znlqj$:= [0.00278820448198090, 0.0392219969145403, 0.199995583924206, 0.999999999237493] **(4.12)**

```
> Znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta*q^2,q)),x)]);
```

$Znlqj$:= [0.00278912574192395, 0.0392240607067740, 0.199995651167449, 0.999999999293630] **(4.13)**

```
> xnm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta,q)),x)]);
```

$xnm1lqj$:= [0.0138618637075470, 0.195940049472153, 0.999973533048939] **(4.14)**

```
> ynm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q,beta,q)),x)])
```

```
;
```

ynm1lqj := [0.0341410964878237, 0.199812355199844, 0.999999780704093] **(4.15)**

```
> Ynm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q^2,beta,q)),x)
  ]);
```

Ynm1lqj := [0.0387983414837290, 0.199992232459703, 0.999999998233398] **(4.16)**

```
> znm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta*q,q)),x)])
  ;
```

znm1lqj := [0.0139759104767839, 0.196190304463930, 0.999980845765026] **(4.17)**

```
> tnm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q,beta*q,q)),x)
  ]);
```

tnm1lqj := [0.0342039042294398, 0.199825298537692, 0.999999842546738] **(4.18)**

```
> Znm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta*q^2,q)),x)
  ]);
```

Znm1lqj := [0.0139989463567174, 0.196240246716721, 0.999982217925692] **(4.19)**

(a)

```
> seq(evalb(xnlqj[i]<ynlqj[i] and ynlqj[i]<ynm1lqj[i] and ynm1lqj
  [i]<xnlqj[i+1] and xnlqj[i+1]<ynlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.20)**

(b)

```
> seq(evalb(xnlqj[i]<znlqj[i] and znlqj[i]<znm1lqj[i] and znm1lqj
  [i]<xnlqj[i+1] and xnlqj[i+1]<znlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.21)**

(c)

```
> seq(evalb(xnlqj[i]<ynlqj[i] and ynlqj[i]<Ynlqj[i] and Ynlqj[i]
  <Ynm1lqj[i] and Ynm1lqj[i]<xnlqj[i+1] and xnlqj[i+1]<ynlqj[i+1]
  and ynlqj[i+1]<Ynlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.22)**

(d)

```
> seq(evalb(xnlqj[i]<znlqj[i] and znlqj[i]<Znm1lqj[i] and Znm1lqj
  [i]<xnlqj[i+1] and xnlqj[i+1]<znlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.23)**

(e)

```
> seq(evalb(znlqj[i]<ynlqj[i] and ynlqj[i]<tnm1lqj[i] and tnm1lqj
  [i]<znlqj[i+1] and znlqj[i+1]<ynlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.24)**

Corollary 13

(a)

```
> seq(evalb(xnlqj[i]<xnm1lqj[i] and xnm1lqj[i]<ynm1lqj[i] and
  ynm1lqj[i]<xnlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.25)**

(b)

```
> seq(evalb(xnlqj[i]<xnm1lqj[i] and xnm1lqj[i]<znm1lqj[i] and
  znm1lqj[i]<xnlqj[i+1]),i=1..n-1 )
```

true, true, true **(4.26)**

(c)

```
> seq(evalb(xnlqj[i]<znlqj[i] and znlqj[i]<ynlqj[i] and ynlqj[i]
  <xnlqj[i+1] and xnlqj[i+1]<znlqj[i+1] and znlqj[i+1]<ynlqj[i+1]
  ),i=1..n-1 )
```

(4.27)

$$true, true, true \quad (4.27)$$

Let us take beta > 0

```
> alpha:=3;beta:=2;q:=0.25;n:=4;
```

$$\alpha := 3$$
$$\beta := 2$$
$$q := 0.25$$
$$n := 4 \quad (4.28)$$

```
> hyp1:= 0 < alpha*q and alpha*q<1 and  beta*q<1 and beta>0;
```

$$hyp1 := true \quad (4.29)$$

```
> xnlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta,q)),x)]);
```

$$xnlqj := [0.00316238988595498, 0.0595569232947782, 0.249961912897355, 0.999999988632735] \quad (4.30)$$

```
> ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha*q,beta,q)),x)]);
```

$$ynlqj := [0.0119897802632498, 0.0622751665870337, 0.249999371988303, 0.999999999955334] \quad (4.31)$$

```
> Ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha*q^2,beta,q)),x)]);
```

$$Ynlqj := [0.0146748425864762, 0.0624850136236804, 0.249999990056842, 0.999999999999821] \quad (4.32)$$

```
> znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta*q,q)),x)]);
```

$$znlqj := [0.00314771527970496, 0.0594865555270670, 0.249956829730581, 0.999999977444477] \quad (4.33)$$

```
> xnm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta,q)),x)]);
```

$$xnm1lqj := [0.0130103104888992, 0.239861445593766, 0.999939979858659] \quad (4.34)$$

```
> ynm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q,beta,q)),x)])
;
```

$$ynm1lqj := [0.0483498166496878, 0.249245479281853, 0.999999037930959] \quad (4.35)$$

```
> Ynm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q^2,beta,q)),x)
]);
```

$$Ynm1lqj := [0.0588245198879855, 0.249950102288371, 0.999999984873851] \quad (4.36)$$

```
> znm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta*q,q)),x)])
;
```

$$znm1lqj := [0.0127681648474317, 0.238782847032166, 0.999883839750643] \quad (4.37)$$

(a)

```
> seq(evalb(xnlqj[i]<ynlqj[i] and ynlqj[i]<ynm1lqj[i] and ynm1lqj
[i]<xnlqj[i+1] and xnlqj[i+1]<ynlqj[i+1]),i=1..n-1 )
```

$$true, true, true \quad (4.38)$$

(b)

```
> seq(evalb(znlqj[i]<xnlqj[i] and xnlqj[i]<znm1lqj[i] and znm1lqj
[i]<znlqj[i+1] and znlqj[i+1]<xnlqj[i+1]),i=1..n-1 )
```

$$true, true, true \quad (4.39)$$

(c)

```
> seq(evalb(xnlqj[i]<ynlqj[i] and ynlqj[i]<Ynlqj[i] and Ynlqj[i]
<Ynm1lqj[i] and Ynm1lqj[i]<xnlqj[i+1] and xnlqj[i+1]<ynlqj[i+1]
and ynlqj[i+1]<Ynlqj[i+1]),i=1..n-1 )
```

$$true, true, true \quad (4.40)$$

Corollary

(a)

```
> seq(evalb(xnlqj[i]<xnm1lqj[i] and xnm1lqj[i]<ynm1lqj[i] and
  ynm1lqj[i]<xnlqj[i+1]),i=1..n-1 )
```

$$true, true, true \quad (4.41)$$

(b)

```
> seq(evalb(xnlqj[i]<znm1lqj[i] and znm1lqj[i]<xnm1lqj[i] and
  xnm1lqj[i]<xnlqj[i+1]),i=1..n-1 )
```

$$true, true, true \quad (4.42)$$

Remark 14-i

We note that the results obtained by Gochhayat et al. work not for the family they defined but for the family given by

```
> LQJ:=(n,x,alpha,beta,q)->1/((-1)^n*q^(-binomial(n,2))*
  qpochhammer(alpha*beta*q^(n+1), q, n)/qpochhammer(alpha*q, q,
  n))*add(qphihyperterm([q^(-n),alpha*beta*q^(n+1)],[alpha*q],q,
  q*q^x,j),j=0..n);
```

$$LQJ := (n, x, \alpha, \beta, q) \rightarrow \left(qpochhammer(\alpha q, q, n)\ add\left(qphihyperterm\left(\left[q^{-n}, \alpha \beta q^{n+1}\right], [\alpha q], q, q q^{x}, j\right), j = 0..n\right)\right) \Big/ \left((-1)^{n} q^{-\mathrm{binomial}(n,2)}\ qpochhammer\left(\alpha \beta q^{n+1}, q, n\right)\right) \quad (4.43)$$

x is now q^x in the definition

```
> alpha:=3;beta:=-2;q:=0.2;n:=4;
```

$$\alpha := 3$$
$$\beta := -2$$
$$q := 0.2$$
$$n := 4 \quad (4.44)$$

```
> hyp1:= 0 < alpha*q and alpha*q<1 and  beta*q<1 and beta<0;
```

$$hyp1 := true \quad (4.45)$$

```
> xnlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta,q)),x)]);
```

$$xnlqj := [6.61566371924548\ 10^{-10}, 1.00001477789451, 2.01236764565406, 3.65593947067016] \quad (4.46)$$

```
> ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha*q,beta,q)),x)]);
```

$$ynlqj := [1.07242409705082\ 10^{-12}, 1.00000012197046, 2.00056404979275, 3.09784113866547] \quad (4.47)$$

```
> Ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha*q^2,beta,q)),x)]);
```

$$Ynlqj := [-6.21334934559609\ 10^{-15}, 1.00000000098182, 2.00002331772528, 3.01883579489364] \quad (4.48)$$

```
> znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta*q,q)),x)]);
```

$$znlqj := [4.73772237126873\ 10^{-10}, 1.00001371946229, 2.01220408614095, 3.65491417824767] \quad (4.49)$$

```
> Znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta*q^2,q)),x)]);
```

$$Znlqj := [4.38892357879883\ 10^{-10}, 1.00001351055483, 2.01217139345372, 3.65470891480848] \quad (4.50)$$

```
> xnm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta,q)),x)]);
```
$$xnm1lqj := [0.0000164450589317238, 1.01274272464125, 2.65845224259399] \quad (4.51)$$
```
> ynm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q,beta,q)),x)])
  ;
```
$$ynm1lqj := [1.36256222965252\ 10^{-7}, 1.00058322498877, 2.09840554807739] \quad (4.52)$$
```
> Ynm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q^2,beta,q)),x)
  ]);
```
$$Ynm1lqj := [1.09765153903244\ 10^{-9}, 1.00002413168932, 2.01895192938316] \quad (4.53)$$
```
> znm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta*q,q)),x)])
  ;
```
$$znm1lqj := [0.0000119013093147111, 1.01194966086290, 2.65336120062478] \quad (4.54)$$
```
> tnm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha*q,beta*q,q)),x)
  ]);
```
$$tnm1lqj := [9.78312199428900\ 10^{-8}, 1.00054297779033, 2.09726355803853] \quad (4.55)$$
```
> Znm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta*q^2,q)),x)
  ]);
```
$$Znm1lqj := [0.0000110487222113862, 1.01179151381526, 2.65233792459722] \quad (4.56)$$

Theorem 2 Gochhayat et al.
```
> seq((xnlqj[i]<znm1lqj[i] and znm1lqj[i]<xnm1lqj[i] and xnm1lqj
  [i]<xnlqj[i+1]),i=1..n-1 )
```
$$true, true, true \quad (4.57)$$

Theorem 4 Gochhayat et al.
```
> seq(evalb(znlqj[i]<xnlqj[i] and xnlqj[i]<Znm1lqj[i] and Znm1lqj
  [i]<znlqj[i+1] and znlqj[i+1]<xnlqj[i+1]),i=1..n-1 )
```
$$true, true, true \quad (4.58)$$

Theorem 6 Gochhayat et al.
```
> seq(evalb(znlqj[i]<ynm1lqj[i] and ynm1lqj[i]<znm1lqj[i] and
  znm1lqj[i]<znlqj[i+1]),i=1..n-1 )
```
$$true, true, true \quad (4.59)$$

Theorem 7 Gochhayat et al.
```
> seq(evalb(xnlqj[i]<ynm1lqj[i] and ynm1lqj[i]<xnm1lqj[i] and
  xnm1lqj[i]<xnlqj[i+1]),i=1..n-1 )
```
$$true, true, true \quad (4.60)$$

In conclusion, the definition of the little q-Jacobi polynomials in Gochhayat et al. should be as in Equation (4.43)
```
> unassign('alpha,beta,q,n')
```

q-Meixner

```
> QM:=(n,x,beta,gamma,q)->1/((-1)^n*q^(n^2)/gamma^n/qpochhammer
  (beta*q, q, n))*add(qphihyperterm([q^(-n),x],[beta*q],q,-q^
  (n+1)/gamma,j),j=0..n);
```
$$QM := (n, x, \beta, \gamma, q) \quad (5.1)$$
$$\rightarrow \frac{1}{(-1)^n q^{n^2}} \Big(\gamma^n\, qpochhammer(\beta q, q, n)\ add\Big(qphihyperterm\Big([q^{-n}, x], [\beta q], q,$$

$$-\frac{q^{n+1}}{\gamma}, j\Big), j = 0..n\Big)\Big)$$

```
> Fqm:=1/((-1)^n*q^(n^2)/gamma^n/qpochhammer(beta*q, q, n))*
  (qphihyperterm([q^(-n),x],[beta*q],q,-q^(n+1)/gamma,k)):
```

Proposition 15

```
> eq7a1:=qMixRec2(Fqm,q,k,M(n),beta,0,gamma,1,0,0):
> eq7a2:=subs(_C1=1,eq7a1):
> eq7a:=lhs(eq7a2)=combine(map(qsimpcomb,rhs(eq7a2)),power)
```

$$eq7a := M(n,\beta,\gamma q) = M(n,\beta,\gamma) + M(n-1,\beta,\gamma)\,(\beta q^n - 1)\,\gamma\,(q^n - 1)\,q^{1-2n} \quad (5.2)$$

```
> eq7b:=qMixRec2(Fqm,q,k,M(n),beta,0,gamma,2,0,0);
```

$$eq7b := M(n,\beta,\gamma q^2) = -\frac{(\beta\gamma q^{n+1} - \gamma\beta q - \gamma q - 1)\,q^n M(n,\beta,\gamma)}{\gamma q + q^n} + \frac{M(n-1,\beta,\gamma)\,(\beta q^n - 1)\,(\gamma\beta q + q^n x + \gamma q + 1)\,\gamma\,(q^n - 1)\,q^{-n+1}}{\gamma q + q^n} \quad (5.3)$$

```
> eq7c:=qMixRec1(Fqm,q,k,M(n),beta,1)
```

$$eq7c := M(n,\beta) = \frac{(\gamma\beta q + q^n x)\,q^{-n} M(n,q\beta)}{\beta\gamma q + x} - \frac{\beta\,(q^n + \gamma)\,\gamma M(n-1,q\beta)\,(q^n - 1)\,q^{-3n+2}}{\beta\gamma q + x} \quad (5.4)$$

```
> eq7d1:=qMixRec2(Fqm,q,k,M(n),beta,1,gamma,1,0,0):
> eq7d2:=subs(_C1=1,eq7d1):
> eq7d:=lhs(eq7d2)=combine(map(qsimpcomb,rhs(eq7d2)),power)
```

$$eq7d := M(n,\beta,\gamma q) = M(n,q\beta,\gamma) - \gamma\,(q^n - 1)\,M(n-1,q\beta,\gamma)\,q^{1-2n} \quad (5.5)$$

