```
> restart;
> read `qsum17.mpl`:
```

```
                Package "q-Hypergeometric Summation", Maple V-17
        Copyright 1998-2013, Harald Boeing & Wolfram Koepf, University of Kassel        (1)
```

```
> Digits:=40:
```

Quasi-orthogonality of the classical O.P. of the q-linear and q-quadratic lattice

```
> _qsum_local_specialsolution:= false:
> qMixRec:=proc(F,q,k,Sn,alpha,beta,shift)
            local zeit,pp,qq,rr,Rat,evalS,z,lo,hi,sigma,sigmasol,
  Poly,K,j,J,\
                      PQR,f,rec,S,n,eq;
     zeit:= time();
       lo:=1; hi:=5;
       S:=op(0,Sn); n:=op(1,Sn);
            sigmasol:= NULL;
            for J from 1 to hi while (sigmasol = NULL) do
          Poly:=qsimpcomb(subs({alpha=alpha*q^(shift), beta=beta*
  q^(shift)},F)-add(sigma[j]*subs(n=n-j,F),j=0..J));
                      Rat:= `power/subs`({q^k=K,q^n=N,q^(-n)=1/N}
  ,qratio(Poly,k));
                      if has(Rat,{k,qpochhammer}) then
                                ERROR(`Algorithm not applicable.`);
                      fi;
                      if (J < lo) then next; fi;
          pp:=1: qq:=numer(Rat): rr:=denom(Rat):
                      PQR:= `qgosper/update`(pp,qq,rr,q,K);
                      f:= `qgosper/findf`(op(PQR),q,K,[seq(sigma
  [j],j=0..J)],'sigmasol');
            od;
            if (sigmasol = NULL) then
                  ERROR(cat(`Found no q-derivative rule of order
  smaller than `,J,`.`));
            fi;
       if beta=0 then
       rec:= subs(sigmasol, add(sigma[j]*S(n-j,alpha), j=0..J-1))
  ;
       else
            rec:= subs(sigmasol, add(sigma[j]*S(n-j,alpha,beta),
  j=0..J-1));
       fi;
      rec:=combine(map(factor, subs({N=q^n,K=q^k},rec)),power);
            if (_qsum_profile) then
                      printf(`CPU-time: %.1f seconds`, time()-
  zeit);
            fi;
       if beta=0 then
         eq:=S(n,alpha*q^(shift))=rec
       else
         eq:=S(n,alpha*q^(shift),beta*q^(shift))=rec
```

```
      fi;
            RETURN(eq);
end:
```

the monic big q-Jacobi

The big q-Jacobi polynomials are orthogonal on (gamma*q, alpha*q) for 0<alpha*q<1, 0<beta*q<1, gamma<0

```
> bqj:=(n,x,alpha,beta,gamma,q)->1/(qpochhammer(alpha*beta*q^
  (n+1),q,n)/(qpochhammer(alpha*q, q, n)*qpochhammer(gamma*q,
  q, n)))*add(qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],
  [alpha*q,gamma*q],q,q,j),j=0..n);
```

$$bqj := (n, x, \alpha, \beta, \gamma, q) \rightarrow \frac{1}{qpochhammer(\alpha\,\beta\,q^{n+1}, q, n)} \big(qpochhammer(\alpha\,q, q, n)\ qpochhammer(\gamma\,q, q, n)\ add\big(qphihyperterm\big([q^{-n}, \alpha\,\beta\,q^{n+1}, x], [\alpha\,q, \gamma\,q], q, q, j\big), j = 0\,..n\big)\big) \tag{1.1.1}$$

The summand of the above sum is

```
> Fbqj:=1/(qpochhammer(alpha*beta*q^(n+1),q,n)/(qpochhammer
  (alpha*q, q, n)*qpochhammer(gamma*q, q, n)))*(qphihyperterm(
  [q^(-n),alpha*beta*q^(n+1),x],[alpha*q,gamma*q],q,q,k))
```

$$Fbqj := \big(qpochhammer(\alpha\,q, q, n)\ qpochhammer(\gamma\,q, q, n)\ qpochhammer(q^{-n}, q, k)\ qpochhammer(\alpha\,\beta\,q^{n+1}, q, k)\ qpochhammer(x, q, k)\ q^{k}\big) \Big/ \big(qpochhammer(\alpha\,\beta\,q^{n+1}, q, n)\ qpochhammer(\alpha\,q, q, k)\ qpochhammer(\gamma\,q, q, k)\ qpochhammer(q, q, k)\big) \tag{1.1.2}$$

Proposition 5

```
> eq7a1:=qMixRec(Fbqj,q,k,P(n),alpha,0,-1):
> eq7a2:=subs(_C1=1,eq7a1):
> eq7a:=lhs(eq7a2)=combine(map(qsimpcomb,rhs(eq7a2)),power)
```

$$eq7a := P\left(n, \frac{\alpha}{q}\right) = P(n, \alpha) + \frac{(q^{n}-1)\,(\beta\,q^{n}-1)\,(\gamma\,q^{n}-1)\,q\,\alpha\,P(n-1, \alpha)}{(\alpha\,\beta\,q^{2n}-1)\,(\alpha\,\beta\,q^{2n}-q)} \tag{1.1.3}$$

```
> Eq7a := P(n, alpha/q,beta,gamma) = P(n, alpha,beta,gamma)+
  (q^n-1)*(beta*q^n-1)*(gamma*q^n-1)*q*alpha*P(n-1, alpha,beta,
  gamma)/((alpha*beta*q^(2*n)-1)*(alpha*beta*q^(2*n)-q))
```

$$Eq7a := P\left(n, \frac{\alpha}{q}, \beta, \gamma\right) = P(n, \alpha, \beta, \gamma) + \frac{(q^{n}-1)\,(\beta\,q^{n}-1)\,(\gamma\,q^{n}-1)\,q\,\alpha\,P(n-1, \alpha, \beta, \gamma)}{(\alpha\,\beta\,q^{2n}-1)\,(\alpha\,\beta\,q^{2n}-q)} \tag{1.1.4}$$

Let us check if the above relation is valid for some values of n

```
> checka:=n->qsimpcomb(-bqj(n,x,alpha/q,beta,gamma,q)+bqj(n,x,
  alpha,beta,gamma,q)+(q^n-1)*(beta*q^n-1)*(gamma*q^n-1)*q*
  alpha*bqj(n-1,x,alpha,beta,gamma,q)/((alpha*beta*q^(2*n)-1)*
  (alpha*beta*q^(2*n)-q)))
```

$$checka := n \rightarrow qsimpcomb\Big(-bqj\left(n, x, \frac{\alpha}{q}, \beta, \gamma, q\right) + bqj(n, x, \alpha, \beta, \gamma, q) \tag{1.1.5}$$

$$+\frac{(q^n-1)\,(\beta q^n-1)\,(\gamma q^n-1)\,q\,\alpha\,bqj(n-1,x,\alpha,\beta,\gamma,q)}{(\alpha\beta q^{2n}-1)\,(\alpha\beta q^{2n}-q)}\Bigg)$$

```
> [seq(checka(n),n=0..10)]
```

$$[0,0,0,0,0,0,0,0,0,0,0] \tag{1.1.6}$$

```
> eq7b1:=qMixRec(Fbqj,q,k,P(n),beta,0,-1):
> eq7b2:=subs(_C1=1,eq7b1):
> eq7b:=lhs(eq7b2)=combine(map(qsimpcomb,rhs(eq7b2)),power)
```

$$eq7b := P\left(n,\frac{\beta}{q}\right)=P(n,\beta)-\frac{P(n-1,\beta)\,\alpha\,(q^n-1)\,(\gamma q^n-1)\,(\alpha q^n-1)\,q^{n+1}\beta}{(\alpha\beta q^{2n}-q)\,(\alpha\beta q^{2n}-1)} \tag{1.1.7}$$

```
> Eq7b := P(n,alpha, beta/q,gamma) = P(n,alpha, beta, gamma)-
  alpha*beta*P(n-1, alpha, beta, gamma)*(alpha*q^n-1)*(q^n-1)*
  q^(n+1)*(gamma*q^n-1)/((alpha*beta*q^(2*n)-q)*(alpha*beta*q^
  (2*n)-1))
```

$$Eq7b := P\left(n,\alpha,\frac{\beta}{q},\gamma\right)=P(n,\alpha,\beta,\gamma)-\frac{\alpha\beta P(n-1,\alpha,\beta,\gamma)\,(\alpha q^n-1)\,(q^n-1)\,q^{n+1}\,(\gamma q^n-1)}{(\alpha\beta q^{2n}-q)\,(\alpha\beta q^{2n}-1)} \tag{1.1.8}$$

```
> eq7c1:=qMixRec(Fbqj,q,k,P(n),beta,gamma,-1):
> eq7c2:=subs(_C1=1,eq7c1):
> eq7c:=lhs(eq7c2)=combine(map(qsimpcomb,rhs(eq7c2)),power)
```

$$eq7c := P\left(n,\frac{\beta}{q},\frac{\gamma}{q}\right)=P(n,\beta,\gamma)-\frac{(q^n-1)\,(\alpha q^n-1)\,(-\alpha\beta q^n+\gamma)\,q\,P(n-1,\beta,\gamma)}{(\alpha\beta q^{2n}-1)\,(\alpha\beta q^{2n}-q)} \tag{1.1.9}$$

Corollary 6

```
> eq81:=subs(beta=beta/q,Eq7a):
> eq82:=subs({Eq7b, subs(n=n-1,Eq7b)}, eq81):
> eq8:=lhs(eq82)=combine(map(qsimpcomb,collect(rhs(eq82),[P(n,
  alpha, beta,gamma),P(n-1, alpha, beta,gamma),P(n-2,alpha,
  beta,gamma)] )),power)
```

$$eq8 := P\left(n,\frac{\alpha}{q},\frac{\beta}{q},\gamma\right)=P(n,\alpha,\beta,\gamma)-\frac{P(n-1,\alpha,\beta,\gamma)\,(\alpha\beta q^{2n}-q^{n+1}\beta-\beta q^n+q)\,\alpha\,(q^n-1)\,q\,(\gamma q^n-1)}{(\alpha\beta q^{2n}-q^2)\,(\alpha\beta q^{2n}-1)}-\Big((\gamma q^n-q)\,(q^n-q)\,\alpha^2\,(q^n-1)\,(\gamma q^n-1)\,(\alpha q^n-q)\,(\beta q^n-q)\,P(n-2,\alpha,\beta,\gamma)\,q^{n+3}\beta\Big)\Big/\Big((\alpha\beta q^{2n}-q^2)^2\,(\alpha\beta q^{2n}-q^3)\,(\alpha\beta q^{2n}-q)\Big) \tag{1.1.10}$$

Theorem 8 (We give some specific values to the parameters and do some simulations)

```
> alpha:=1.2;beta:=1.9;gam:=-0.5;q:=0.2;n:=7;
```

$$\alpha := 1.2$$

```
                              β := 1.9
                           gam := -0.5
                              q := 0.2
                              n := 7                                    (1.1.11)
```

```
> hyp1:=alpha>1 and beta>1 and  0 < alpha*q and alpha*q<1 and
  0<= beta*q and beta*q<1 and gam <0 ;
```

```
                          hyp1 := true                                  (1.1.12)
```

x_{n,i} zeros of P_n(alpha, beta, gam)

```
> xnbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha,beta,gam,q)),x)])
  ;
```

```
xnbqj := [-0.09999999999998580653791289890001858341165,            (1.1.13)
    -0.01999995947012877041200347700287100231468,
    -0.003885620221658069627810813425820985589473,
    0.001281746403552302595800449666893326127597,
    0.009596217473257513484289640403872046138906,
    0.04799999999491449471082258018353886272212 3,
    0.2399999999999999998443132524247872421828 1]
```

x_{n-1,i} zeros of P_{n-1}(alpha, beta, gam)

```
> xnm1bqj:= sort([solve(qsimpcomb(bqj(n-1,x,alpha,beta,gam,q)),
  x)]);
```

```
xnm1bqj := [-0.09999999999076334007376757807155547590888 8,        (1.1.14)
    -0.01998962908626907589972734990500008178454,
    -0.002428697809971220933080874503190631136209,
    0.009380326114825333434642687282332142676292,
    0.04799992404337514306910401707651916723028,
    0.2399999999999941627841447439425099180103 1]
```

y_{n,i} zeros of p_n(alpha/q, beta, gam)

```
> ynbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha/q,beta,gam,q)),x)
  ]);
```

```
ynbqj := [-0.09999999999147345830195982153628344361082 1,          (1.1.15)
    -0.01998979552763828422344906664587266648991,
    -0.002433279056457934769963661359961743535023,
    0.009378676517260657930699617405291794558664,
    0.04799992089381603146407689100893030007597,
    0.2399999999999927039380884049730745938981 6,
    1.2000000000000000000001570856810288625 17]
```

z_{n,i} zeros of p_n(alpha,beta/q, gam)

```
> znbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha,beta/q,gam,q)),x)
  ]);
```

```
znbqj := [-0.1000000000000127785652291960762861429623,             (1.1.16)
    -0.01999997482692167106089505773061321380532,
    -0.003893743835336782596387715055170675918373,
    0.001261375639529531464192469785492431258347,
    0.009595543889065026591630303960788324823111,
```

```
    0.047999999902836750102685741500881002299884,
    0.23999999999999999134382216743853894443047 ]
```

v_{n,i} zeros of p_n(alpha/q,beta/q, gam)

```
> vnbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha/q,beta/q,gam,q)),
  x)]);
vnbqj := [ -0.10000000007686432787529506050701831457 69,                    (1.1.17)
    -0.019993618653548516072150942098470615476 61,
    -0.0025380369900527247845767242607402957401 15,
    0.0093414214454540827485208778115864981664 43,
    0.047999849326438819443222807494773743822 60,
    0.23999999999959468434423236424629472253 35,
    1.200000000000000000003735178977440261271 ]
```

w_{n,i} zeros of p_n(alpha,beta/q, gam/q)

```
> wnbqj:= sort([solve(qsimpcomb(bqj(n,x,alpha,beta/q,gam/q,q)),
  x)]);
wnbqj := [ -0.50000000000000000417018470916085376041 11,                    (1.1.18)
    -0.099999999884559592915543860085729954994 08,
    -0.019989206516569756655894358765385118288 78,
    -0.0024176315845903836045177858530496166164 12,
    0.0093841870553486949584082061704286630002 76,
    0.047999930667404850896228787646703252812 88,
    0.23999999999996055512954810458629852449 72 ]
```