Theorem 16

```
> gam:=5;beta:=3;q:=0.01;n:=5;
```

$$gam := 5$$
$$\beta := 3$$
$$q := 0.01$$
$$n := 5 \quad (5.6)$$

```
> hyp1:= 0 < beta*q and beta*q<1 and  gam>0;
```

$$hyp1 := true \quad (5.7)$$

```
> xnqm:= sort([solve(qsimpcomb(QM(n,x,beta,gam,q)),x)]);
```

$$xnqm := [481.186622846061, 4.99811867696362\ 10^{6}, 4.99998613724130\ 10^{10}, 5.00049495890780\ 10^{14}, 5.05000494908515\ 10^{18}] \quad (5.8)$$

```
> ynqm:= sort([solve(qsimpcomb(QM(n,x,beta*q,gam,q)),x)]);
```

$$ynqm := [495.890688616056, 4.99958911813709\ 10^{6}, 5.00000084166755\ 10^{10}, 5.00049510596648\ 10^{14}, 5.05000495057028\ 10^{18}] \quad (5.9)$$

```
> Ynqm:= sort([solve(qsimpcomb(QM(n,x,beta*q^2,gam,q)),x)]);
```

$$Ynqm := [496.037729702919, 4.99960382254892\ 10^{6}, 5.00000098871177\ 10^{10}, 5.00049510743711\ 10^{14}, 5.05000495058508\ 10^{18}] \quad (5.10)$$

```
> znqm:= sort([solve(qsimpcomb(QM(n,x,beta,gam*q,q)),x)]);
```

$znqm := [5.79246647801432, 50079.2161849776, 5.00008416674918\ 10^{8}, 5.00049593929898\ 10^{12}, 5.05000495898613\ 10^{16}]$ **(5.11)**

```
> Znqm:= sort([solve(qsimpcomb(QM(n,x,beta,gam*q^2,q)),x)]);
```

$Znqm := [1.04006608754240, 598.821953423781, 5.00988711754259\ 10^{6}, 5.00059397841759\ 10^{10}, 5.05000594908319\ 10^{14}]$ **(5.12)**

```
> xnm1qm:= sort([solve(qsimpcomb(QM(n-1,x,beta,gam,q)),x)]);
```

$xnm1qm := [481.186627564105, 4.99812357659483\ 10^{6}, 5.00047633147525\ 10^{10}, 5.05000476096667\ 10^{14}]$ **(5.13)**

```
> ynm1qm:= sort([solve(qsimpcomb(QM(n-1,x,beta*q,gam,q)),x)]);
```

$ynm1qm := [495.890693477150, 4.99959401920976\ 10^{6}, 5.00049103734306\ 10^{10}, 5.05000490948121\ 10^{14}]$ **(5.14)**

```
> Ynm1qm:= sort([solve(qsimpcomb(QM(n-1,x,beta*q^2,gam,q)),x)]);
```

$Ynm1qm := [496.037734565449, 4.99960872363592\ 10^{6}, 5.00049118440177\ 10^{10}, 5.05000491096638\ 10^{14}]$ **(5.15)**

```
> znm1qm:= sort([solve(qsimpcomb(QM(n-1,x,beta,gam*q,q)),x)]);
```

$znm1qm := [5.79246653440939, 50079.2652773479, 5.00057437059382\ 10^{8}, 5.05000575106374\ 10^{12}]$ **(5.16)**

```
> Znm1qm:= sort([solve(qsimpcomb(QM(n-1,x,beta,gam*q^2,q)),x)]);
```

$Znm1qm := [1.04006609581947, 598.822540221044, 5.01037828243655\ 10^{6}, 5.05010476077036\ 10^{10}]$ **(5.17)**

(a)

```
> seq(evalb(znqm[i]<xnqm[i] and xnqm[i]<xnm1qm[i] and xnm1qm[i]
  <znqm[i+1] and znqm[i+1]<xnqm[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* **(5.18)**

(b)

```
> seq(evalb(Znqm[i]<xnqm[i] and xnqm[i]<xnm1qm[i] and xnm1qm[i]
  <Znqm[i+1] and Znqm[i+1]<xnqm[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* **(5.19)**

(c)

```
> seq(evalb(znqm[i]<xnqm[i] and xnqm[i]<ynqm[i] and ynqm[i]
  <ynm1qm[i] and ynm1qm[i]<znqm[i+1] and znqm[i+1]<xnqm[i+1] and
  xnqm[i+1]<ynqm[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* **(5.20)**

Corollary 17

(a)

```
> seq(evalb(znqm[i]<znm1qm[i] and znm1qm[i]<xnm1qm[i] and xnm1qm
  [i]<ynm1qm[i] and ynm1qm[i]<znqm[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* **(5.21)**

(b)

```
> seq(evalb(Znqm[i]<Znm1qm[i] and Znm1qm[i]<xnm1qm[i] and xnm1qm
  [i]<Znqm[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* **(5.22)**

(c)

```
> seq(evalb(znqm[i]<xnqm[i] and xnqm[i]<xnm1qm[i] and xnm1qm[i]
  <ynm1qm[i] and ynm1qm[i]<znqm[i+1] and znqm[i+1]<xnqm[i+1]),i=
  1..n-1 )
```

true, true, true, true **(5.23)**

```
> unassign('gam,beta,q,n')
```

quantum q-Krawtchouk (see Remark 10-ii)

```
> QQK:=(n,x,p,N,q)->1/(p^n*q^(n^2)/qpochhammer(q^(-N), q, n))*add
  (qphihyperterm([q^(-n),x],[q^(-N)],q,p*q^(n+1),j),j=0..n);
```

$$QQK := (n, x, p, N, q) \rightarrow \frac{1}{p^n q^{n^2}} \left(qpochhammer\left(q^{-N}, q, n\right) add\left(qphihyperterm\left(\left[q^{-n}, x\right], \left[q^{-N}\right], q, p\, q^{n+1}, j\right), j=0..n\right)\right) \quad \textbf{(6.1)}$$

```
> Fqqk:=1/(p^n*q^(n^2)/qpochhammer(q^(-NN), q, n))*(qphihyperterm
  ([q^(-n),x],[q^(-NN)],q,p*q^(n+1),k)):
```

Proposition

```
> eq81:=qMixRec1(Fqqk,q,k,K(n),p,1):
> eq82:=subs({_C1=1,NN=N},eq81):
> eq8:=lhs(eq82)=combine(map(qsimpcomb,rhs(eq82)),power)
```

$$eq8 := K(n, p) = K(n, p\, q) - \frac{\left(-q^{N+1} + q^n\right) K(n-1, p\, q) \left(q^n - 1\right) q^{-2n-N-1}}{p} \quad \textbf{(6.2)}$$

Theorem

```
> p:=7;N:=15;q:=0.9;n:=10;
```

$$p := 7$$
$$N := 15$$
$$q := 0.9$$
$$n := 10 \quad \textbf{(6.3)}$$

```
> hyp1:= p>q^(-N) and n<=N;
```

$hyp1 := true$ **(6.4)**

```
> xnqqk:= sort([solve(qsimpcomb(QQK(n,x,p,N,q)),x)]);
```

xnqqk := [1.05855833485928, 1.22885550620686, 1.42214494893254, 1.64516329516691, 1.90520740225176, 2.21137524454909, 2.57584252258899, 3.01606958616932, 3.55982193097764, 4.26164875558221] **(6.5)**

```
> ynqqk:= sort([solve(qsimpcomb(QQK(n,x,p*q,N,q)),x)]);
```

ynqqk := [1.08847724420947, 1.26872664618378, 1.47040854325419, 1.70166300191046, 1.97016161254304, 2.28505065719231, 2.65827849017821, 3.10659923236205, 3.65586179565906, 4.35423137158672] **(6.6)**

```
> xnm1qqk:= sort([solve(qsimpcomb(QQK(n-1,x,p,N,q)),x)]);
```

xnm1qqk := [1.07775855054121, 1.26190292375868, 1.47154934374926, 1.71561169303587, 2.00387992789383, 2.34928083984289, 2.77075390195520, 3.29980639646173, 4.00360770528275] **(6.7)**

```
> ynm1qqk:= sort([solve(qsimpcomb(QQK(n-1,x,p*q,N,q)),x)]);
```

$ynm1qqk := [1.11089433729327, 1.30524494290594, 1.52385273759608, 1.77694863428410, 2.07468496011171, 2.43002246836809, 2.86157645443847, 3.39969225257442, 4.10718762351627]$ **(6.8)**

```
> seq(evalb(xnqqk[i]<ynqqk[i] and ynqqk[i]<ynm1qqk[i] and ynm1qqk[i]<xnqqk[i+1] and xnqqk[i+1]<ynqqk[i+1]),i=1..n-1 )
```

true, true, true, true, true, true, true, true, true **(6.9)**

Corollary

```
> seq(evalb(xnqqk[i]<xnm1qqk[i] and xnm1qqk[i]<ynm1qqk[i] and ynm1qqk[i]<xnqqk[i+1]),i=1..n-1 )
```

true, true, true, true, true, true, true, true, true **(6.10)**

```
> unassign('p,N,q,n')
```

q-Krawtchouk

```
> QK:=(n,x,p,N,q)->1/(qpochhammer(-p*q^n, q, n)/qpochhammer(q^(-N), q, n))*add(qphihyperterm([q^(-n),x,-p*q^n],[q^(-N),0],q,q,j),j=0..n);
```

$$QK := (n, x, p, N, q) \rightarrow \frac{1}{qpochhammer(-p\,q^n, q, n)} \left(qpochhammer(q^{-N}, q, n)\right) add\left(qphihyperterm([q^{-n}, x, -p\,q^n], [q^{-N}, 0], q, q, j), j = 0..n\right) \tag{7.1}$$

```
> Fqk:=1/(qpochhammer(-p*q^n, q, n)/qpochhammer(q^(-NN), q, n))*(qphihyperterm([q^(-n),x,-p*q^n],[q^(-NN),0],q,q,k)):
```

Proposition 19

```
> eq9a1:=qMixRec1(Fqk,q,k,K(n),p,1):
> eq9a2:=subs({_C1=1,NN=N},eq9a1):
> eq9a:=lhs(eq9a2)=combine(map(qsimpcomb,rhs(eq9a2)),power)
```

$$eq9a := K(n, p) = K(n, p\,q) - \frac{p\,(-q^{N+1} + q^n)\,K(n-1, p\,q)\,(q^n - 1)\,q^{n-N}}{(p\,q^{2n} + q)\,(1 + p\,q^{2n})} \tag{7.2}$$

```
> eq9b:=subs(NN=N,qMixRec1(Fqk,q,k,K(n),p,2))
```

$$eq9b := K(n, p) = \frac{(1 + p\,q^{2n})\,(p\,q^{N+1} + 1)\,K(n, p\,q^2)}{(p\,q^n + 1)\,(p\,q^{n+N+1} + 1)} - \left(p\,(q^n - 1)\,(-q^{N+1} + q^n)\,(p\,q^{2n+N+1}x + p\,q^{n+N+1} + x\,q^N + q^n)\,K(n-1, p\,q^2)\,q^{-N}\right) \Big/ \left((p\,q^n + 1)\,(p\,q^{n+N+1} + 1)\,(1 + p\,q^{1+2n})\right) \tag{7.3}$$

Theorem 20

```
> p:=7;N:=15;q:=0.9;n:=10;
```

$p := 7$

$N := 15$

$q := 0.9$

$n := 10$ **(7.4)**

```
> hyp1:= p>0 and n<=N;
```

$hyp1 := true$ **(7.5)**

```
> xnqk:= sort([solve(qsimpcomb(QK(n,x,p,N,q)),x)]);
```
xnqk := [1.02241534426124, 1.18912392803786, 1.40050453558640, 1.66052005308520, 1.97620812442526, 2.35647025630761, 2.81124348517741, 3.35010030649856, 3.97848867846124, 4.68517171621682] **(7.6)**

```
> ynqk:= sort([solve(qsimpcomb(QK(n,x,p*q,N,q)),x)]);
```
ynqk := [1.02918032551211, 1.20470837259552, 1.42513363099841, 1.69441338526835, 2.01939608205169, 2.40837771809778, 2.87007499236576, 3.41182478563571, 4.03501531007680, 4.72131042006417] **(7.7)**

```
> Ynqk:= sort([solve(qsimpcomb(QK(n,x,p*q^2,N,q)),x)]);
```
Ynqk := [1.03719446001906, 1.22190988559946, 1.45146507706954, 1.72992335517728, 2.06389277153883, 2.46097208050341, 2.92854749538212, 3.47162440447511, 4.08759609791331, 4.75221503006528] **(7.8)**

```
> znqk:= sort([solve(qsimpcomb(QK(n,x,p,N+1,q)),x)]);
```
znqk := [1.03689008985937, 1.22327697668543, 1.45751979560348, 1.74509320326102, 2.09479565620602, 2.51751924115180, 3.02560394433899, 3.63164822313605, 4.34503893476018, 5.15946280703187] **(7.9)**

```
> Znqk:= sort([solve(qsimpcomb(QK(n,x,p,N+2,q)),x)]);
```
Znqk := [1.05606373059752, 1.26411512710617, 1.52313127336623, 1.84065598627673, 2.22743770821656, 2.69653487640336, 3.26290558329242, 3.94245715640273, 4.74893553368193, 5.68194794555356] **(7.10)**

```
> xnm1qk:= sort([solve(qsimpcomb(QK(n-1,x,p,N,q)),x)]);
```
xnm1qk := [1.03417067082768, 1.21876231333393, 1.45323224156902, 1.74402000859400, 2.10156952315693, 2.53937055806088, 3.07373135249781, 3.72311879940881, 4.50417404766329] **(7.11)**

```
> ynm1qk:= sort([solve(qsimpcomb(QK(n-1,x,p*q,N,q)),x)]);
```
ynm1qk := [1.04320302524019, 1.23790829739922, 1.48269761884854, 1.78423971231393, 2.15275442838750, 2.60089929072753, 3.14313749848121, 3.79402796330771, 4.56097646027617] **(7.12)**

```
> Ynm1qk:= sort([solve(qsimpcomb(QK(n-1,x,p*q^2,N,q)),x)]);
```
Ynm1qk := [1.05368713652632, 1.25886628071143, 1.51410305469643, 1.82634436190665, 2.20549804959239, 2.66324980750439, 3.21203561946868, 3.86232099901248, 4.61249538424057] **(7.13)**

```
> znm1qk:= sort([solve(qsimpcomb(QK(n-1,x,p,N+1,q)),x)]);
```
znm1qk := [1.05191447098257, 1.25762807087498, 1.51663914209278, 1.83752677067053, 2.23296846619113, 2.71914392940747, 3.31594439597285, 4.04709890188181, 4.93854122610044] **(7.14)**

```
> Znm1qk:= sort([solve(qsimpcomb(QK(n-1,x,p,N+2,q)),x)]);
```
Znm1qk := [1.07442010436772, 1.30317383637925, 1.58883038560556, 1.94252490699319, 2.37937830828049, 2.91850405675391, 3.58370987518796, 4.40449269179586, 5.41709994110206] **(7.15)**

(a)

```
> seq(evalb(xnqk[i]<ynqk[i] and ynqk[i]<ynm1qk[i] and ynm1qk[i]
  <xnqk[i+1] and xnqk[i+1]<ynqk[i+1]),i=1..n-1 )
```
true, *true*, *true*, *true*, *true*, *true*, *true*, *true*, *true* **(7.16)**

(b)