8-(i)

```
> evalb(gam*q<xnbqj[1]);seq(evalb(xnbqj[i]<ynbqj[i] and ynbqj
  [i]<xnm1bqj[i] and xnm1bqj[i]<xnbqj[i+1] and  xnbqj[i+1]
  <ynbqj[i+1]) ,i=1..n-1)
                              true
                  true, true, true, true, true, true                        (1.1.19)
```

8--(ii)

```
> znbqj[1]<gam*q and gam*q<xnbqj[1] and xnbqj[1]<xnm1bqj[1];
  seq(evalb(xnm1bqj[i]<znbqj[i+1] and znbqj[i+1]<xnbqj[i+1]) ,
  i=1..n-1);
                              true
                  true, true, true, true, true, true                        (1.1.20)
```

8--(iii)

```
> seq(evalb(wnbqj[i]<xnbqj[i] and xnbqj[i]<xnm1bqj[i] and
  xnm1bqj[i]<wnbqj[i+1] and wnbqj[i+1]<xnbqj[i+1]) ,i=1..n-1);
  evalb(xnbqj[n]<alpha*q);
                  true, true, true, true, true, true
                              true                                          (1.1.21)
```

8--(iV)

```
> evalb(vnbqj[1]<gam*q)
                              true                                          (1.1.22)
```

```
> unassign('alpha,beta,q,n, gamm')
```

the monic q-Hahn

The q-Hahn polynomials are defined by

```
> QH:=(n,x,alpha,beta,N,q)->1/(qpochhammer(alpha*beta*q^(n+1),
  q, n)/(qpochhammer(alpha*q, q, n)*qpochhammer(q^(-N), q, n)))
  *add(qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],[alpha*q,q^
  (-N)],q,q,j),j=0..n);
```

$$QH := (n, x, \alpha, \beta, N, q) \rightarrow \frac{1}{qpochhammer(\alpha \beta q^{n+1}, q, n)} \left(qpochhammer(\alpha q, q, n) \, qpochhammer(q^{-N}, q, n) \, add\left(qphihyperterm\left([q^{-n}, \alpha \beta q^{n+1}, x], [\alpha q, q^{-N}], q, q, j\right), j = 0 ..n\right)\right) \tag{1.2.1}$$

The summand of the above sum is

```
> Fqh:=1/(qpochhammer(alpha*beta*q^(n+1), q, n)/(qpochhammer
  (alpha*q, q, n)*qpochhammer(q^(-NN), q, n)))*(qphihyperterm(
  [q^(-n),alpha*beta*q^(n+1),x],[alpha*q,q^(-NN)],q,q,k)):
```

They are orthogonal for 0<alpha*q<1, 0<beta*q<1 on (1, q^(-N))

Equations (12)

```
> eq12a1:=qMixRec(Fqh,q,k,Q(n),alpha,0,-1):
```

```
> eq12a2:=subs({_C1=1,NN=N},eq12a1):
```

```
> eq12a:=lhs(eq12a2)=combine(map(qsimpcomb,rhs(eq12a2)),power)
```

$$eq12a := Q\left(n, \frac{\alpha}{q}\right) = Q(n, \alpha) + \frac{(q^n - 1) \, (\beta q^n - 1) \, (-q^{N+1} + q^n) \, \alpha \, Q(n-1, \alpha) \, q^{-N}}{(\alpha \beta q^{2n} - 1) \, (\alpha \beta q^{2n} - q)} \tag{1.2.2}$$

```
> Eq12a := Q(n, alpha/q,beta) = Q(n, alpha,beta)+(q^n-1)*(beta*
  q^n-1)*(-q^(N+1)+q^n)*alpha*Q(n-1, alpha, beta)*q^(-N)/(
  (alpha*beta*q^(2*n)-1)*(alpha*beta*q^(2*n)-q))
```

$$Eq12a := Q\left(n, \frac{\alpha}{q}, \beta\right) = Q(n, \alpha, \beta) + \frac{(q^n - 1) \, (\beta q^n - 1) \, (-q^{N+1} + q^n) \, \alpha \, Q(n-1, \alpha, \beta) \, q^{-N}}{(\alpha \beta q^{2n} - 1) \, (\alpha \beta q^{2n} - q)} \tag{1.2.3}$$

```
> eq12b1:=qMixRec(Fqh,q,k,Q(n),beta,0,-1):
```

```
> eq12b2:=subs({_C1=1,NN=N},eq12b1):
```

```
> eq12b:=lhs(eq12b2)=combine(map(qsimpcomb,rhs(eq12b2)),power)
```

$$eq12b := Q\left(n, \frac{\beta}{q}\right) = Q(n, \beta) - \frac{\alpha \, (q^n - 1) \, Q(n-1, \beta) \, (\alpha q^n - 1) \, (-q^{N+1} + q^n) \, q^{n-N} \beta}{(\alpha \beta q^{2n} - q) \, (\alpha \beta q^{2n} - 1)} \tag{1.2.4}$$

```
> Eq12b := Q(n,alpha, beta/q) = Q(n,alpha, beta)+(q^n-1)*alpha*
  (alpha*q^n-1)*beta*Q(n-1,alpha, beta)*(q^(N+1)-q^n)*q^(n-N)/(
  (alpha*beta*q^(2*n)-q)*(alpha*beta*q^(2*n)-1))
```

$$Eq12b := Q\left(n, \alpha, \frac{\beta}{q}\right) = Q(n, \alpha, \beta) \tag{1.2.5}$$

$$+\frac{(q^n-1)\,\alpha\,(\alpha q^n-1)\,\beta\,Q(n-1,\alpha,\beta)\,(q^{N+1}-q^n)\,q^{n-N}}{(\alpha\beta q^{2n}-q)\,(\alpha\beta q^{2n}-1)}$$

Corollary 10

```
> eq131:=subs(beta=beta/q,Eq12a):
```

```
> eq132:=subs({Eq12b, subs(n=n-1,Eq12b)}, eq131):
```

```
> eq13:=lhs(eq132)=combine(map(qsimpcomb,collect(rhs(eq132),[Q
  (n, alpha, beta),Q(n-1, alpha, beta),Q(n-2, alpha, beta)] )),
  power)
```

$$eq13 := Q\left(n,\frac{\alpha}{q},\frac{\beta}{q}\right)=Q(n,\alpha,\beta) \tag{1.2.6}$$

$$-\frac{Q(n-1,\alpha,\beta)\,(\alpha\beta q^{2n}-q^{n+1}\beta-\beta q^n+q)\,\alpha\,(q^n-1)\,(-q^{N+1}+q^n)\,q^{-N}}{(\alpha\beta q^{2n}-q^2)\,(\alpha\beta q^{2n}-1)}$$

$$-\Big(\big(q^n-q\big)\,\alpha^2\,(q^n-1)\,(-q^{N+2}+q^n)\,(\alpha q^n-q)\,(\beta q^n-q)\,(-q^{N+1}$$
$$+q^n)\,Q(n-2,\alpha,\beta)\,q^{n+1-2N}\beta\big)\Big/\Big((\alpha\beta q^{2n}-q^2)^2\,(\alpha\beta q^{2n}-q^3)\,(\alpha\beta q^{2n}$$
$$-q)\Big)$$