```
> seq(evalb(xnqk[i]<Ynqk[i] and Ynqk[i]<Ynm1qk[i] and Ynm1qk[i]
  <xnqk[i+1] and xnqk[i+1]<Ynqk[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \quad (7.17)$$

Corollary 21

(a)

```
> seq(evalb(xnqk[i]<xnm1qk[i] and xnm1qk[i]<ynm1qk[i] and ynm1qk
  [i]<xnqk[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \quad (7.18)$$

(b)

```
> seq(evalb(xnqk[i]<ynqk[i] and ynqk[i]<Ynqk[i] and Ynqk[i]
  <Ynm1qk[i] and Ynm1qk[i]<xnqk[i+1] and xnqk[i+1]<ynqk[i+1] and
  ynqk[i+1]<Ynqk[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \quad (7.19)$$

```
> unassign('p,N,q,n')
```

Affine q-Krawtchouk (see Remark 10-i)

```
> AQK:=(n,x,p,N,q)->1/(1/(qpochhammer(p*q, q, n)*qpochhammer(q^(-
  N), q, n)))*add(qphihyperterm([q^(-n),x,0],[p*q,q^(-N)],q,q,j),
  j=0..n);
```

$$AQK := (n, x, p, N, q) \rightarrow qpochhammer(p\, q, q, n)\; qpochhammer(q^{-N}, q, n)\; add(qphihyperterm([q^{-n}, x, 0], [p\, q, q^{-N}], q, q, j), j = 0..n) \quad (8.1)$$

```
> Faqk:=1/(1/(qpochhammer(p*q, q, n)*qpochhammer(q^(-NN), q, n)))
  *(qphihyperterm([q^(-n),x,0],[p*q,q^(-NN)],q,q,k)):
```

Proposition

```
> eq10a1:=qMixRec1(Faqk,q,k,K(n),p,1):
> eq10a2:=subs({_C1=1,NN=N},eq10a1):
> eq10a:=lhs(eq10a2)=combine(map(qsimpcomb,rhs(eq10a2)),power)
```

$$eq10a := K(n, p) = K(n, p\, q) - p\,(q^n - 1)\,(-q^{N+1} + q^n)\, K(n-1, p\, q)\, q^{-N} \quad (8.2)$$

```
> eq10b1:=qMixRec2(subs(q^(-NN)=alpha,Faqk),q,k,K(n),p,0,alpha,
  -1,0,0):
> eq10b2:=subs({_C1=1,alpha=q^(-N)},eq10b1):
> eq10b:=lhs(eq10b2)=combine(map(qsimpcomb,rhs(eq10b2)),power)
```

$$eq10b := K\left(n, p, \frac{q^{-N}}{q}\right) = K(n, p, q^{-N}) - K(n-1, p, q^{-N})\,(p\, q^n - 1)\,(q^n - 1)\, q^{-N-1} \quad (8.3)$$

Note that in the above relation, the shift is on N which appears in Fqak as a power of q, that is why we did the substitution subs(q^(-NN)=alpha,Faqk)

```
> eq10c:=subs(NN=N,qMixRec2(Faqk,q,k,K(n),NN,0,p,0,1,1))
```

$$eq10c := K(n, N, p) = \frac{(p\, q^{n+N+2} - p\, q^{N+2} + p\, q^{n+1} - q^n)\, q^{-n} K(n, N, p\, q)}{p\, q^{n+1} - 1} - \frac{p\, K(n-1, N, p\, q^2)\,(-q^{N+1} + q^n)\,(p\, q^2 - x)\,(q^n - 1)\, q^{-n+1}}{p\, q^{n+1} - 1} \quad (8.4)$$

Theorem

```
> p:=1;N:=10;q:=0.9;n:=9;
```

$$p := 1$$
$$N := 10$$
$$q := 0.9$$
$$n := 9 \tag{8.5}$$

```
> hyp1:= 0<p*q and p*q<1 and n<=N;
```

$$hyp1 := true \tag{8.6}$$

```
> xnaqk:= sort([solve(qsimpcomb(AQK(n,x,p,N,q)),x)]);
```

xnaqk := [1.00018930062079, 1.11666770448010, 1.26846276746177, 1.46872135338665, 1.71333259629191, 1.98946833507715, 2.28012832302565, 2.57124565755309, 2.86716934680518] **(8.7)**

```
> ynaqk:= sort([solve(qsimpcomb(AQK(n,x,p*q,N,q)),x)]);
```

ynaqk := [1.00094474604262, 1.12384297881846, 1.28742833723485, 1.49654590297421, 1.74333569225585, 2.01450712981827, 2.29474239300883, 2.57574347153419, 2.86761707422617] **(8.8)**

```
> Ynaqk:= sort([solve(qsimpcomb(AQK(n,x,p*q^2,N,q)),x)]);
```

Ynaqk := [1.00263759408670, 1.13356611195619, 1.30740925719718, 1.52261809786614, 1.76937245031484, 2.03445795533731, 2.30492141281513, 2.57829678951076, 2.86781816391920] **(8.9)**

```
> znaqk:= sort([solve(qsimpcomb(AQK(n,x,p,N+1,q)),x)]);
```

znaqk := [1.00059958820093, 1.12417029554534, 1.29869903368088, 1.53496114896923, 1.82528254509913, 2.15447848710669, 2.50202282663187, 2.84643906013371, 3.18452904219032] **(8.10)**

```
> Znaqk:= sort([solve(qsimpcomb(AQK(n,x,p,N+2,q)),x)]);
```

Znaqk := [1.00140256944538, 1.13508208829548, 1.33584206956362, 1.61095108604711, 1.95010505127057, 2.33598739366433, 2.74491925598106, 3.14911725814915, 3.53643819164790] **(8.11)**

```
> xnm1aqk:= sort([solve(qsimpcomb(AQK(n-1,x,p,N,q)),x)]);
```

xnm1aqk := [1.00081027536242, 1.12724288929471, 1.30919012619358, 1.55522377526191, 1.85484522940237, 2.18899396401927, 2.53172566462801, 2.86053506907088] **(8.12)**

```
> ynm1aqk:= sort([solve(qsimpcomb(AQK(n-1,x,p*q,N,q)),x)]);
```

ynm1aqk := [1.00338660642622, 1.14326359570179, 1.34189425562623, 1.59766111660774, 1.89755915177239, 2.22242994874585, 2.54889988867428, 2.86401981991314] **(8.13)**

```
> Ynm1aqk:= sort([solve(qsimpcomb(AQK(n-1,x,p*q^2,N,q)),x)]);
```

Ynm1aqk := [1.00819502322163, 1.16238818562213, 1.37431275438417, 1.63628069959274, 1.93404270904362, 2.24879219442135, 2.56066777325803, 2.86592769513510] **(8.14)**

```
> znm1aqk:= sort([solve(qsimpcomb(AQK(n-1,x,p,N+1,q)),x)]);
```

znm1aqk := [1.00181836611801, 1.13979760825616, 1.35013826888443, 1.63761275906207, 1.98903506886262, 2.38274867057775, 2.78821275116762, 3.17284482066370] **(8.15)**

```
> Znm1aqk:= sort([solve(qsimpcomb(AQK(n-1,x,p,N+2,q)),x)]);
```
$Znm1aqk := [1.00340868888532, 1.15582773415505, 1.39776361762036, 1.73032924089955, 2.13798841185532, 2.59640271748539, 3.07058565920914, 3.51839259943694]$ **(8.16)**

(a)

```
> seq(evalb(xnaqk[i]<ynaqk[i] and ynaqk[i]<ynm1aqk[i] and ynm1aqk[i]<xnaqk[i+1] and xnaqk[i+1]<ynaqk[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true **(8.17)**

(b)

```
> seq(evalb(xnaqk[i]<znaqk[i] and znaqk[i]<xnm1aqk[i] and xnm1aqk[i]<xnaqk[i+1] and xnaqk[i+1]<znaqk[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true **(8.18)**

(c)

```
> seq(evalb(xnaqk[i]<ynaqk[i] and ynaqk[i]<Ynm1aqk[i] and Ynm1aqk[i]<xnaqk[i+1] and xnaqk[i+1]<ynaqk[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true **(8.19)**

Corollary

(a)

```
> seq(evalb(znaqk[i]<xnm1aqk[i] and xnm1aqk[i]<znm1aqk[i] and znm1aqk[i]<znaqk[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true **(8.20)**

(b)

```
> seq(evalb(xnaqk[i]<xnm1aqk[i] and xnm1aqk[i]<ynm1aqk[i] and ynm1aqk[i]<xnaqk[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true **(8.21)**

```
> unassign('p,N,q,n')
```

little q-Laguerre / Wall (see Remark 14-ii)

```
> LQLW:=(n,x,alpha,q)->1/((-1)^n*q^(-binomial(n, 2))/qpochhammer(alpha*q, q, n))*add(qphihyperterm([q^(-n),0],[alpha*q],q,q*x,j),j=0..n);
```
$$LQLW := (n, x, \alpha, q) \rightarrow \frac{qpochhammer(\alpha q, q, n)\ add(qphihyperterm([q^{-n}, 0], [\alpha q], q, q x, j), j = 0..n)}{(-1)^n q^{-\mathrm{binomial}(n, 2)}}$$ **(9.1)**

```
> Flqlw:=1/((-1)^n*q^(-binomial(n, 2))/qpochhammer(alpha*q, q, n))*(qphihyperterm([q^(-n),0],[alpha*q],q,q*x,k)):
```

Proposition

```
> eq11a1:=qMixRec1(Flqlw,q,k,p(n),alpha,1):
> eq11a2:=subs(_C1=1,eq11a1):
> eq11a:=lhs(eq11a2)=combine(map(qsimpcomb,rhs(eq11a2)),power)
```
$$eq11a := p(n, \alpha) = p(n, \alpha q) - q^n \alpha (q^n - 1)\ p(n-1, \alpha q)$$ **(9.2)**

```
> eq11b:=qMixRec1(Flqlw,q,k,p(n),alpha,2)
```
(9.3)

$$eq11b := p(n, \alpha) = \frac{(\alpha q - 1)\, p(n, \alpha q^2)}{\alpha q^{n+1} - 1} - \frac{\alpha q\, (q^n - 1)\, (\alpha q^{n+1} - q^n - x)\, p(n-1, \alpha q^2)}{\alpha q^{n+1} - 1} \quad \textbf{(9.3)}$$

Theorem

```
> alpha:=0.8;q:=0.9;n:=5;
```

$$\alpha := 0.8$$
$$q := 0.9$$
$$n := 5 \quad \textbf{(9.4)}$$

```
> hyp1:= 0<alpha*q and alpha*q<1;
```

$$hyp1 := true \quad \textbf{(9.5)}$$

```
> xnlqlw:= sort([solve(qsimpcomb(LQLW(n,x,alpha,q)),x)]);
```

$xnlqlw :=$ [0.0817276057390994, 0.212210080211859, 0.391962468297108, 0.612846706987492, 0.861860659564460] **(9.6)**

```
> ynlqlw:= sort([solve(qsimpcomb(LQLW(n,x,alpha*q,q)),x)]);
```

$ynlqlw :=$ [0.112476212325451, 0.255238302503982, 0.439279456838318, 0.656017164342872, 0.891045632709406] **(9.7)**

```
> Ynlqlw:= sort([solve(qsimpcomb(LQLW(n,x,alpha*q^2,q)),x)]);
```

$Ynlqlw :=$ [0.143298216216850, 0.295477696830761, 0.481675029692871, 0.693156709510878, 0.914553439596651] **(9.8)**

```
> xnm1lqlw:= sort([solve(qsimpcomb(LQLW(n-1,x,alpha,q)),x)]);
```

$xnm1lqlw :=$ [0.102397916312504, 0.266124445200725, 0.492414808876872, 0.773000509609921] **(9.9)**

```
> ynm1lqlw:= sort([solve(qsimpcomb(LQLW(n-1,x,alpha*q,q)),x)]);
```

$ynm1lqlw :=$ [0.139355037608029, 0.316309329114693, 0.544628849025715, 0.814150696251568] **(9.10)**

```
> Ynm1lqlw:= sort([solve(qsimpcomb(LQLW(n-1,x,alpha*q^2,q)),x)]);
```

$Ynm1lqlw :=$ [0.175855790581066, 0.362454972389906, 0.590401232057860, 0.848187525771178] **(9.11)**

(a)

```
> seq(evalb(xnlqlw[i]<ynlqlw[i] and ynlqlw[i]<ynm1lqlw[i] and
  ynm1lqlw[i]<xnlqlw[i+1] and xnlqlw[i+1]<ynlqlw[i+1]),i=1..n-1 )
```

true, true, true, true **(9.12)**

(b)

```
> seq(evalb(xnlqlw[i]<Ynlqlw[i] and Ynlqlw[i]<Ynm1lqlw[i] and
  Ynm1lqlw[i]<xnlqlw[i+1] and xnlqlw[i+1]<Ynlqlw[i+1]),i=1..n-1 )
```

true, true, true, true **(9.13)**

Corollary

(a)

```
> seq(evalb(xnlqlw[i]<xnm1lqlw[i] and xnm1lqlw[i]<ynm1lqlw[i] and
  ynm1lqlw[i]<xnlqlw[i+1]),i=1..n-1 )
```

true, true, true, true **(9.14)**

(b)

```
> seq(evalb(xnlqlw[i]<ynlqlw[i] and ynlqlw[i]<Ynlqlw[i] and
  Ynlqlw[i]<Ynm1lqlw[i] and Ynm1lqlw[i]<xnlqlw[i+1] and xnlqlw
  [i+1]<ynlqlw[i+1] and ynlqlw[i+1]<Ynlqlw[i+1]),i=1..n-1 )
```

$$true, true, true, true \tag{9.15}$$

(c)

```
> seq(evalb(xnlqlw[i]<xnm1lqlw[i] and xnm1lqlw[i]<Ynm1lqlw[i] and
  Ynm1lqlw[i]<xnlqlw[i+1]),i=1..n-1 )
```

$$true, true, true, true \tag{9.16}$$

```
> unassign('alpha,q,n')
```

q-Laguerre

```
> QL:=(n,x,alpha,q)->1/((-1)^n*q^(n*(n+alpha))/qpochhammer(q,q,n)
  )*qpochhammer(q^(alpha+1),q,n)/qpochhammer(q,q,n)*add
  (qphihyperterm([q^(-n)],[q^(alpha+1)],q,-q^(n+alpha+1)*x,j) ,j=
  0..n);
```

$$QL := (n, x, \alpha, q) \rightarrow \frac{1}{(-1)^n q^{n(n+\alpha)}\ qpochhammer(q, q, n)} \Big(1\ qpochhammer(q, q, n)\ qpochhammer\big(q^{\alpha+1}, q, n\big)\ add\Big(qphihyperterm\big([q^{-n}], [q^{\alpha+1}], q, -q^{n+\alpha+1} x, j\big), j = 0..n\Big)\Big) \tag{10.1}$$

```
> Fql:=1/((-1)^n*q^(n^2)*(q^(alpha))^n/qpochhammer(q,q,n))*
  qpochhammer(q^(alpha)*q,q,n)/qpochhammer(q,q,n)*(qphihyperterm(
  [q^(-n)],[q^(alpha)*q],q,-q^(n+1)*q^alpha*x,k))
```

$$Fql := \left(qpochhammer\big(q^{\alpha} q, q, n\big)\ qpochhammer\big(q^{-n}, q, k\big)\ \big(-q^{n+1} q^{\alpha} x\big)^k (-1)^k q^{\frac{1}{2} k(k-1)}\right) \Big/ \left((-1)^n q^{n^2} \big(q^{\alpha}\big)^n\ qpochhammer\big(q^{\alpha} q, q, k\big)\ qpochhammer(q, q, k)\right) \tag{10.2}$$

Proposition 22

```
> eq12a1:=qMixRec1(subs(q^alpha=alpha,Fql),q,k,L(n),alpha,1):
> eq12a2:=subs({_C1=1,alpha=q^alpha},eq12a1):
> eq12a:=lhs(eq12a2)=combine(map(qsimpcomb,rhs(eq12a2)),power)
```

$$eq12a := L\big(n, q^{\alpha}\big) = L\big(n, q^{\alpha+1}\big) - L\big(n-1, q^{\alpha+1}\big)\ \big(q^n - 1\big)\ q^{-2n-\alpha} \tag{10.3}$$

Note that we did the substitution subs(q^alpha=alpha,Fql) since alpha is a power of q.