Theorem 12

```
> alpha:=1.5;beta:=1.9;N:=10;q:=0.3;n:=7;
```

$$\alpha := 1.5$$
$$\beta := 1.9$$
$$N := 10$$
$$q := 0.3$$
$$n := 7 \tag{1.2.7}$$

```
> hyp1:= alpha>1 and beta>1 and 0 < alpha*q and alpha*q<1 and
  0< beta*q and beta*q<1 and n <=N;
```

$$hyp1 := true \tag{1.2.8}$$

zeros of Q(n,alpha, beta)

```
> xnqh:= sort([solve(qsimpcomb(QH(n,x,alpha,beta,N,q)),x)]);
```

$$xnqh := [57.722588817450312989370205961515727314 31,$$
$$398.79303250546653482708349155527932979 46,$$
$$1371.5338473662950265900073711205641453 69,$$
$$4572.4734638555365719617718559141283323 15,$$
$$15241.579027564451842398617684372748499 43,$$
$$50805.263425290859979513097425647648846 33,$$
$$1.6935087808430286711036593030567357595 17\ 10^5] \tag{1.2.9}$$

zeros of Q(n-1,alpha, beta)

```
> xnm1qh:= sort([solve(qsimpcomb(QH(n-1,x,alpha,beta,N,q)),x)])
  ;
```

$$xnm1qh := [191.33237301392575455108976977762766472 74, \tag{1.2.10}$$
$$1329.2672805170317728612249574679827659 56,$$
$$4571.7894142345372951043338675322066442 86,$$

```
    15241.57826827690287658257119147149008493,
    50805.26342523295323316088395646194989914,
    1.693508780843028669547542405392142642043 10^5]
```

zeros of Q(n,alpha/q, beta)

```
> ynqh:= sort([solve(qsimpcomb(QH(n,x,alpha/q,beta,N,q)),x)]);
ynqh := [-35.20314879465939054195602223073730375478,                    (1.2.11)
    314.3342452550018039358514392944702514225,
    1365.377925114486001864377711340149245914,
    4572.445108537253801404793792787833749449,
    15241.57901834101758537099135174359376873,
    50805.26342529065023118977420705731870024,
    1.693508780843028671101971295534509108119 10^5]
```

zeros of Q(n,alpha, beta/q)

```
> znqh:= sort([solve(qsimpcomb(QH(n,x,alpha,beta/q,N,q)),x)]);
znqh := [57.75150285832352999087749514620657536899,                    (1.2.12)
    398.8374140543768391825190150602181735850,
    1371.536824503312910327423492323197994944,
    4572.473476148766347458014432450919513491,
    15241.57902756833194451963294166980618496,
    50805.26342529086006699445023739199492260,
    1.693508780843028671103660005297964123852 10^5]
```

zeros of Q(n,alpha/q, beta/q)

```
> Znqh:= sort([solve(qsimpcomb(QH(n,x,alpha/q,beta/q,N,q)),x)])
  ;
Znqh := [-35.25095571681310598018704164582760607786,                    (1.2.13)
    314.5536457390290933388627912656832903087,
    1365.459021600899896347854627899691942929,
    4572.446505985168543862222852985559026350,
    15241.57901990283135814073174050932340061,
    50805.26342529076964932486568946016049126,
    1.693508780843028671105183009961057245657 10^5]
```

12--(i)

```
> evalb(ynqh[1]<1 and 1<xnqh[1] and xnqh[1]<xnm1qh[1]);
  seq(evalb(xnm1qh[i]<ynqh[i+1] and  ynqh[i+1]<xnqh[i+1]),i=1..
  n-1);
  evalb(xnqh[n]<q^(-N) )
                              true
                  true, true, true, true, true, true
                              true                                    (1.2.14)
```

12--(ii)

```
> evalb(1<xnqh[1]);
  seq(evalb(xnqh[i]<znqh[i] and  znqh[i]<xnm1qh[i] and xnm1qh
  [i]<xnqh[i+1]),i=1..n-1);
  evalb(xnqh[n]<q^(-N) and q^(-N)<znqh[n] )
```

true

true, true, true, true, true, true

true **(1.2.15)**

Theorem 14

```
> evalb(Znqh[1]<1 and q^(-N)<Znqh[n])
```

true **(1.2.16)**

```
> unassign('alpha,beta,N,q,n')
```

the monic little q-jacobi

The little q-Jacobi polynomials are given by

```
> LQJ:=(n,x,alpha,beta,q)->1/((-1)^n*q^(-binomial(n,2))*
  qpochhammer(alpha*beta*q^(n+1), q, n)/qpochhammer(alpha*q, q,
  n))*add(qphihyperterm([q^(-n),alpha*beta*q^(n+1)],[alpha*q],
  q,q*x,j),j=0..n);
```

$$LQJ := \left(n, x, \alpha, \beta, q\right) \rightarrow \left(qpochhammer\left(\alpha\, q, q, n\right)\, add\left(qphihyperterm\left(\left[q^{-n}, \alpha\,\beta\, q^{n+1}\right], \left[\alpha\, q\right], q, q\, x, j\right), j=0\,..n\right)\right) \Big/ \left(\left(-1\right)^{n} q^{-\mathrm{binomial}(n,\, 2)}\, qpochhammer\left(\alpha\,\beta\, q^{n+1}, q, n\right)\right) \tag{1.3.1}$$

```
> Flqj:=1/((-1)^n*q^(-binomial(n,2))*qpochhammer(alpha*beta*q^
  (n+1), q, n)/qpochhammer(alpha*q, q, n))*(qphihyperterm([q^(-
  n),alpha*beta*q^(n+1)],[alpha*q],q,q*x,k)):
```

orthogonal for 0<alpha*q<1 and beta*q<1 on (0,1)