```
> eq12b:=combine(subs(alpha=q^alpha, qMixRec1(subs(q^alpha=alpha,
  Fql),q,k,L(n),alpha,2)), power)
```

$$eq12b := L\big(n, q^{\alpha}\big) = \frac{q^n \big(q^{\alpha+1} - 1\big)\ L\big(n, q^{\alpha+2}\big)}{q^{\alpha+n+1} - 1} + \frac{\big(q^{\alpha+n+1} x - q^{\alpha+1} + 1\big)\ \big(q^n - 1\big)\ q^{-\alpha-n-1}\ L\big(n-1, q^{\alpha+2}\big)}{q^{\alpha+n+1} - 1} \tag{10.4}$$

Theorem 23

```
> alpha:=5;q:=0.15;n:=10;
```

$$\alpha := 5$$
$$q := 0.15$$
$$n := 10 \tag{10.5}$$

```
> hyp1:= evalb(-1<alpha);
```
$$hyp1 := true \tag{10.6}$$

```
> xnql:= sort([solve(qsimpcomb(QL(n,x,alpha,q)),x)]);
```
$$xnql := [76141.2898543456, 3.82419831386058\ 10^{6}, 1.72897700949113\ 10^{8}, 7.70390735446771\ 10^{9}, 3.42529183431091\ 10^{11}, 1.52252575036698\ 10^{13}, 6.76941535367995\ 10^{14}, 3.01628872829097\ 10^{16}, 1.36370726176920\ 10^{18}, 6.84915452407741\ 10^{19}] \tag{10.7}$$

```
> ynql:= sort([solve(qsimpcomb(QL(n,x,alpha+1,q)),x)]);
```
$$ynql := [5.07613600549379\ 10^{5}, 2.54946888337357\ 10^{7}, 1.15265156238950\ 10^{9}, 5.13593838479805\ 10^{10}, 2.28352789943973\ 10^{12}, 1.01501716757130\ 10^{14}, 4.51294356956009\ 10^{15}, 2.01085915222341\ 10^{17}, 9.09138174514796\ 10^{18}, 4.56610301605313\ 10^{20}] \tag{10.8}$$

```
> Ynql:= sort([solve(qsimpcomb(QL(n,x,alpha+2,q)),x)]);
```
$$Ynql := [3.38409567185048\ 10^{6}, 1.69964625632902\ 10^{8}, 7.68434397199207\ 10^{9}, 3.42395893804732\ 10^{11}, 1.52235193394973\ 10^{13}, 6.76678111780207\ 10^{14}, 3.00862904641740\ 10^{16}, 1.34057276815187\ 10^{18}, 6.06092116343401\ 10^{19}, 3.04406867736886\ 10^{21}] \tag{10.9}$$

```
> xnm1ql:= sort([solve(qsimpcomb(QL(n-1,x,alpha,q)),x)]);
```
$$xnm1ql := [76141.2920503621, 3.82419905058239\ 10^{6}, 1.72897923013364\ 10^{8}, 7.70397331943079\ 10^{9}, 3.42548737331163\ 10^{11}, 1.52310541194688\ 10^{13}, 6.78664092144924\ 10^{14}, 3.06834074782797\ 10^{16}, 1.54105972346788\ 10^{18}] \tag{10.10}$$

```
> ynm1ql:= sort([solve(qsimpcomb(QL(n-1,x,alpha+1,q)),x)]);
```
$$ynm1ql := [5.07613615189613\ 10^{5}, 2.54946937452208\ 10^{7}, 1.15265304281814\ 10^{9}, 5.13598236144122\ 10^{10}, 2.28365825877416\ 10^{12}, 1.01540360862482\ 10^{14}, 4.52442728140737\ 10^{15}, 2.04556049858192\ 10^{17}, 1.02737314898084\ 10^{19}] \tag{10.11}$$

```
> Ynm1ql:= sort([solve(qsimpcomb(QL(n-1,x,alpha+2,q)),x)]);
```
$$Ynm1ql := [3.38409576945216\ 10^{6}, 1.69964658376143\ 10^{8}, 7.68435384151665\ 10^{9}, 3.42398825580955\ 10^{11}, 1.52243884017275\ 10^{13}, 6.76935739149235\ 10^{14}, 3.01628485431572\ 10^{16}, 1.36370699905760\ 10^{18}, 6.84915432654118\ 10^{19}] \tag{10.12}$$

(a)

```
> seq(evalb(xnql[i]<ynql[i] and ynql[i]<ynm1ql[i] and ynm1ql[i]
  <xnql[i+1] and xnql[i+1]<ynql[i+1]),i=1..n-1 )
```
$$true, true, true, true, true, true, true, true, true \tag{10.13}$$

(b)

```
> seq(evalb(xnql[i]<Ynql[i] and Ynql[i]<Ynm1ql[i] and Ynm1ql[i]
  <xnql[i+1] and xnql[i+1]<Ynql[i+1]),i=1..n-1 )
```
$$true, true, true, true, true, true, true, true, true \tag{10.14}$$

Corollary 24

(a)

```
> seq(evalb(xnql[i]<xnm1ql[i] and xnm1ql[i]<ynm1ql[i] and ynm1ql
```

```
[i]<xnql[i+1]),i=1..n-1 )
```

true, true, true, true, true, true, true, true, true **(10.15)**

(b)

```
> seq(evalb(xnql[i]<ynql[i] and ynql[i]<Ynql[i] and Ynql[i]
  <Ynm1ql[i] and Ynm1ql[i]<xnql[i+1] and xnql[i+1]<ynql[i+1] and
  ynql[i+1]<Ynql[i+1]),i=1..n-1 )
```

true, true, true, true, true, true, true, true, true **(10.16)**

```
> unassign('alpha,q,n')
```

alternative q-Charlier

```
> AQC:=(n,x,alpha,q)->1/((-1)^n*q^(-binomial(n,2))*qpochhammer(-
  alpha*q^n, q, n))*add(qphihyperterm([q^(-n),-alpha*q^(n)],[0],
  q,q*x,j),j=0..n);
```

$$AQC := (n, x, \alpha, q) \rightarrow \frac{add\left(qphihyperterm\left([q^{-n}, -\alpha q^{n}], [0], q, q\,x, j\right), j=0..n\right)}{(-1)^{n}\, q^{-\mathrm{binomial}(n,\,2)}\, qpochhammer\left(-\alpha q^{n}, q, n\right)} \quad \textbf{(11.1)}$$

```
> Faqc:=1/((-1)^n*q^(-binomial(n,2))*qpochhammer(-alpha*q^n, q,
  n))*(qphihyperterm([q^(-n),-alpha*q^(n)],[0],q,q*x,k)):
```

Proposition 26

```
> eq13a1:=qMixRec1(Faqc,q,k,y(n),alpha,1):
> eq13a2:=subs(_C1=0,eq13a1):
> eq13a:=lhs(eq13a2)=combine(map(qsimpcomb,rhs(eq13a2)),power)
```

$$eq13a := y(n, \alpha) = y(n, \alpha q) - \frac{\alpha q^{2n} (q^{n}-1)\, y(n-1, \alpha q)}{(\alpha q^{2n}+q)\,(1+\alpha q^{2n})} \quad \textbf{(11.2)}$$

```
> eq14a:=qMixRec1(Faqc,q,k,y(n),alpha,2)
```

$$eq14a := y(n, \alpha) = \frac{(1+\alpha q^{2n})\, y(n, \alpha q^{2})}{\alpha q^{n}+1} - \frac{q^{n} \alpha (q^{n}-1)\,(\alpha q^{1+2n} x + q^{n} + x)\, y(n-1, \alpha q^{2})}{(1+\alpha q^{1+2n})\,(\alpha q^{n}+1)} \quad \textbf{(11.3)}$$

Theorem 27

```
> alpha:=5;q:=0.45;n:=5;
```

$$\alpha := 5$$
$$q := 0.45$$
$$n := 5 \quad \textbf{(11.4)}$$

```
> hyp1:= evalb(0<alpha);
```

hyp1 := *true* **(11.5)**

```
> xnaqc:= sort([solve(qsimpcomb(AQC(n,x,alpha,q)),x)]);
```

xnaqc := [0.0353665676523125, 0.0900967157697371, 0.202442455600504, 0.449998985589962, 0.999999994959251] **(11.6)**

```
> znaqc:= sort([solve(qsimpcomb(AQC(n,x,alpha*q,q)),x)]);
```

znaqc := [0.0382201709341356, 0.0908837749812385, 0.202494123353513, 0.449999954174421, 0.999999999898184] **(11.7)**

```
> Znaqc:= sort([solve(qsimpcomb(AQC(n,x,alpha*q^2,q)),x)]);
```
$Znaqc := [0.0396930765094522, 0.0910725104186227, 0.202499435890089, 0.449999998035423, 0.999999999998020]$ **(11.8)**

```
> xnm1aqc:= sort([solve(qsimpcomb(AQC(n-1,x,alpha,q)),x)]);
```
$xnm1aqc := [0.0680756489950872, 0.194503075530361, 0.449092202944757, 0.999974590301318]$ **(11.9)**

```
> znm1aqc:= sort([solve(qsimpcomb(AQC(n-1,x,alpha*q,q)),x)]);
```
$znm1aqc := [0.0788251832549846, 0.200368282428927, 0.449895392760289, 0.999998729256557]$ **(11.10)**

```
> Znm1aqc:= sort([solve(qsimpcomb(AQC(n-1,x,alpha*q^2,q)),x)]);
```
$Znm1aqc := [0.0850618988568233, 0.202001828800380, 0.449989357550514, 0.999999942785226]$ **(11.11)**

(a)

```
> seq(evalb(xnaqc[i]<znaqc[i] and znaqc[i]<znm1aqc[i] and znm1aqc
  [i]<xnaqc[i+1] and xnaqc[i+1]<znaqc[i+1]),i=1..n-1 )
```
true, true, true, true **(11.12)**

(b)

```
> seq(evalb(xnaqc[i]<Znaqc[i] and Znaqc[i]<Znm1aqc[i] and Znm1aqc
  [i]<xnaqc[i+1] and xnaqc[i+1]<Znaqc[i+1]),i=1..n-1 )
```
true, true, true, true **(11.13)**

Corollary 28

(a)

```
> seq(evalb(xnaqc[i]<xnm1aqc[i] and xnm1aqc[i]<znm1aqc[i] and
  znm1aqc[i]<Znm1aqc[i] and Znm1aqc[i]<xnaqc[i+1]),i=1..n-1 )
```
true, true, true, true **(11.14)**

(b)

```
> seq(evalb(xnaqc[i]<znaqc[i] and znaqc[i]<Znaqc[i] and Znaqc[i]
  <Znm1aqc[i] and Znm1aqc[i]<xnaqc[i+1] and xnaqc[i+1]<znaqc[i+1]
  and znaqc[i+1]<Znaqc[i+1]),i=1..n-1 )
```
true, true, true, true **(11.15)**

```
> unassign('alpha,q,n')
```

q-Charlier (see Remark 18-ii)

```
> QC:=(n,x,alpha,q)->1/((-1)^n*q^(n^2)/alpha^n)*add(qphihyperterm
  ([q^(-n),x],[0],q,-q^(n+1)/alpha,j),j=0..n);
```
$$QC := (n, x, \alpha, q) \to \frac{\alpha^n \, add\left(qphihyperterm\left([q^{-n}, x], [0], q, -\frac{q^{n+1}}{\alpha}, j\right), j=0..n\right)}{(-1)^n q^{n^2}}$$ **(12.1)**

```
> Fqc:=1/((-1)^n*q^(n^2)/alpha^n)*(qphihyperterm([q^(-n),x],[0],
  q,-q^(n+1)/alpha,k)):
```

Proposition

```
> eq14a1:=qMixRec2(Fqc,q,k,C(n),a,0,alpha,1,0,0):
```

```
> eq14a2:=subs({_C1=1,NN=N},eq14a1):
> eq14a:=lhs(eq14a2)=combine(map(qsimpcomb,rhs(eq14a2)),power)
```

$$eq14a := C(n, a, \alpha q) = C(n, a, \alpha) - \alpha C(n-1, a, \alpha) (q^n - 1) q^{1-2n} \quad (12.2)$$

In (12.2) and (12.3), the variable a was given arbitrary to fit the number of parameters of qMixRec2

```
> eq14b:=qMixRec2(Fqc,q,k,C(n),a,0,alpha,2,0,0)
```

$$eq14b := C(n, a, \alpha q^2) = \frac{(\alpha q + 1) q^n C(n, a, \alpha)}{\alpha q + q^n} - \frac{(q^n x + \alpha q + 1) \alpha C(n-1, a, \alpha) (q^n - 1) q^{-n+1}}{\alpha q + q^n} \quad (12.3)$$

Theorem

```
> alpha:=0.15;q:=0.9;n:=10;
```

$$\alpha := 0.15$$
$$q := 0.9$$
$$n := 10 \quad (12.4)$$

```
> hyp1:= 0<alpha and alpha<q^(2*n)/(q*(1-q^n));
```

$$hyp1 := true \quad (12.5)$$

```
> xnqc:= sort([solve(qsimpcomb(QC(n,x,alpha,q)),x)]);
```

$xnqc :=$ [1.00044237488698, 1.11684770412359, 1.26292535990893, 1.45503428162685, 1.70952192415461, 2.04859976560654, 2.50805407426554, 3.15067627385567, 4.10185543060771, 5.69013404948536] (12.6)

```
> ynqc:= sort([solve(qsimpcomb(QC(n,x,alpha*q,q)),x)]);
```

$ynqc :=$ [1.00022983788084, 1.11466426510887, 1.25485899093403, 1.43723645775360, 1.67840363804124, 1.99949089356504, 2.43394444375273, 3.04034425296436, 3.93562739448890, 5.42605673189309] (12.7)

```
> Ynqc:= sort([solve(qsimpcomb(QC(n,x,alpha*q^2,q)),x)]);
```

$Ynqc :=$ [1.00011527705395, 1.11323118937731, 1.24867620174512, 1.42235134751727, 1.65127391667225, 1.95580177781076, 2.36730637206896, 2.94054006459745, 3.78474461251723, 5.18590524809739] (12.8)

```
> xnm1qc:= sort([solve(qsimpcomb(QC(n-1,x,alpha,q)),x)]);
```

$xnm1qc :=$ [1.00087539921324, 1.12071423927258, 1.27616894476230, 1.48454100804820, 1.76487720336587, 2.14651772911441, 2.68119578481859, 3.47261406590352, 4.79280824436452] (12.9)

```
> ynm1qc:= sort([solve(qsimpcomb(QC(n-1,x,alpha*q,q)),x)]);
```

$ynm1qc :=$ [1.00049091617819, 1.11748559558390, 1.26589858689703, 1.46353781691985, 1.72923610557575, 2.09061930008483, 2.59603963326739, 3.34237146862856, 4.58365924638323] (12.10)

```
> Ynm1qc:= sort([solve(qsimpcomb(QC(n-1,x,alpha*q^2,q)),x)]);
```

$Ynm1qc :=$ [1.00026656693334, 1.11521212140455, 1.25765928163513, 1.44553904528852, 1.69775116716933, 2.04049713043808, 2.51907386069202, 3.22413420064468, 4.39332874090307] (12.11)

(a)

```
> seq(evalb(ynqc[i]<xnqc[i] and xnqc[i]<xnm1qc[i] and xnm1qc[i]
```

```
<ynqc[i+1] and ynqc[i+1]<xnqc[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \tag{12.12}$$

(b)

```
> seq(evalb(Ynqc[i]<xnqc[i] and xnqc[i]<xnm1qc[i] and xnm1qc[i]
  <Ynqc[i+1] and Ynqc[i+1]<xnqc[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \tag{12.13}$$

Corollary

(a)

```
> seq(evalb(ynqc[i]<ynm1qc[i] and ynm1qc[i]<xnm1qc[i] and xnm1qc
  [i]<ynqc[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \tag{12.14}$$

(b)

```
> seq(evalb(Ynqc[i]<Ynm1qc[i] and Ynm1qc[i]<xnm1qc[i] and xnm1qc
  [i]<Ynqc[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \tag{12.15}$$

```
> unassign('alpha,q,n')
```

Al-Salam-Carlitz I

```
> ASCI:=(n,x,alpha,q)->(-alpha)^n*q^(binomial(n,2))*add
  (qphihyperterm([q^(-n),1/x],[0],q,q*x/alpha,j),j=0..n);
```

$$ASCI := (n, x, \alpha, q) \rightarrow (-\alpha)^n q^{\mathrm{binomial}(n,2)}\, add\left(qphihyperterm\left(\left[q^{-n}, \frac{1}{x}\right], [0], q, \frac{q\,x}{\alpha}, j\right), j = 0..n\right) \tag{13.1}$$

```
> Fasc1:=(-alpha)^n*q^(binomial(n,2))*(qphihyperterm([q^(-n),
  1/x],[0],q,q*x/alpha,k)):
```

Proposition 29

```
> eq151:=qMixRec1(Fasc1,q,k,U(n),alpha,1):
> eq152:=subs(_C1=1,eq151):
> eq15:=lhs(eq152)=combine(map(qsimpcomb,rhs(eq152)),power)
```

$$eq15 := U(n, \alpha) = U(n, \alpha\, q) + \alpha\, (q^n - 1)\, U(n-1, \alpha\, q) \tag{13.2}$$

Theorem 30

```
> alpha:=-1;q:=0.9;n:=5;
```

$$\alpha := -1$$
$$q := 0.9$$
$$n := 5 \tag{13.3}$$

```
> hyp1:= evalb(alpha<0);
```

$$hyp1 := true \tag{13.4}$$

```
> xnasc1:= sort([solve(qsimpcomb(ASCI(n,x,alpha,q)),x)]);
```

$$xnasc1 := [-0.767107798732687, -0.390843606988264, -7.89840733857423\ 10^{-13}, 0.390843606989804, 0.767107798731897] \tag{13.5}$$

```
> ynasc1:= sort([solve(qsimpcomb(ASCI(n,x,alpha*q,q)),x)]);
```

$ynasc1 := [-0.650603818873362, -0.286593516168726, 0.0863006164308967, 0.455157444041616, 0.805249274569536]$ **(13.6)**

```
> xnm1asc1:= sort([solve(qsimpcomb(ASCI(n-1,x,alpha,q)),x)]);
```

$xnm1asc1 := [-0.663852907695327, -0.223180010181403, 0.223180010181457, 0.663852907695254]$ **(13.7)**

```
> ynm1asc1:= sort([solve(qsimpcomb(ASCI(n-1,x,alpha*q,q)),x)]);
```

$ynm1asc1 := [-0.546909801164284, -0.122831716456566, 0.300712916042884, 0.712928601577946]$ **(13.8)**

```
> seq(evalb(xnasc1[i]<ynasc1[i] and ynasc1[i]<ynm1asc1[i] and
  ynm1asc1[i]<xnasc1[i+1] and xnasc1[i+1]<ynasc1[i+1]),i=1..n-1 )
```

$$true, true, true, true \quad \textbf{(13.9)}$$

Corollary 31

```
> seq(evalb(xnasc1[i]<xnm1asc1[i] and xnm1asc1[i]<ynm1asc1[i] and
  ynm1asc1[i]<xnasc1[i+1]),i=1..n-1 )
```

$$true, true, true, true \quad \textbf{(13.10)}$$

```
> unassign('alpha,q,n')
```

Al-Salam-Carlitz II

```
> ASCII:=(n,x,alpha,q)->(-alpha)^n*q^(-binomial(n,2))*add
  (qphihyperterm([q^(-n),x],[],q,q^(n)/alpha,j),j=0..n);
```

$$ASCII := (n, x, \alpha, q) \rightarrow (-\alpha)^n q^{-\mathrm{binomial}(n,2)} \, add\left(qphihyperterm\left([q^{-n}, x], [\,], q, \frac{q^n}{\alpha}, j\right), j = 0..n\right) \quad \textbf{(14.1)}$$

```
> Fasc2:=(-alpha)^n*q^(-binomial(n,2))*(qphihyperterm([q^(-n),x],
  [],q,q^(n)/alpha,k)):
```

Proposition 32

```
> eq16a1:=qMixRec2(Fasc2,q,k,V(n),a,0,alpha,1,0,0):
> eq16a2:=subs({_C1=1,NN=N},eq16a1):
> eq16a:=lhs(eq16a2)=combine(map(qsimpcomb,rhs(eq16a2)),power)
```

$$eq16a := V(n, a, \alpha q) = V(n, a, \alpha) - \alpha (q^n - 1) q^{-n+1} V(n-1, a, \alpha) \quad \textbf{(14.2)}$$

In (14.2) and (14.3), the variable a was given arbitrary to fit the number of parameters of qMixRec2

```
> eq16b:=qMixRec2(Fasc2,q,k,V(n),a,0,alpha,2,0,0)
```

$$eq16b := V(n, a, \alpha q^2) = (\alpha q^{n+1} - \alpha q + 1) V(n, a, \alpha) - (q^n x - \alpha q + 1) \alpha (q^n - 1) q^{-n+1} V(n-1, a, \alpha) \quad \textbf{(14.3)}$$

Theorem 33

```
> alpha:=0.35;q:=0.9;n:=10;
```

$$\alpha := 0.35$$
$$q := 0.9$$
$$n := 10 \quad \textbf{(14.4)}$$

```
> hyp1:= 0<alpha*q and alpha*q<1 and alpha*q<q^n;
```
$hyp1 := true$ (14.5)

```
> xnasc2:= sort([solve(qsimpcomb(ASCII(n,x,alpha,q)),x)]);
```
$xnasc2 := [1.00608175657218, 1.13892044932534, 1.30480434291955, 1.50932417694615, 1.76119541902787, 2.07411151592838, 2.46970689800811, 2.98441528822168, 3.68825613491887, 4.75904370626005]$ (14.6)

```
> ynasc2:= sort([solve(qsimpcomb(ASCII(n,x,alpha*q,q)),x)]);
```
$ynasc2 := [1.00339378459700, 1.12965723378822, 1.28698675332243, 1.48178950390534, 1.72249417404782, 2.02210299973345, 2.40122829816966, 2.89467861697119, 3.56943519393867, 4.59568195255488]$ (14.7)

```
> Ynasc2:= sort([solve(qsimpcomb(ASCII(n,x,alpha*q^2,q)),x)]);
```
$Ynasc2 := [1.00180669951709, 1.12300595056079, 1.27271782335808, 1.45845806011292, 1.68865804208274, 1.97575007725711, 2.33940289446675, 2.81289982665047, 3.46036585989841, 4.44481321773458]$ (14.8)

```
> xnm1asc2:= sort([solve(qsimpcomb(ASCII(n-1,x,alpha,q)),x)]);
```
$xnm1asc2 := [1.01042638044455, 1.15288428645563, 1.33228201856829, 1.55517200814122, 1.83313678169712, 2.18521101736912, 2.64380966838043, 3.27142913480612, 4.22692242345294]$ (14.9)

```
> ynm1asc2:= sort([solve(qsimpcomb(ASCII(n-1,x,alpha*q,q)),x)]);
```
$ynm1asc2 := [1.00635544808007, 1.14089743854807, 1.31097244342285, 1.52340617790383, 1.78916376315952, 2.12631458886844, 2.56578453611182, 3.16733213311946, 4.08297713071167]$ (14.10)

```
> Ynm1asc2:= sort([solve(qsimpcomb(ASCII(n-1,x,alpha*q^2,q)),x)])
  ;
```
$Ynm1asc2 := [1.00372529306556, 1.13174342028657, 1.29328330375075, 1.49590445876757, 1.75017501311912, 2.07329735773162, 2.49480443943798, 3.07188156628005, 3.95012575403586]$ (14.11)

(a)

```
> seq(evalb(ynasc2[i]<xnasc2[i] and xnasc2[i]<xnm1asc2[i] and
  xnm1asc2[i]<ynasc2[i+1] and ynasc2[i+1]<xnasc2[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true, true (14.12)

(b)

```
> seq(evalb(Ynasc2[i]<xnasc2[i] and xnasc2[i]<xnm1asc2[i] and
  xnm1asc2[i]<Ynasc2[i+1] and Ynasc2[i+1]<xnasc2[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true, true (14.13)

Corollary 34

(a)

```
> seq(evalb(ynasc2[i]<ynm1asc2[i] and ynm1asc2[i]<xnm1asc2[i] and
  xnm1asc2[i]<ynasc2[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true, true (14.14)

(b)

```
> seq(evalb(Ynasc2[i]<ynasc2[i] and ynasc2[i]<xnasc2[i] and
  xnasc2[i]<xnm1asc2[i] and xnm1asc2[i]<Ynasc2[i+1] and Ynasc2
  [i+1]<ynasc2[i+1] and ynasc2[i+1]<xnasc2[i+1]),i=1..n-1 )
```
true, true, true, true, true, true, true, true, true (14.15)

```
> unassign('alpha,q,n')
```

Askey-Wilson

```
> AW:=(n,x,a,b,c,d,q)->qpochhammer(a*b,q,n)*qpochhammer(a*c,q,n)*
  qpochhammer(a*d,q,n)/(2*a)^n /qpochhammer(a*b*c*d*q^(n-1),q,n)
  *add(qpochhammer(q^(-n),q,k)*qpochhammer(a*b*c*d*q^(n-1),q,k)*
  mul(1-2*a*x*q^j+a^2*q^(2*j),j=0..k-1)*q^k/(qpochhammer(a*b,q,k)
  *qpochhammer(a*c,q,k)*qpochhammer(a*d,q,k)*qpochhammer(q,q,k)
  ), k=0..n);
```

$$AW := (n, x, a, b, c, d, q) \to \frac{1}{(2\,a)^n\, qpochhammer(a\,b\,c\,d\,q^{n-1}, q, n)} \Big(qpochhammer(a\,b, q, n)\, qpochhammer(a\,c, q, n)\, qpochhammer(a\,d, q, n)\, add\Big(\big(qpochhammer(q^{-n}, q, k)\, qpochhammer(a\,b\,c\,d\,q^{n-1}, q, k)\, mul(1 - 2\,a\,x\,q^j + a^2\,q^{2j}, j = 0..k-1)\, q^k\big) \Big/ \big(qpochhammer(a\,b, q, k)\, qpochhammer(a\,c, q, k)\, qpochhammer(a\,d, q, k)\, qpochhammer(q, q, k)\big), k = 0..n\Big)\Big) \tag{15.1}$$

```
> FAW:=qpochhammer(a*b,q,n)*qpochhammer(a*c,q,n)*qpochhammer(a*d,
  q,n)/(2*a)^n /qpochhammer(a*b*c*d*q^(n-1),q,n) * qhyperterm([q^
  (-n),a*b*c*d*q^(n-1),a*exp(I*theta),a*exp(-I*theta) ],[a*b,a*c,
  a*d],q,q,k)
```

$$FAW := \Big(qpochhammer(a\,b, q, n)\, qpochhammer(a\,c, q, n)\, qpochhammer(a\,d, q, n)\, qpochhammer(q^{-n}, q, k)\, qpochhammer(a\,b\,c\,d\,q^{n-1}, q, k)\, qpochhammer(a\,e^{I\theta}, q, k)\, qpochhammer(a\,e^{-I\theta}, q, k)\, q^k\Big) \Big/ \Big((2\,a)^n\, qpochhammer(a\,b\,c\,d\,q^{n-1}, q, n)\, qpochhammer(a\,b, q, k)\, qpochhammer(a\,c, q, k)\, qpochhammer(a\,d, q, k)\, qpochhammer(q, q, k)\Big) \tag{15.2}$$

Proposition 36

```
> Awreca:=qMixRec1(FAW,q,k,P(n),a,1)
```

$$Awreca := P(n, a) = P(n, a\,q) - \frac{1}{2}\, \frac{a\,(q^n - 1)\,(c\,d\,q^n - q)\,(q^n\,b\,d - q)\,(q^n\,b\,c - q)\,P(n-1, a\,q)}{(q^{2n}\,a\,b\,c\,d - q^2)\,(q^{2n}\,a\,b\,c\,d - q)} \tag{15.3}$$

```
> Awrecb:=qMixRec2(FAW,q,k,P(n),a,1,b,1,0,1)
```

$$Awrecb := P(n, a, b\,q) = P(n, a\,q, b) + \frac{1}{2}\, \frac{(q^n - 1)\,(c\,d\,q^n - q)\,(a - b)\,P(n-1, a\,q, b\,q)}{q^{2n}\,a\,b\,c\,d - q} \tag{15.4}$$

Theorem 37

(a)

First case -1<a<1

```
> a:=-0.35;b:=-0.68;c:=0.75;d:=-0.9;q:=0.8;n:=10;
```

$$a := -0.35$$
$$b := -0.68$$
$$c := 0.75$$

$d := -0.9$

$q := 0.8$

$n := 10$ **(15.5)**

```
> hyp1:= -1<a and a<0 and -1<b and b<1 and -1<c and c<1 and -1<d
  and d<1 ;
```