Equations (14)

```
> eq14a1:=qMixRec(Flqj,q,k,p(n),alpha,0,-1):
> eq14a2:=subs(_C1=1,eq14a1):
> eq14a:=lhs(eq14a2)=combine(map(qsimpcomb,rhs(eq14a2)),power)
```

$$eq14a := p\left(n, \frac{\alpha}{q}\right) = p\left(n, \alpha\right) + \frac{p\left(n-1, \alpha\right)\,\left(q^{n}-1\right)\,\left(\beta\, q^{n}-1\right)\,\alpha\, q^{n}}{\left(\alpha\,\beta\, q^{2\,n}-1\right)\,\left(\alpha\,\beta\, q^{2\,n}-q\right)} \tag{1.3.2}$$

```
> Eq14a  := p(n, alpha/q,beta) = p(n, alpha,beta)+p(n-1, alpha,
  beta)*(q^n-1)*(beta*q^n-1)*alpha*q^n/((alpha*beta*q^(2*n)-1)*
  (alpha*beta*q^(2*n)-q))
```

$$Eq14a := p\left(n, \frac{\alpha}{q}, \beta\right) = p\left(n, \alpha, \beta\right) + \frac{p\left(n-1, \alpha, \beta\right)\,\left(q^{n}-1\right)\,\left(\beta\, q^{n}-1\right)\,\alpha\, q^{n}}{\left(\alpha\,\beta\, q^{2\,n}-1\right)\,\left(\alpha\,\beta\, q^{2\,n}-q\right)} \tag{1.3.3}$$

```
> eq14b1:=qMixRec(Flqj,q,k,p(n),beta,0,-1):
> eq14b2:=subs(_C1=1,eq14b1):
> eq14b:=lhs(eq14b2)=combine(map(qsimpcomb,rhs(eq14b2)),power)
```

$$eq14b := p\left(n, \frac{\beta}{q}\right) = p\left(n, \beta\right) - \frac{p\left(n-1, \beta\right)\,\left(q^{n}-1\right)\,\left(\alpha\, q^{n}-1\right)\,\alpha\,\beta\, q^{2\,n}}{\left(\alpha\,\beta\, q^{2\,n}-1\right)\,\left(\alpha\,\beta\, q^{2\,n}-q\right)} \tag{1.3.4}$$

```
> Eq14b  := p(n,alpha, beta/q)  = p(n,alpha, beta)-p(n-1,alpha,
  beta)*(q^n-1)*(alpha*q^n-1)*alpha*beta*q^(2*n)/((alpha*beta*
```

```
q^(2*n)-1)*(alpha*beta*q^(2*n)-q));
```

$$Eq14b := p\left(n, \alpha, \frac{\beta}{q}\right) = p(n, \alpha, \beta) - \frac{p(n-1, \alpha, \beta)\,(q^n-1)\,(\alpha q^n-1)\,\alpha \beta q^{2n}}{(\alpha \beta q^{2n}-1)\,(\alpha \beta q^{2n}-q)} \tag{1.3.5}$$

```
> eq151:=subs(beta=beta/q,Eq14a):
> eq152:=subs({Eq14b, subs(n=n-1,Eq14b)}, eq151):
> eq15:=lhs(eq152)=combine(map(qsimpcomb,collect(rhs(eq152),[p
  (n, alpha, beta),p(n-1, alpha, beta),p(n-2, alpha, beta)] )),
  power)
```

$$eq15 := p\left(n, \frac{\alpha}{q}, \frac{\beta}{q}\right) = p(n, \alpha, \beta) - \frac{p(n-1, \alpha, \beta)\,(\alpha \beta q^{2n} - q^{n+1}\beta - \beta q^n + q)\,(q^n-1)\,\alpha q^n}{(\alpha \beta q^{2n}-q^2)\,(\alpha \beta q^{2n}-1)} - \frac{p(n-2, \alpha, \beta)\,(q^n-q)\,\alpha^2\,(q^n-1)\,(\alpha q^n-q)\,(\beta q^n-q)\,q^{3n+1}\beta}{(\alpha \beta q^{2n}-q^2)^2\,(\alpha \beta q^{2n}-q^3)\,(\alpha \beta q^{2n}-q)} \tag{1.3.6}$$

```
> eqA1:=subs(alpha=alpha/q,eq14a):
> eqA2:=subs({eq14a, subs(n=n-1,eq14a)}, eqA1):
> eq6A:=lhs(eqA2)=combine(map(qsimpcomb,collect(rhs(eqA2),[p(n,
  alpha),p(n-1, alpha),p(n-2, alpha)] )),power)
```

$$eq6A := p\left(n, \frac{\alpha}{q^2}\right) = p(n, \alpha) + \frac{p(n-1, \alpha)\,(q+1)\,(q^n-1)\,(\beta q^n-1)\,\alpha q^n}{(\alpha \beta q^{2n}-q^2)\,(\alpha \beta q^{2n}-1)} + \frac{(q^n-q)\,\alpha^2\,(q^n-1)\,(\beta q^n-1)\,(\beta q^n-q)\,p(n-2, \alpha)\,q^{2n+2}}{(\alpha \beta q^{2n}-q^2)^2\,(\alpha \beta q^{2n}-q^3)\,(\alpha \beta q^{2n}-q)} \tag{1.3.7}$$

Theorem 18 and 19

```
> alpha:=3;beta:=2;q:=0.2;n:=7;
```

$$\alpha := 3$$

$$\beta := 2$$

$$q := 0.2$$

$$n := 7 \tag{1.3.8}$$

```
> hyp1:= alpha>1 and beta>1 and 0 < alpha*q and alpha*q<1 and
  0<beta*q and  beta*q<1;
```

$$hyp1 := true \tag{1.3.9}$$

zeros of P(n, alpha, beta)

```
> xnlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta,q)),x)]);
```

$$\begin{aligned} xnlqj := [&0.000022292347606588274994959811882225678204 32,\\ &0.00031374586475753210366908746158341286366 94,\\ &0.00159996363800502486350903846172431474191 4,\\ &0.00799999999276799664693722863536520490542 8,\\ &0.03999999999999999453627820148215955497917 0,\\ &0.19999999999999999999998515306002252990 12,\\ &0.99999999999999999999999999999999912402269 9] \end{aligned} \tag{1.3.10}$$

zeros of P(n-1, alpha, beta)

```
> xnm1lqj:= sort([solve(qsimpcomb(LQJ(n-1,x,alpha,beta,q)),x)])
  ;
xnm1lqj := [0.0001114727937494773936397869979311360177457,              (1.3.11)
    0.0015687542806908459464374196615062063592 90,
    0.0079998190329989919317740581957218233924 74,
    0.0399999999646977383120822525834162360280 6,
    0.1999999999999999758684936419011100025055 7,
    0.9999999999999999999999643618691962886320]
```

zeros of P_n(alpha/q, beta)

```
> ynlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha/q,beta,q)),x)]);
ynlqj := [-0.0000656620904153779588663555473532809153 2038,               (1.3.12)
    0.0002135743121307239382725776645799864820626,
    0.0015961013830601715831972393029444697640 01,
    0.0079999956101360136204143857189470029415 24,
    0.0399999999998302231410236969734053674434 2,
    0.1999999999999999999768257132444726961679 0,
    0.9999999999999999999999999931572159139865]
```

zeros of P_n(alpha, beta/q)

```
> znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha,beta/q,q)),x)]);
znlqj := [0.0000222938217073478256830851217548566230 1969,               (1.3.13)
    0.0003137491929757458596009569105032492846880,
    0.0015999637505194417603932427367232511345 76,
    0.0079999999928829752430679219466923999315 55,
    0.0399999999999999949728605357404032182063 89,
    0.1999999999999999999999910909235241571312,
    1.000000000000000000000000000000087606701]
```

zeros of P_n(alpha/q, beta/q)

```
> Znlqj:= sort([solve(qsimpcomb(LQJ(n,x,alpha/q,beta/q,q)),x)])
  ;
Znlqj := [-0.0000656778257020198502774914613287267763 4377,              (1.3.14)
    0.0002136037148332339828581704128640499012752,
    0.0015961122181462906541579984945130740818 84,
    0.0079999567858812848369966712705398823081 4,
    0.0399999999999843741186257944481222701688 41,
    0.1999999999999999998608972873292667080335,
    1.000000000000000000000000000684558810628 2]
```