$hyp1 := true$ **(15.6)**

```
> xnAW:= sort([solve(qsimpcomb(AW(n,x,a,b,c,d,q)),x)]);
```

$xnAW := [-0.975360044798646, -0.910197481829320, -0.808673096390221, -0.670722371315580, -0.497451886025852, -0.291579077556014, -0.0572938267069724, 0.197873204594961, 0.463865389366667, 0.726916273315463]$ **(15.7)**

```
> xnaAW:= sort([solve(qsimpcomb(AW(n,x,a*q,b,c,d,q)),x)]);
```

$xnaAW := [-0.973871750262613, -0.903468902910997, -0.799097396115280, -0.653224104874130, -0.476636599897251, -0.266645765745763, -0.0310754816643127, 0.222692523381193, 0.484533391778522, 0.740375714612959]$ **(15.8)**

```
> xnm1aAW:= sort([solve(qsimpcomb(AW(n-1,x,a*q,b,c,d,q)),x)]);
```

$xnm1aAW := [-0.969434966479056, -0.890589833032633, -0.768631629521036, -0.604592638531495, -0.400923631658960, -0.161809084555329, 0.105691134381277, 0.391243794364910, 0.681031169849449]$ **(15.9)**

```
> seq(evalb(xnAW[i]<xnaAW[i] and xnaAW[i]<xnm1aAW[i] and
  xnm1aAW[i]<xnAW[i+1] and xnAW[i+1]<xnaAW[i+1]),i=1..n-1 )
```

true, true, true, true, true, true, true, true, true **(15.10)**

Second case 0<a<1

```
> a:=0.65;b:=-0.68;c:=0.75;d:=-0.9;q:=0.8;n:=10;
```

$a := 0.65$

$b := -0.68$

$c := 0.75$

$d := -0.9$

$q := 0.8$

$n := 10$ **(15.11)**

```
> hyp1:= 0<a and a<1 and -1<b and b<1 and -1<c and c<1 and -1<d
  and d<1 ;
```

$hyp1 := true$ **(15.12)**

```
> xnAW:= sort([solve(qsimpcomb(AW(n,x,a,b,c,d,q)),x)]);
```

$xnAW := [-0.950271870240780, -0.830878816620128, -0.656042889318941, -0.437508287238490, -0.189702113728362, 0.0710532083445337, 0.327586667915700, 0.563010690458352, 0.761833339265765, 0.911004763724854]$ **(15.13)**

```
> xnaAW:= sort([solve(qsimpcomb(AW(n,x,a*q,b,c,d,q)),x)]);
```

$xnaAW := [-0.953557600934383, -0.841180002267172, -0.675863267267604, -0.467924809213566, -0.230032233660789, 0.0233207018671522, 0.276717468377055, 0.514709256456492, 0.722731744013601, 0.887864171842630]$ **(15.14)**

```
> xnm1aAW:= sort([solve(qsimpcomb(AW(n-1,x,a*q,b,c,d,q)),x)]);
```

$xnm1aAW := [-0.944333701486362, -0.812218577901680, -0.619764321834123,$ **(15.15)**

-0.380958932790839, -0.112927459015943, 0.165133198992997,
0.433302676253992, 0.672340884437236, 0.865031805473536]

```
> seq(evalb(xnaAW[i]<xnAW[i] and xnAW[i]<xnm1aAW[i] and xnm1aAW
  [i]<xnaAW[i+1]  and xnaAW[i+1]<xnAW[i+1]),i=1..n-1 )
```

(b)

First case b>a

```
> a:=-0.35;b:=0.68;c:=0.75;d:=-0.9;q:=0.8;n:=10;
```

$$a := -0.35$$
$$b := 0.68$$
$$c := 0.75$$
$$d := -0.9$$
$$q := 0.8$$
$$n := 10 \quad (15.16)$$

```
> hyp1:= -1<a and a<1 and -1<b and b<1 and -1<c and c<1 and -1<d
  and d<1 and a<b;
```

$$hyp1 := true \quad (15.17)$$

```
> xnaAW:= sort([solve(qsimpcomb(AW(n,x,a*q,b,c,d,q)),x)]);
```

xnaAW := [-0.895179833702638, -0.721065521243075, -0.508869507509511, **(15.18)**
-0.272827489482000, -0.0264894560383913, 0.217063766103260,
0.445976931689340, 0.647493243132775, 0.812849361880361, 0.932717114834274]

```
> xnbAW:= sort([solve(qsimpcomb(AW(n,x,a,b*q,c,d,q)),x)]);
```

xnbAW := [-0.912283284096839, -0.756542841896406, -0.560432933668217, **(15.19)**
-0.336647679050732, -0.0975557645645418, 0.144780146560137,
0.377642709509306, 0.589906194904960, 0.770300076657464, 0.909988631582766]

```
> xnm1abAW:= sort([solve(qsimpcomb(AW(n-1,x,a*q,b*q,c,d,q)),x)]);
```

xnm1abAW := [-0.881081623266461, -0.690253372974096, -0.460284619415269, **(15.20)**
-0.207272830753832, 0.0537091661717550, 0.308006477539469,
0.541643086536343, 0.741876073547249, 0.897699302070536]

```
> seq(evalb(xnbAW[i]<xnaAW[i] and xnaAW[i]<xnm1abAW[i] and
  xnm1abAW[i]<xnbAW[i+1] and xnbAW[i+1]<xnaAW[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true*, *true*, *true*, *true*, *true*, *true* **(15.21)**

Second case b<a

```
> a:=-0.35;b:=-0.68;c:=0.75;d:=-0.9;q:=0.8;n:=10;
```

$$a := -0.35$$
$$b := -0.68$$
$$c := 0.75$$
$$d := -0.9$$
$$q := 0.8$$
$$n := 10 \quad (15.22)$$

```
> hyp1:= -1<a and a<1 and -1<b and b<1 and -1<c and c<1 and -1<d
  and d<1 and b<a;
```

$$hyp1 := true \quad (15.23)$$

```
> xnaAW:= sort([solve(qsimpcomb(AW(n,x,a*q,b,c,d,q)),x)]);
```

xnaAW := [-0.973871750262613, -0.903468902910997, -0.799097396115280, **(15.24)**
-0.653224104874130, -0.476636599897251, -0.266645765745763,

-0.0310754816643127, 0.222692523381193, 0.484533391778522, 0.740375714612959]

```
> xnbAW:= sort([solve(qsimpcomb(AW(n,x,a,b*q,c,d,q)),x)]);
```

$xnbAW := [-0.968106104280185, -0.891925606022522, -0.779709802420604, -0.633003241149678, -0.453545984080181, -0.244479555207483, -0.0108137686077971, 0.239911353871059, 0.497406091790753, 0.748103895295964]$ **(15.25)**

```
> xnm1abAW:= sort([solve(qsimpcomb(AW(n-1,x,a*q,b*q,c,d,q)),x)]);
```

$xnm1abAW := [-0.959838757403108, -0.867648103544712, -0.733757942544885, -0.560694477929661, -0.351511242435859, -0.111177162307216, 0.152883327661063, 0.430058591247051, 0.706469327084572]$ **(15.26)**

```
> seq(evalb(xnaAW[i]<xnbAW[i] and xnbAW[i]<xnm1abAW[i] and xnm1abAW[i]<xnaAW[i+1] and xnaAW[i+1]<xnbAW[i+1]),i=1..n-1 )
```

Corollary 38

(a)

```
> a:=-0.35;b:=0.58;c:=0.75;d:=-0.9;q:=0.8;n:=10;
```

$$a := -0.35$$
$$b := 0.58$$
$$c := 0.75$$
$$d := -0.9$$
$$q := 0.8$$
$$n := 10$$ **(15.27)**

```
> hyp1:= -1<a and a<0 and 0<b and b<1 and -1<c and c<1 and -1<d and d<1;
```

$$hyp1 := true$$ **(15.28)**

```
> xnAW:= sort([solve(qsimpcomb(AW(n,x,a,b,c,d,q)),x)]);
```

$xnAW := [-0.910369258803934, -0.752144256615583, -0.553334690746130, -0.327027302781208, -0.0858148183025449, 0.157726192870576, 0.390778076091187, 0.601812682608217, 0.779570396616921, 0.915241942757094]$ **(15.29)**

```
> xnaAW:= sort([solve(qsimpcomb(AW(n,x,a*q,b,c,d,q)),x)]);
```

$xnaAW := [-0.901183952896553, -0.734318894765660, -0.529430382383441, -0.300312006649612, -0.0592777896037803, 0.181390281269358, 0.410081223697957, 0.615310715655363, 0.787886085891509, 0.918712989992952]$ **(15.30)**

```
> xnbAW:= sort([solve(qsimpcomb(AW(n,x,a,b*q,c,d,q)),x)]);
```

$xnbAW := [-0.916508592824650, -0.766321256404015, -0.576175854812490, -0.358118265946765, -0.123578585813227, 0.115870259471000, 0.348563352773538, 0.563417673365438, 0.749805037681922, 0.898333639839050]$ **(15.31)**

```
> xnm1abAW:= sort([solve(qsimpcomb(AW(n-1,x,a*q,b*q,c,d,q)),x)]);
```

$xnm1abAW := [-0.887229811000399, -0.703375462197884, -0.480318986188890, -0.233234878326037, 0.0237174265315495, 0.276771400893211, 0.512570885169816, 0.718776702609196, 0.884215047705834]$ **(15.32)**

```
> seq(evalb(xnbAW[i]<xnAW[i] and xnAW[i]<xnaAW[i] and xnaAW[i]<xnm1abAW[i] and xnm1abAW[i]<xnbAW[i+1] and xnbAW[i+1]<xnAW[i+1] and xnAW[i+1]<xnaAW[i+1]),i=1..n-1 )
```

(15.33)

$$true, true, true, true, true, true, true, true, true \quad (15.33)$$

(b)

```
> a:=0.35;b:=-0.58;c:=0.75;d:=-0.9;q:=0.8;n:=10;
```

$$a := 0.35$$
$$b := -0.58$$
$$c := 0.75$$
$$d := -0.9$$
$$q := 0.8$$
$$n := 10 \quad (15.34)$$

```
> hyp1:= -1<b and b<0 and 0<a and a<1 and -1<c and c<1 and -1<d
  and d<1;
```

$$hyp1 := true \quad (15.35)$$

```
> xnAW:= sort([solve(qsimpcomb(AW(n,x,a,b,c,d,q)),x)]);
```

$xnAW := [-0.947249911403763, -0.831851848305783, -0.670373652091643, -0.471410696374092, -0.245729219451283, -0.00452151109261131, 0.239650057697559, 0.474193281669741, 0.686623175452655, 0.864819742191130]$ (15.36)

```
> xnaAW:= sort([solve(qsimpcomb(AW(n,x,a*q,b,c,d,q)),x)]);
```

$xnaAW := [-0.949639653086034, -0.837838857785260, -0.682631670981840, -0.487717323733346, -0.268345397249481, -0.0296855591021263, 0.212772860933251, 0.449269731974880, 0.666627029470102, 0.852915294031766]$ (15.37)

```
> xnbAW:= sort([solve(qsimpcomb(AW(n,x,a,b*q,c,d,q)),x)]);
```

$xnbAW := [-0.934845165731067, -0.805753825883004, -0.634100309795144, -0.429658248620646, -0.202872071928161, 0.0354164242248303, 0.273651546998203, 0.500236760271640, 0.703838591802881, 0.873383616010974]$ (15.38)

```
> xnm1abAW:= sort([solve(qsimpcomb(AW(n-1,x,a*q,b*q,c,d,q)),x)]);
```

$xnm1abAW := [-0.924989089768442, -0.780553393474852, -0.589926146228430, -0.364618867386639, -0.116662569783481, 0.140875231339606, 0.394431839455211, 0.630155869226597, 0.834262704555137]$ (15.39)

```
> seq(evalb(xnaAW[i]<xnAW[i] and xnAW[i]<xnbAW[i] and xnbAW[i]
  <xnm1abAW[i] and xnm1abAW[i]<xnaAW[i+1] and xnaAW[i+1]<xnAW
  [i+1] and xnAW[i+1]<xnbAW[i+1]),i=1..n-1 )
```

$$true, true, true, true, true, true, true, true, true \quad (15.40)$$

```
> unassign('a,b,c,d,q,n')
```

The other mixed recurrence equations from which we can derive similar results are given here

```
> Awrec2:=subs(_C1=1 ,qMixRec1(FAW,q,k,P(n),b,1)):
> lhs(Awrec2)=combine(map(qsimpcomb,rhs(Awrec2)),power)
```

$$P(n, b) = P(n, b\,q) - \frac{1}{2}\,\frac{(q^n-1)\,(c\,d\,q^n-q)\,(a\,d\,q^n-q)\,(a\,c\,q^n-q)\,b\,P(n-1, b\,q)}{(q^{2n}\,a\,b\,c\,d-q^2)\,(q^{2n}\,a\,b\,c\,d-q)} \quad (15.41)$$

```
> Awrec3:=subs(_C1=1 ,qMixRec1(FAW,q,k,P(n),c,1)):
> lhs(Awrec3)=combine(map(qsimpcomb,rhs(Awrec3)),power)
```

$$P(n, c) = P(n, c\,q) - \frac{1}{2}\,\frac{(q^n-1)\,(q^n\,b\,d-q)\,(a\,d\,q^n-q)\,(a\,b\,q^n-q)\,c\,P(n-1, c\,q)}{(q^{2n}\,a\,b\,c\,d-q^2)\,(q^{2n}\,a\,b\,c\,d-q)} \quad (15.42)$$

```
> Awrec4:=subs(_C1=1 ,qMixRec1(FAW,q,k,P(n),d,1)):
> lhs(Awrec4)=combine(map(qsimpcomb,rhs(Awrec4)),power)
```

$$P(n, d) = P(n, d\,q) - \frac{1}{2}\,\frac{(q^n-1)\,(q^n b\,c-q)\,(a\,c\,q^n-q)\,(a\,b\,q^n-q)\,d\,P(n-1, d\,q)}{(q^{2\,n} a\,b\,c\,d-q^2)\,(q^{2\,n} a\,b\,c\,d-q)} \tag{15.43}$$

```
> qMixRec2(FAW,q,k,P(n),a,1,c,1,0,1)
```

$$P(n, a, c\,q) = P(n, a\,q, c) + \frac{1}{2}\,\frac{(q^n-1)\,(q^n b\,d-q)\,(a-c)\,P(n-1, a\,q, c\,q)}{q^{2\,n} a\,b\,c\,d-q} \tag{15.44}$$

```
> qMixRec2(FAW,q,k,P(n),a,1,d,1,0,1)
```

$$P(n, a, d\,q) = P(n, a\,q, d) + \frac{1}{2}\,\frac{(q^n-1)\,(q^n b\,c-q)\,(a-d)\,P(n-1, a\,q, d\,q)}{q^{2\,n} a\,b\,c\,d-q} \tag{15.45}$$