18--(i)

```
> evalb(ynlqj[1]<0 and 0<xnlqj[1] and xnlqj[n]<1);
  seq(evalb(xnlqj[i]<xnm1lqj[i] and xnm1lqj[i]<ynlqj[i+1] and
  ynlqj[i+1]<xnlqj[i+1]),i=1..n-1)
                              true
                true, true, true, true, true, true                 (1.3.15)
```

18--(ii)

```
> evalb( 0<xnlqj[1] and xnlqj[n]<1 and 1<znlqj[n]);
```

```
seq(evalb(xnlqj[i]<znlqj[i] and znlqj[i]<xnm1lqj[i] and
xnm1lqj[i]<xnlqj[i+1]),i=1..n-1)
```

$$\mathit{true}$$

$$\mathit{true}, \mathit{true}, \mathit{true}, \mathit{true}, \mathit{true}, \mathit{true} \tag{1.3.16}$$

Theo. 19

```
> evalb(Znlqj[1]<0 and 1<Znlqj[n])
```

$$\mathit{true} \tag{1.3.17}$$

```
> unassign('alpha,beta,q,n')
```

the monic q-Laguerre

```
> QL:=(n,x,alpha,q)->1/((-1)^n*q^(n*(n+alpha))/qpochhammer(q,q,
  n))*qpochhammer(q^(alpha+1),q,n)/qpochhammer(q,q,n)*add
  (qphihyperterm([q^(-n)],[q^(alpha+1)],q,-q^(n+alpha+1)*x,j) ,
  j=0..n);
```

$$QL := (n, x, \alpha, q) \rightarrow \frac{1}{(-1)^n q^{n(n+\alpha)}\, \mathit{qpochhammer}(q, q, n)} \Big(1\, \mathit{qpochhammer}(q, q, n)\, \mathit{qpochhammer}\big(q^{\alpha+1}, q, n\big)\, \mathit{add}\Big(\mathit{qphihyperterm}\big([q^{-n}], [q^{\alpha+1}], q, -q^{n+\alpha+1} x, j\big), j = 0..n\Big)\Big) \tag{1.4.1}$$

```
> Fql:=1/((-1)^n*q^(n^2)*(q^(alpha))^n/qpochhammer(q,q,n))*
  qpochhammer(q^(alpha)*q,q,n)/qpochhammer(q,q,n)*
  (qphihyperterm([q^(-n)],[q^(alpha)*q],q,-q^(n+1)*q^alpha*x,k)
  )
```

$$Fql := \Big(\mathit{qpochhammer}\big(q^{\alpha} q, q, n\big)\, \mathit{qpochhammer}\big(q^{-n}, q, k\big)\, \big(-q^{n+1} q^{\alpha} x\big)^k (-1)^k q^{\frac{1}{2} k(k-1)}\Big) \Big/ \Big((-1)^n q^{n^2} \big(q^{\alpha}\big)^n\, \mathit{qpochhammer}\big(q^{\alpha} q, q, k\big)\, \mathit{qpochhammer}(q, q, k)\Big) \tag{1.4.2}$$

Orthogonal for alpha>-1 on (0,infinity)

Equation (16)

```
> eq16a1:=qMixRec(subs(q^alpha=alpha,Fql),q,k,L(n),alpha,0,-1):
> eq16a2:=subs({_C1=1,alpha=q^alpha},eq16a1):
> eq16a:=combine(lhs(eq16a2),power)=combine(map(qsimpcomb,rhs
  (eq16a2)),power)
```

$$eq16a := L\big(n, q^{\alpha-1}\big) = L\big(n, q^{\alpha}\big) - \big(q^n - 1\big)\, L\big(n-1, q^{\alpha}\big)\, q^{1-2n-\alpha} \tag{1.4.3}$$

Note that we did the substitution subs(q^alpha=alpha,Fql) since alpha is a power of q.

Theorem 22

```
> alpha:=-0.7;q:=0.15;n:=10;
```

$$\alpha := -0.7$$

$$q := 0.15$$

$$n := 10 \tag{1.4.4}$$

```
> hyp1:= evalb(-1<alpha and alpha<0);
```

$$\mathit{hyp1} := \mathit{true} \tag{1.4.5}$$

zeros of P_n(alpha)

```
> xnql:= sort([solve(qsimpcomb(QL(n,x,alpha,q)),x)]);
```

```
xnql := [0.6578899019948919167004144100965197621680,                    (1.4.6)
    71.06425805988116912060150240548515513829,
    3440.136124311764226002403829511443194323,
    1.547734972385400953181510893480156043499 10^5,
    6.891403547664555681647582314788984446940 10^6,
    3.063855753542152113436847732273396583070 10^8,
    1.362287784995561322513506834576046772064 10^10,
    6.070056260440730649479934927895159787123 10^11,
    2.744361221686144088352892738683878770035 10^13,
    1.378342453283118046851755172188793924527 10^15]
```

zeros of P_{n-1}(alpha)

```
> xnm1ql:= sort([solve(qsimpcomb(QL(n-1,x,alpha,q)),x)]);
```

```
xnm1ql := [0.6578899223317822601976374999933582897918,                  (1.4.7)
    71.06427176219330563458972532863058300386,
    3440.140542792981743241278058003204972463,
    1.547748224922997229437295754273305249098 10^5,
    6.891796955947361759922816797231005939414 10^6,
    3.065022235928457542802550583476022119832 10^8,
    1.365754285332968907610002380408792670765 10^10,
    6.174807070507187375842184963762187053552 10^11,
    3.101270092929366394161910643249561601702 10^13]
```

zeros of P_n(alpha-1)

```
> ynql:= sort([solve(qsimpcomb(QL(n,x,alpha-1,q)),x)]);
```

```
ynql := [-0.5615546448834409798648978655455933900822,                   (1.4.8)
    5.618297398415111113697270177200419840639,
    482.6086188428850340489145522166910617559,
    22993.29500428060601182944409725564965392,
    1.032225657802078298582141137089443095248 10^6,
    4.594793657604071056612825081654451611305 10^7,
    2.043365654186998767077452924803492630582 10^9,
    9.105040263159158057328882943039137001082 10^10,
    4.116538840045772770966349591958222575829 10^12,
    2.067513454916011967627127069082170175683 10^14]
```