```
> Awrec5:=subs(_C1=1,qMixRec2(FAW,q,k,P(n),b,1,c,1,0,1)):
> lhs(Awrec5)=combine(map(qsimpcomb,rhs(Awrec5)),power)
```

$$P(n, b, c\,q) = P(n, b\,q, c) + \frac{1}{2}\,\frac{(b-c)\,(q^n-1)\,(a\,d\,q^n-q)\,P(n-1, b\,q, c\,q)}{q^{2\,n} a\,b\,c\,d-q} \tag{15.46}$$

```
> Awrec6:=subs(_C1=1,qMixRec2(FAW,q,k,P(n),b,1,d,1,0,1)):
> lhs(Awrec6)=combine(map(qsimpcomb,rhs(Awrec6)),power)
```

$$P(n, b, d\,q) = P(n, b\,q, d) + \frac{1}{2}\,\frac{(b-d)\,(q^n-1)\,(a\,c\,q^n-q)\,P(n-1, b\,q, d\,q)}{q^{2\,n} a\,b\,c\,d-q} \tag{15.47}$$

```
> Awrec7:=subs(_C1=1,qMixRec2(FAW,q,k,P(n),c,1,d,1,0,1)):
> lhs(Awrec7)=combine(map(qsimpcomb,rhs(Awrec7)),power)
```

$$P(n, c, d\,q) = P(n, c\,q, d) + \frac{1}{2}\,\frac{(c-d)\,(q^n-1)\,(a\,b\,q^n-q)\,P(n-1, c\,q, d\,q)}{q^{2\,n} a\,b\,c\,d-q} \tag{15.48}$$

q-Racah

```
> QR:=(n,x,alpha,beta,gamma,delta,q)-> qpochhammer(alpha*q, q, n)
  *qpochhammer(beta*delta*q, q, n)*qpochhammer(gamma*q, q, n)
  /qpochhammer(alpha*beta*q^(n+1), q, n)*add(qpochhammer(q^(-n),
  q,m)*qpochhammer(alpha*beta*q^(n+1),q,m)*mul(1-x*q^k+gamma*
  delta*q^(2*k+1),k=0..m-1)  *q^m /(qpochhammer(alpha*q,q,m)*
  qpochhammer(beta*delta*q,q,m)*qpochhammer(gamma*q,q,m)*
  qpochhammer(q,q,m) )  ,m=0..n);
```

$$QR := (n, x, \alpha, \beta, \gamma, \delta, q) \rightarrow \frac{1}{qpochhammer(\alpha\,\beta\,q^{n+1}, q, n)}\Big(qpochhammer(\alpha\,q, q, n)\;qpochhammer(\beta\,\delta\,q, q, n)\;qpochhammer(\gamma\,q, q, n)\;add\big(\big(qpochhammer(q^{-n}, q, m)\;qpochhammer(\alpha\,\beta\,q^{n+1}, q, m)\;mul(1-x\,q^k+\gamma\,\delta\,q^{2\,k+1}, k=0..m-1)\;q^m\big) \big/ \big(qpochhammer(\alpha\,q, q, m)\;qpochhammer(\beta\,\delta\,q, q, m)\;qpochhammer(\gamma\,q, q, m)\;qpochhammer(q, q, m)\big), m=0..n\big)\Big) \tag{16.1}$$

```
> FQR:=qpochhammer(alpha*q, q, n)*qpochhammer(beta*delta*q, q, n)
  *qpochhammer(gamma*q, q, n)/qpochhammer(alpha*beta*q^(n+1), q,
```

```
n)*qhyperterm([q^(-n),alpha*beta*q^(n+1),q^(-x),gamma*delta*q^
(x+1)],[alpha*q,beta*delta*q,gamma*q],q,q,k)
```

$$FQR := \left(qpochhammer(\alpha q, q, n)\, qpochhammer(\beta\delta q, q, n)\, qpochhammer(\gamma q, q, n)\, qpochhammer(q^{-n}, q, k)\, qpochhammer(\alpha\beta q^{n+1}, q, k)\, qpochhammer(q^{-x}, q, k)\, qpochhammer(\gamma\delta q^{x+1}, q, k)\, q^k\right) \Big/ \left(qpochhammer(\alpha\beta q^{n+1}, q, n)\, qpochhammer(\alpha q, q, k)\, qpochhammer(\beta\delta q, q, k)\, qpochhammer(\gamma q, q, k)\, qpochhammer(q, q, k)\right) \quad (16.2)$$

Proposition 41

```
> QRreca1:=subs(_C1=1,qMixRec1(FQR,q,k,P(n),alpha,1)):
> QRreca:=lhs(QRreca1)=combine(map(qsimpcomb,rhs(QRreca1)),power)
```

$$QRreca := P(n, \alpha) = P(n, \alpha q) - \frac{(q^n-1)(\beta q^n-1)(\beta\delta q^n-1)(\gamma q^n-1)\, q\, \alpha\, P(n-1, \alpha q)}{(\alpha\beta q^{1+2n}-1)(\alpha\beta q^{2n}-1)} \quad (16.3)$$

```
> QRrecb1:=subs(_C1=1,qMixRec1(FQR,q,k,P(n),beta,1)):
> QRrecb:=lhs(QRrecb1)=combine(map(qsimpcomb,rhs(QRrecb1)),power)
```

$$QRrecb := P(n, \beta) = P(n, q\beta) - \frac{(q^n-1)(\alpha q^n-1)(\alpha q^n-\delta)(\gamma q^n-1)\, q\, \beta\, P(n-1, q\beta)}{(\alpha\beta q^{1+2n}-1)(\alpha\beta q^{2n}-1)} \quad (16.4)$$

```
> QRrecc1:=subs(_C1=1,qMixRec2(FQR,q,k,P(n),alpha,1,beta,1,0,1)):
> QRrecc:=lhs(QRrecc1)=combine(map(qsimpcomb,rhs(QRrecc1)),power)
```

$$QRrecc := P(n, \alpha, q\beta) = P(n, \alpha q, \beta) + \frac{q\,(-\beta\delta+\alpha)(q^n-1)(\gamma q^n-1)\, P(n-1, \alpha q, q\beta)}{\alpha\beta q^{1+2n}-1} \quad (16.5)$$

Theorem 42

(a)

```
> beta:=-0.68;gam:=0.25;delta:=0.9;q:=0.8;n:=10;N:=15;
```

$$\beta := -0.68$$
$$gam := 0.25$$
$$\delta := 0.9$$
$$q := 0.8$$
$$n := 10$$
$$N := 15 \quad (16.6)$$

```
> hyp1:= gam*q<1 and 0<delta*q and delta*q<1 and beta*q<1 and
beta*delta*q<1;
```

$$hyp1 := true \quad (16.7)$$

```
> xnQR:= sort([solve(qsimpcomb(QR(n,x,q^(-N-1),beta,gam,delta,
q)),x)]);
```

$xnQR$:= [1.63619638275228, 2.36249012181753, 3.40098395780668, 4.86212148611183, 6.87282659635875, 9.56632290504587, 13.0526580345733, 17.3694807546631, 22.4661434041240, 28.3901445556595] (16.8)

```
> xnalphaQR:= sort([solve(qsimpcomb(QR(n,x,q^(-N-1)*q,beta,gam,
```

```
delta,q)),x)]);
```

xnalphaQR := [1.58005921495486, 2.23425432808419, 3.15078014821478, 4.40968222067769, 6.09446127240602, 8.28024692734268, 11.0111331585002, 14.2875794143803, 18.1337745902291, 22.7391948455660] **(16.9)**

```
> xnm1alphaQR:= sort([solve(qsimpcomb(QR(n-1,x,q^(-N-1)*q,beta,
  gam,delta,q)),x)]);
```

xnm1alphaQR := [1.64831661668255, 2.38999367011792, 3.45388423113376, 4.95508298257057, 7.02432794498232, 9.79246797225212, 13.3475448738720, 17.6666658397671, 22.6424067471768] **(16.10)**

```
> seq(evalb(xnalphaQR[i]<xnQR[i] and xnQR[i]<xnm1alphaQR[i] and
  xnm1alphaQR[i]<xnalphaQR[i+1] and xnalphaQR[i+1]<xnQR[i+1]),i=
  1..n-1 )
```

true, true, true, true, true, true, true, true, true **(16.11)**

(b)

```
> alpha:=-0.68;gam:=0.25;delta:=0.9;q:=0.8;n:=10;N:=15;
```

$$\alpha := -0.68$$
$$gam := 0.25$$
$$\delta := 0.9$$
$$q := 0.8$$
$$n := 10$$
$$N := 15 \quad \textbf{(16.12)}$$

```
> hyp1:= gam*q<1 and 0<delta*q and delta*q<1 and alpha*q<1 and
  alpha*q^n<delta;
```

hyp1 := true **(16.13)**

```
> xnQR:= sort([solve(qsimpcomb(QR(n,x,alpha,q^(-N-1)/delta,gam,
  delta,q)),x)]);
```

xnQR := [1.60891769498104, 2.30773404861691, 3.30979601079623, 4.72618901261793, 6.68839594971397, 9.34066308320062, 12.8130686042065, 17.1691245974234, 22.3607114458937, 28.3683085686166] **(16.14)**

```
> xnbetaQR:= sort([solve(qsimpcomb(QR(n,x,alpha,q^(-N-1)/delta*q,
  gam,delta,q)),x)]);
```

xnbetaQR := [1.55789135196433, 2.19088430744977, 3.08096164658336, 4.31011549774280, 5.96723084326476, 8.13754124817107, 10.8792527173951, 14.2004633750533, 18.1020800049461, 22.7350032850685] **(16.15)**

```
> xnm1betaQR:= sort([solve(qsimpcomb(QR(n-1,x,alpha,q^(-N-1)
  /delta*q,gam,delta,q)),x)]);
```

xnm1betaQR := [1.62079881359989, 2.33481586534991, 3.36225825884596, 4.81942114778617, 6.84295906611073, 9.57755900628272, 13.1357189388949, 17.5202777961578, 22.5972769003039] **(16.16)**

```
> seq(evalb(xnbetaQR[i]<xnQR[i] and xnQR[i]<xnm1betaQR[i] and
  xnm1betaQR[i]<xnbetaQR[i+1] and xnbetaQR[i+1]<xnQR[i+1]),i=1..
  n-1 )
```

true, true, true, true, true, true, true, true, true **(16.17)**

```
> unassign('alpha,beta,gam,delta,q,n,N')
```

Continuous dual q-Hahn (see Remark 39-i)

```
> CDQH:=(n,x,a,b,c,q)->qpochhammer(a*b,q,n)*qpochhammer(a*c,q,n)/
  (2*a)^n*add(qpochhammer(q^(-n),q,k)*mul(1-2*a*x*q^j+a^2*q^(2*
  j),j=0..k-1)*q^k/(qpochhammer(a*b,q,k)*qpochhammer(a*c,q,k)*
  qpochhammer(q,q,k) ), k=0..n);
```

$$CDQH := (n,x,a,b,c,q) \to \frac{1}{(2\,a)^n}\left(qpochhammer(a\,b,q,n)\;qpochhammer(a\,c,q,n)\;add\left(\frac{qpochhammer(q^{-n},q,k)\;mul(1-2\,a\,x\,q^j+a^2\,q^{2j},j=0..k-1)\;q^k}{qpochhammer(a\,b,q,k)\;qpochhammer(a\,c,q,k)\;qpochhammer(q,q,k)},k=0..n\right)\right) \quad (17.1)$$

```
> FCDQH:=qpochhammer(a*b,q,n)*qpochhammer(a*c,q,n)/(2*a)^n*
  qhyperterm([q^(-n),a*exp(I*theta),a*exp(-I*theta)],[a*b,a*c],q,
  q,k)
```

$$FCDQH := \left(qpochhammer(a\,b,q,n)\;qpochhammer(a\,c,q,n)\;qpochhammer(q^{-n},q,k)\;qpochhammer(a\,e^{I\theta},q,k)\;qpochhammer(a\,e^{-I\theta},q,k)\;q^k\right)\Big/\left((2\,a)^n\;qpochhammer(a\,b,q,k)\;qpochhammer(a\,c,q,k)\;qpochhammer(q,q,k)\right) \quad (17.2)$$

```
> CDQHrec1:=qMixRec1(FCDQH,q,k,P(n),a,1)
```

$$CDQHrec1 := P(n,a) = P(n,a\,q) - \frac{1}{2}\,\frac{a\,(q^n\,b\,c-q)\,(q^n-1)\,P(n-1,a\,q)}{q} \quad (17.3)$$

```
> CDQHrec2:=subs(_C1=1 ,qMixRec1(FCDQH,q,k,P(n),b,1)):
> lhs(CDQHrec2)=combine(map(qsimpcomb,rhs(CDQHrec2)),power)
```

$$P(n,b) = P(n,b\,q) - \frac{1}{2}\,\frac{b\,(q^n-1)\,(a\,c\,q^n-q)\,P(n-1,b\,q)}{q} \quad (17.4)$$

```
> CDQHrec3:=subs(_C1=1 ,qMixRec1(FCDQH,q,k,P(n),c,1)):
> lhs(CDQHrec3)=combine(map(qsimpcomb,rhs(CDQHrec3)),power)
```

$$P(n,c) = P(n,c\,q) - \frac{1}{2}\,\frac{c\,(q^n-1)\,(a\,b\,q^n-q)\,P(n-1,c\,q)}{q} \quad (17.5)$$

```
> CDQHrec4:= qMixRec2(FCDQH,q,k,P(n),a,1,b,1,0,1)
```

$$CDQHrec4 := P(n,a,b\,q) = P(n,a\,q,b) + \frac{1}{2}\,(a-b)\,(q^n-1)\,P(n-1,a\,q,b\,q) \quad (17.6)$$

```
> CDQHrec5:=qMixRec2(FCDQH,q,k,P(n),a,1,c,1,0,1)
```

$$CDQHrec5 := P(n,a,c\,q) = P(n,a\,q,c) + \frac{1}{2}\,(a-c)\,(q^n-1)\,P(n-1,a\,q,c\,q) \quad (17.7)$$

```
> CDQHrec6:=subs(_C1=1,qMixRec2(FCDQH,q,k,P(n),b,1,c,1,0,1)):
> lhs(CDQHrec6)=combine(map(qsimpcomb,rhs(CDQHrec6)),power)
```

$$P(n,b,c\,q) = P(n,b\,q,c) + \frac{1}{2}\,(b-c)\,(q^n-1)\,P(n-1,b\,q,c\,q) \quad (17.8)$$

Continuous q-Hahn (see Remark 39-iv)

```
> CQH:=(n,x,a,b,c,d,theta,q)->qpochhammer(a*b*exp(2*I*phi),q,n)*
  qpochhammer(a*c,q,n)*qpochhammer(a*d,q,n)/(a*exp(I*phi))^n*add
```