Theorem 22

```
> evalb(ynql[1]<0 and 0<xnql[1]);
  seq(evalb(xnql[i]<xnm1ql[i] and xnm1ql[i]<ynql[i+1] and ynql
```

```
[i+1]<xnql[i+1]), i=1..n-1)
```

$$true$$

$$true, true, true, true, true, true, true, true, true \tag{1.4.9}$$

```
> unassign('alpha,q,n')
```

the monic Al-Salam-Carlitz I

```
> ASCI:=(n,x,alpha,q)->(-alpha)^n*q^(binomial(n,2))*add
  (qphihyperterm([q^(-n),1/x],[0],q,q*x/alpha,j),j=0..n);
```

$$ASCI := (n, x, \alpha, q) \rightarrow (-\alpha)^n q^{\mathrm{binomial}(n,\,2)} \, add\left(qphihyperterm\left(\left[q^{-n}, \frac{1}{x}\right], [0], q, \frac{q\,x}{\alpha}, j\right), j = 0 .. n\right) \tag{1.5.1}$$

```
> Fasc1:=(-alpha)^n*q^(binomial(n,2))*(qphihyperterm([q^(-n),
  1/x],[0],q,q*x/alpha,k)):
```

Orthogonal for alpha<0 on (alpha,1) subset of (alpha/q, 1)

Equations (17) and (18)

```
> eq171:=qMixRec(Fasc1,q,k,U(n),alpha,0,-1):
> eq172:=subs(_C1=1,eq171):
> eq17:=lhs(eq172)=combine(map(qsimpcomb,rhs(eq172)),power)
```

$$eq17 := U\left(n, \frac{\alpha}{q}\right) = U(n, \alpha) + \frac{\alpha\,(q^n - 1)\,U(n-1, \alpha)}{q} \tag{1.5.2}$$

```
> eq181:=subs(alpha=alpha/q,eq17):
> eq182:=subs({eq17, subs(n=n-1,eq17)}, eq181):
> eq18:=lhs(eq182)=combine(map(qsimpcomb,collect(rhs(eq182),[U
  (n, alpha),U(n-1, alpha),U(n-2, alpha)] )),power)
```

$$eq18 := U\left(n, \frac{\alpha}{q^2}\right) = U(n, \alpha) + \frac{\alpha\,(q^n - 1)\,(q+1)\,U(n-1, \alpha)}{q^2} + \frac{\alpha^2\,(q^n - 1)\,(q^n - q)\,U(n-2, \alpha)}{q^4} \tag{1.5.3}$$

Theorem 25

```
> alpha:=-1;q:=0.9;n:=7;
```

$$\alpha := -1$$

$$q := 0.9$$

$$n := 7 \tag{1.5.4}$$

```
> hyp1:= evalb(alpha<0 and alpha<q^n/(q^n-1));
```

$$hyp1 := true \tag{1.5.5}$$

zeros of P_n(alpha)

```
> xnasc1:= sort([solve(qsimpcomb(ASCI(n,x,alpha,q)),x)]);
```

$$xnasc1 := [-0.895898604242127682380288719632339755680 3, \tag{1.5.6}$$

$$-0.6197941772155604635718920413472472029792,$$

```
    -0.31590011667724010101532058202239491782 10,
    1.5925159975957603821281138947356241504 73 10^-37,
    0.31590011667724010101532058202239491731 48,
    0.61979417721556046357189204134724720367 41,
    0.89589860424212768238028871963233975532 63 ]
```

zeros of P_{n-1}(alpha)

```
> xnm1asc1:= sort([solve(qsimpcomb(ASCI(n-1,x,alpha,q)),x)]);
xnm1asc1 := [-0.84197863850921585162002640349509257110 09,                (1.5.7)
    -0.51978070377798047866741808160422097068 20,
    -0.17547821310417577733840668273521041599 97,
    0.17547821310417577733840668273521041586 70,
    0.51978070377798047866741808160422097100 45,
    0.84197863850921585162002640349509257090 61 ]
```

zeros of P_n(alpha/q)

```
> ynasc1:= sort([solve(qsimpcomb(ASCI(n,x,alpha/q,q)),x)]);
ynasc1 := [-1.0195244764234855575820078262969079607 47,                   (1.5.8)
    -0.73664317715837606447190826092880597046 81,
    -0.42054176452326337672366646600327043185 22,
    -0.08882549202285206278786551033753892377 424,
    0.24561335010260756323399078754438325012 59,
    0.57036798756998057635856835759172414956 32,
    0.86988346134427781086177780731930477603 59 ]
```

zeros of P_n(alpha/q^2)

```
> znasc1:= sort([solve(qsimpcomb(ASCI(n,x,alpha/q^2,q)),x)]);
znasc1 := [-1.1549083717217889203173869822445823199 72,                   (1.5.9)
    -0.86494085871547276853124454794027882491 84,
    -0.53599767955367818535021437548490637959 90,
    -0.18749027777459546004636909004463907115 92,
    0.16675802252926035141654380418498335331 16,
    0.51394147499322158682407987227437671303 44,
    0.83888967789737438365891230690936751695 07 ]
```

25--(i)

```
> evalb(ynasc1[1]<alpha and alpha<xnasc1[1] and xnasc1[n]<1);
  seq(evalb(xnasc1[i]<xnm1asc1[i] and xnm1asc1[i]<ynasc1[i+1]),
  i=1..n-1)
                              true
                 true, true, true, true, true, true                      (1.5.10)
```

```
> unassign('alpha,q,n')
```

the monic Askey-Wilson

```
> AW:=(n,x,a,b,c,d,q)->qpochhammer(a*b,q,n)*qpochhammer(a*c,q,
  n)*qpochhammer(a*d,q,n)/(2*a)^n /qpochhammer(a*b*c*d*q^(n-1),
  q,n)  *add(qpochhammer(q^(-n),q,k)*qpochhammer(a*b*c*d*q^
  (n-1),q,k)*mul(1-2*a*x*q^j+a^2*q^(2*j),j=0..k-1)*q^k/
  (qpochhammer(a*b,q,k)*qpochhammer(a*c,q,k)*qpochhammer(a*d,q,
  k)*qpochhammer(q,q,k) ), k=0..n);
```

$$AW := (n, x, a, b, c, d, q) \rightarrow \frac{1}{(2\,a)^{n}\,\mathit{qpochhammer}\left(a\,b\,c\,d\,q^{n-1}, q, n\right)} \Big(\mathit{qpochhammer}(a\,b, q, n)\,\mathit{qpochhammer}(a\,c, q, n)\,\mathit{qpochhammer}(a\,d, q, n)\,\mathit{add}\Big(\left(\mathit{qpochhammer}\left(q^{-n}, q, k\right)\,\mathit{qpochhammer}\left(a\,b\,c\,d\,q^{n-1}, q, k\right)\,\mathit{mul}\left(1-2\,a\,x\,q^{j}+a^{2}\,q^{2j}, j=0\,..k-1\right)\,q^{k}\right) \Big/ \left(\mathit{qpochhammer}(a\,b, q, k)\,\mathit{qpochhammer}(a\,c, q, k)\,\mathit{qpochhammer}(a\,d, q, k)\,\mathit{qpochhammer}(q, q, k)\right), k=0\,..n\Big)\Big) \tag{1.6.1}$$

```
> FAW:=qpochhammer(a*b,q,n)*qpochhammer(a*c,q,n)*qpochhammer(a*
  d,q,n)/(2*a)^n /qpochhammer(a*b*c*d*q^(n-1),q,n)  * qhyperterm
  ([q^(-n),a*b*c*d*q^(n-1),a*exp(I*theta),a*exp(-I*theta) ],[a*
  b,a*c,a*d],q,q,k)
```