```
(qpochhammer(q^(-n),q,k)*qpochhammer(a*b*c*d*q^(n-1),q,k)*mul(1
-2*a*exp(I*theta)*x*q^j+a^2*exp(2*I*theta)*q^(2*j),j=0..k-1)*
q^k/(qpochhammer(a*b*exp(2*I*theta),q,k)*qpochhammer(a*c,q,k)*
qpochhammer(a*d,q,k)*qpochhammer(q,q,k)    ),k=0..n  ):
> FCQH:=qpochhammer(a*b*exp(2*I*phi),q,n)*qpochhammer(a*c,q,n)*
qpochhammer(a*d,q,n)/(2*a*exp(I*phi))^n/qpochhammer(a*b*c*d*q^
(n-1), q, n) *qhyperterm([q^(-n),a*b*c*d*q^(n-1),a*exp(I*
(theta+2*phi)),a*exp(-I*theta)],[a*b*exp(2*I*phi),a*c,a*d],q,q,
k)
```

$$FCQH := \Big(qpochhammer\big(a\, b\, e^{2 I \phi}, q, n\big)\, qpochhammer(a\, c, q, n)\, qpochhammer(a\, d, q, n)\, qpochhammer\big(q^{-n}, q, k\big)\, qpochhammer\big(a\, b\, c\, d\, q^{n-1}, q, k\big)\, qpochhammer\big(a\, e^{I(\theta+2\phi)}, q, k\big)\, qpochhammer\big(a\, e^{-I\theta}, q, k\big)\, q^k\Big) \Big/ \Big(\big(2\, a\, e^{I\phi}\big)^n\, qpochhammer\big(a\, b\, c\, d\, q^{n-1}, q, n\big)\, qpochhammer\big(a\, b\, e^{2 I\phi}, q, k\big)\, qpochhammer(a\, c, q, k)\, qpochhammer(a\, d, q, k)\, qpochhammer(q, q, k)\Big) \tag{18.1}$$

```
> CQHrec1:=qMixRec1(FCQH,q,k,P(n),a,1)
```

$$CQHrec1 := P(n, a) = P(n, a\, q) - \frac{1}{2}\, \frac{a\, (q^n - 1)\, (q^n\, b\, d - q)\, (q^n\, b\, c - q)\, \big(c\, d\, q^n - e^{2 I\phi}\, q\big)\, P(n-1, a\, q)\, e^{-I\phi}}{(q^{2n}\, a\, b\, c\, d - q^2)\, (q^{2n}\, a\, b\, c\, d - q)} \tag{18.2}$$

```
> CQHrec2:=subs(_C1=1 ,qMixRec1(FCQH,q,k,P(n),b,1)):
> lhs(CQHrec2)=combine(map(qsimpcomb,rhs(CQHrec2)),power)
```

$$P(n, b) = P(n, b\, q) - \frac{1}{2}\, \frac{P(n-1, b\, q)\, (q^n - 1)\, (a\, d\, q^n - q)\, (q^n\, a\, c - q)\, \big(c\, d\, q^n - e^{2 I\phi}\, q\big)\, b\, e^{-I\phi}}{(q^{2n}\, a\, b\, c\, d - q^2)\, (q^{2n}\, a\, b\, c\, d - q)} \tag{18.3}$$

```
> CQHrec3:=subs(_C1=1 ,qMixRec1(FCQH,q,k,P(n),c,1)):
> lhs(CQHrec3)=combine(map(qsimpcomb,rhs(CQHrec3)),power)
```

$$P(n, c) = P(n, c\, q) - \frac{1}{2}\, \frac{(q^n - 1)\, (q^n\, b\, d - q)\, (a\, d\, q^n - q)\, \big(a\, b\, e^{2 I\phi}\, q^n - q\big)\, c\, P(n-1, c\, q)\, e^{-I\phi}}{(q^{2n}\, a\, b\, c\, d - q^2)\, (q^{2n}\, a\, b\, c\, d - q)} \tag{18.4}$$

```
> CQHrec4:=subs(_C1=1 ,qMixRec1(FCQH,q,k,P(n),d,1)):
> lhs(CQHrec4)=combine(map(qsimpcomb,rhs(CQHrec4)),power)
```

$$P(n, d) = P(n, d\, q) - \frac{1}{2}\, \frac{(q^n - 1)\, (q^n\, b\, c - q)\, (q^n\, a\, c - q)\, \big(a\, b\, e^{2 I\phi}\, q^n - q\big)\, d\, P(n-1, d\, q)\, e^{-I\phi}}{(q^{2n}\, a\, b\, c\, d - q^2)\, (q^{2n}\, a\, b\, c\, d - q)} \tag{18.5}$$

```
> CQHrec5:=qMixRec2(FCQH,q,k,P(n),a,1,b,1,0,1):
> lhs(CQHrec5)=combine(map(qsimpcomb,rhs(CQHrec5)),power)
```

$$P(n, a, b\, q) = P(n, a\, q, b) + \frac{1}{2}\, \frac{(q^n - 1)\, \big(c\, d\, q^n - e^{2 I\phi}\, q\big)\, (a - b)\, P(n-1, a\, q, b\, q)\, e^{-I\phi}}{q^{2n}\, a\, b\, c\, d - q} \tag{18.6}$$

```
> CQHrec7:=qMixRec2(FCQH,q,k,P(n),a,1,c,1,0,1):
> lhs(CQHrec7)=combine(map(qsimpcomb,rhs(CQHrec7)),power)
```

$$P(n, a, c\, q) = P(n, a\, q, c) \tag{18.7}$$

$$+\frac{1}{2}\frac{(q^n-1)(q^n b d-q)(e^{2I\phi}a-c)P(n-1,aq,cq)e^{-I\phi}}{q^{2n}abcd-q}$$

```
> CQHrec9:= qMixRec2(FCQH,q,k,P(n),a,1,d,1,0,1) :
> lhs(CQHrec9)=combine(map(qsimpcomb,rhs(CQHrec9)),power)
```

$$P(n,a,dq)=P(n,aq,d)+\frac{1}{2}\frac{(q^n-1)(q^n bc-q)(e^{2I\phi}a-d)P(n-1,aq,dq)e^{-I\phi}}{q^{2n}abcd-q} \tag{18.8}$$

```
> CQHrec11:= subs(_C1=1,qMixRec2(FCQH,q,k,P(n),b,1,c,1,0,1))  :
> lhs(CQHrec11)=combine(map(qsimpcomb,rhs(CQHrec11)),power)
```

$$P(n,b,cq)=P(n,bq,c)+\frac{1}{2}\frac{P(n-1,bq,cq)(e^{2I\phi}b-c)(q^n-1)(adq^n-q)e^{-I\phi}}{q^{2n}abcd-q} \tag{18.9}$$

```
> CQHrec12:= subs(_C1=1,qMixRec2(FCQH,q,k,P(n),b,1,d,1,0,1))   :
> lhs(CQHrec12)=combine(map(qsimpcomb,rhs(CQHrec12)),power)
```

$$P(n,b,dq)=P(n,bq,d)+\frac{1}{2}\frac{P(n-1,bq,dq)(e^{2I\phi}b-d)(q^n-1)(q^n ac-q)e^{-I\phi}}{q^{2n}abcd-q} \tag{18.10}$$

```
> CQHrec6:= subs(_C1=1,qMixRec2(FCQH,q,k,P(n),c,1,d,1,0,1)):
> lhs(CQHrec6)=combine(map(qsimpcomb,rhs(CQHrec6)),power)
```

$$P(n,c,dq)=P(n,cq,d)+\frac{1}{2}\frac{(c-d)(q^n-1)(abe^{2I\phi}q^n-q)P(n-1,cq,dq)e^{-I\phi}}{q^{2n}abcd-q} \tag{18.11}$$

Dual q-Hahn (see Remark 43-i)

```
> DQH:=(n,x,gamma,delta,N,q)->qpochhammer(gamma*q,q,n)*
  qpochhammer(q^(-N),q,n)*add(qpochhammer(q^(-n),q,m)*mul(1-x*
  q^k+gamma*delta*q^(2*k+1),k=0..m-1)  *q^m/(qpochhammer(gamma*q,
  q,m)*qpochhammer(q^(-N),q,m)*qpochhammer(q,q,m)) ,m=0..n  );
```

$$DQH:=(n,x,\gamma,\delta,N,q)\rightarrow qpochhammer(\gamma q,q,n)\,qpochhammer(q^{-N},q,n)\,add\left(\frac{qpochhammer(q^{-n},q,m)\,mul(1-x\,q^k+\gamma\delta q^{2k+1},k=0..m-1)\,q^m}{qpochhammer(\gamma q,q,m)\,qpochhammer(q^{-N},q,m)\,qpochhammer(q,q,m)},m=0..n\right) \tag{19.1}$$

```
> FDQH:=qpochhammer(gamma*q,q,n)*qpochhammer(q^(-NN),q,n)*
  qhyperterm([q^(-n),q^(-x),gamma*delta*q^(x+1)],[gamma*q,q^(-NN)
  ],q,q,k)
```

$$FDQH:=\left(qpochhammer(\gamma q,q,n)\,qpochhammer(q^{-NN},q,n)\,qpochhammer(q^{-n},q,k)\,qpochhammer(q^{-x},q,k)\,qpochhammer(\gamma\delta q^{x+1},q,k)\,q^k\right)\Big/\left(qpochhammer(\gamma q,q,k)\,qpochhammer(q^{-NN},q,k)\,qpochhammer(q,q,k)\right) \tag{19.2}$$

The equation obtained directly by substitution and the deduced interlacing property

```
> subs({beta=0,gamma=q^(-N-1),delta=alpha*delta*q^(N+1)} ,QRreca)
```

$$P(n,\alpha)=P(n,\alpha q)-(q^n-1)(q^{-N-1}q^n-1)\,q\,\alpha\,P(n-1,\alpha q) \quad \textbf{(19.3)}$$

```
> alpha:=0.68;delta:=0.9;q:=0.8;n:=5;N:=12;
```

$$\alpha := 0.68$$
$$\delta := 0.9$$
$$q := 0.8$$
$$n := 5$$
$$N := 12 \quad \textbf{(19.4)}$$

```
> hyp1:= 0<alpha*q and alpha*q<1 and 0<delta*q and delta*q<1 ;
```

$$hyp1 := true \quad \textbf{(19.5)}$$

```
> xnDQH:= sort([solve(qsimpcomb(DQH(n,x,alpha,delta,N,q)),x)]);
```

$$xnDQH := [2.73558240129661, 5.00192649628977, 7.97297187474108, 11.2418817469526, 14.5399420934921] \quad \textbf{(19.6)}$$

```
> xnalphaDQH:= sort([solve(qsimpcomb(DQH(n,x,alpha*q,delta,N,q)),
  x)]);
```

$$xnalphaDQH := [3.10638192815087, 5.49778590093648, 8.38467063036222, 11.4281786699460, 14.5603702071576] \quad \textbf{(19.7)}$$

```
> xnm1alphaDQH:= sort([solve(qsimpcomb(DQH(n-1,x,alpha*q,delta,N,
  q)),x)]);
```

$$xnm1alphaDQH := [3.76067786712151, 6.95389441743998, 10.6887750587670, 14.4230177919045] \quad \textbf{(19.8)}$$

```
> seq(evalb(xnDQH[i]<xnalphaDQH[i] and xnalphaDQH[i]
  <xnm1alphaDQH[i] and xnm1alphaDQH[i]<xnDQH[i+1] and xnDQH[i+1]
  <xnalphaDQH[i+1]),i=1..n-1 )
```

$$true, true, true, true \quad \textbf{(19.9)}$$

```
> unassign('alpha,delta,q,n,N')
```

Proposition 44

```
> DQHreca1:=subs({NN=N,_C1=1},qMixRec2(subs(gamma=alpha,FDQH),q,
  k,P(n),alpha,1,delta,1,0,0)):
> DQHreca:=lhs(DQHreca1)=combine(map(qsimpcomb,rhs(DQHreca1)),
  power)
```

$$DQHreca := P(n,\alpha,\delta q)=P(n,\alpha q,\delta)-\alpha(q^n-1)(-q^{N+1}+q^n)\,P(n-1,\alpha q,\delta)\,q^{-N} \quad \textbf{(19.10)}$$

```
> DQHrecb:=subs(NN=N,qMixRec2(subs(gamma=alpha,FDQH),q,k,P(n),
  alpha,1,delta,1,1,0))
```

$$DQHrecb := P(n,\alpha,\delta q)=-\frac{(-q^{N+3+x}\alpha\delta+q^n)\,q^{-n}\,P(n,\alpha q,\delta q)}{q^{N+3+x}\alpha\delta-1}+\frac{\alpha(-q^{N+2}\delta+q^n)(q^n-1)\,q^{-N-n}\,P(n-1,\alpha q,\delta q)(-q^{N+1}+q^n)}{q^{N+3+x}\alpha\delta-1} \quad \textbf{(19.11)}$$

Theorem 45

```
> alpha:=0.68;delta:=0.9;q:=0.8;n:=5;N:=12;
```

$$\alpha := 0.68$$
$$\delta := 0.9$$

$$q := 0.8$$
$$n := 5$$
$$N := 12 \quad (19.12)$$

```
> hyp1:= 0<alpha*q and alpha*q<1 and 0<delta*q and delta*q<1 ;
```

hyp1 := *true* (19.13)

```
> xndeltaDQH:= sort([solve(qsimpcomb(DQH(n,x,alpha,delta*q,N,q)),
  x)]);
```

xndeltaDQH := [2.61551475793966, 4.90381119933209, 7.90547475609453, 11.2087708584143, 14.5295651689916] (19.14)

```
> xnalphaDQH:= sort([solve(qsimpcomb(DQH(n,x,alpha*q,delta,N,q)),
  x)]);
```

xnalphaDQH := [3.10638192815087, 5.49778590093648, 8.38467063036222, 11.4281786699460, 14.5603702071576] (19.15)

```
> xnm1alphaDQH:= sort([solve(qsimpcomb(DQH(n-1,x,alpha*q,delta,N,
  q)),x)]);
```

xnm1alphaDQH := [3.76067786712151, 6.95389441743998, 10.6887750587670, 14.4230177919045] (19.16)

```
> xnalphadeltaDQH:= sort([solve(qsimpcomb(DQH(n,x,alpha*q,delta*
  q,N,q)),x)]);
```

xnalphadeltaDQH := [3.00352512117531, 5.41753206026659, 8.33347993711190, 11.4061392992799, 14.5533766211195] (19.17)

```
> xnm1alphadeltaDQH:= sort([solve(qsimpcomb(DQH(n-1,x,alpha*q,
  delta*q,N,q)),x)]);
```

xnm1alphadeltaDQH := [3.66002249339528, 6.87907349099463, 10.6464934160999, 14.4095278627431] (19.18)

(a)

```
> seq(evalb(xndeltaDQH[i]<xnalphaDQH[i] and xnalphaDQH[i]
  <xnm1alphaDQH[i] and xnm1alphaDQH[i]<xndeltaDQH[i+1] and
  xndeltaDQH[i+1]<xnalphaDQH[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* (19.19)

(b)

```
> seq(evalb(xndeltaDQH[i]<xnalphadeltaDQH[i] and
  xnalphadeltaDQH[i]<xnm1alphadeltaDQH[i] and
  xnm1alphadeltaDQH[i]<xndeltaDQH[i+1] and xndeltaDQH[i+1]
  <xnalphadeltaDQH[i+1]),i=1..n-1 )
```

true, *true*, *true*, *true* (19.20)

```
> unassign('alpha,delta,q,n,N')
```