$$FAW := \Big(\mathit{qpochhammer}(a\,b, q, n)\,\mathit{qpochhammer}(a\,c, q, n)\,\mathit{qpochhammer}(a\,d, q, n)\,\mathit{qpochhammer}\left(q^{-n}, q, k\right)\,\mathit{qpochhammer}\left(a\,b\,c\,d\,q^{n-1}, q, k\right)\,\mathit{qpochhammer}\left(a\,\mathrm{e}^{\mathrm{I}\theta}, q, k\right)\,\mathit{qpochhammer}\left(a\,\mathrm{e}^{-\mathrm{I}\theta}, q, k\right)\,q^{k}\Big) \Big/ \Big((2\,a)^{n}\,\mathit{qpochhammer}\left(a\,b\,c\,d\,q^{n-1}, q, n\right)\,\mathit{qpochhammer}(a\,b, q, k)\,\mathit{qpochhammer}(a\,c, q, k)\,\mathit{qpochhammer}(a\,d, q, k)\,\mathit{qpochhammer}(q, q, k)\Big) \tag{1.6.2}$$

orthogonal on (-1,1) for max(|a|, |b|, |c|, |d|)<1.

```
> Awrec1:=qMixRec(FAW,q,k,P(n),a,0,-1)
```

$$\mathit{Awrec1} := P\left(n, \frac{a}{q}\right) = P(n, a) - \frac{1}{2}\,\frac{q\,a\,(q^{n}-1)\,(c\,d\,q^{n}-q)\,(q^{n}\,b\,d-q)\,(q^{n}\,b\,c-q)\,P(n-1, a)}{(a\,b\,c\,d\,q^{2n}-q^{3})\,(a\,b\,c\,d\,q^{2n}-q^{2})} \tag{1.6.3}$$

```
> Awrec2:=subs(_C1=1 ,qMixRec(FAW,q,k,P(n),b,0,-1)):
> lhs(Awrec2)=combine(map(qsimpcomb,rhs(Awrec2)),power)
```

$$P\left(n, \frac{b}{q}\right) = P(n, b) - \frac{1}{2}\,\frac{(q^{n}-1)\,(c\,d\,q^{n}-q)\,(a\,d\,q^{n}-q)\,(a\,c\,q^{n}-q)\,q\,b\,P(n-1, b)}{(a\,b\,c\,d\,q^{2n}-q^{3})\,(a\,b\,c\,d\,q^{2n}-q^{2})} \tag{1.6.4}$$

```
> Awrec3:=subs(_C1=1 ,qMixRec(FAW,q,k,P(n),c,0,-1)):
> lhs(Awrec3)=combine(map(qsimpcomb,rhs(Awrec3)),power)
```

$$P\left(n, \frac{c}{q}\right) = P(n, c) - \frac{1}{2}\,\frac{(q^{n}-1)\,(q^{n}\,b\,d-q)\,(a\,d\,q^{n}-q)\,(a\,b\,q^{n}-q)\,q\,c\,P(n-1, c)}{(a\,b\,c\,d\,q^{2n}-q^{3})\,(a\,b\,c\,d\,q^{2n}-q^{2})} \tag{1.6.5}$$

```
> Awrec4:=subs(_C1=1 ,qMixRec(FAW,q,k,P(n),d,0,-1)):
> lhs(Awrec4)=combine(map(qsimpcomb,rhs(Awrec4)),power)
```

(1.6.6)

$$P\left(n, \frac{d}{q}\right) = P(n, d) - \frac{1}{2} \frac{\left(q^n - 1\right) \left(q^n b c - q\right) \left(a c q^n - q\right) \left(a b q^n - q\right) q d P(n-1, d)}{\left(a b c d q^{2n} - q^3\right) \left(a b c d q^{2n} - q^2\right)} \quad \textbf{(1.6.6)}$$

Theorem 27

(i)

```
> a:=0.95;b:=-0.68;c:=0.75;d:=-0.9;q:=0.8;n:=10;
                                  a := 0.95
                                  b := -0.68
                                  c := 0.75
                                  d := -0.9
                                  q := 0.8
                                  n := 10                                    (1.6.7)
> hyp1:= q<abs(a) and abs(a)<1 and abs(b)<1 and abs(c)<1 and
  abs(d)<1 ;
                                hyp1 := true                                 (1.6.8)
```

zeros of P(n,a,b,c,d)

```
> xnAW:= sort([solve(qsimpcomb(AW(n,x,a,b,c,d,q)),x)]);
xnAW := [-0.9425413838594203121535161976763897983078,                          (1.6.9)
    -0.8067846865499503866302295417194399208386,
    -0.6097854765030745328965309285711740295026,
    -0.3667253931676475264675533550526031444816,
    -0.09610895079636025104321695376444652220819,
    0.1814888151792465252702485684657414144575,
    0.4449683561574630874812979936000289905766,
    0.6743056767075273527576099093689180380549,
    0.8520751617738030772670578293491709305034,
    0.9647818480041536937885633154065789622065]
```

zeros of P(n-1,a,b,c,d)

```
> xnm1AW:= sort([solve(qsimpcomb(AW(n-1,x,a,b,c,d,q)),x)]);
xnm1AW := [-0.9299681494072114003646970856026568281517,                     (1.6.10)
    -0.7680730943202930137756280897964190836595,
    -0.5363102006064754850890187580843927494130,
    -0.2560073340344280838522722417832480716865,
    0.04722777916713101569277077201542788557385,
    0.3457149809900441642139037690107093384422,
    0.6122131709447061401496635819131562021795,
    0.8224056136260775920370564268053588160975,
    0.9571381793081005544074732371837556664977]
```

zeros of P(n,a/q,b,c,d)

```
> ynAW:= sort([solve(qsimpcomb(AW(n,x,a/q,b,c,d,q)),x)]);
ynAW := [-0.9362424006737725721236944276657156249330,                       (1.6.11)
```

```
     -0.78727484295142913643313451319886963028886,
     -0.57247458098420186476175167048085073847775,
     -0.30988516886137022679811783232441161200011,
     -0.021359006647054154573606244512258131683 17,
     0.26911822086994691419740798811809724454 77,
     0.53740560928624796659322508662378729899 30,
     0.76121417729714395893195473286184260767 85,
     0.92199464165199219677628100065159095735 14,
     1.01018043693341049539882273477577116693 3]
```

```
> seq(evalb(xnAW[i]<ynAW[i] and ynAW[i]<xnm1AW[i] and xnm1AW[i]
  <xnAW[i+1] and xnAW[i+1]<ynAW[i+1]),i=1..n-1 )
                 true, true, true, true, true, true, true, true, true          (1.6.12)
```

(ii)

```
> a:=-0.85;b:=-0.68;c:=0.75;d:=-0.9;q:=0.8;n:=10;
                              a := -0.85
                              b := -0.68
                              c := 0.75
                              d := -0.9
                              q := 0.8
                              n := 10                                         (1.6.13)
```

```
> hyp1:= q<abs(a) and abs(a)<1 and abs(b)<1 and abs(c)<1 and
  abs(d)<1 ;
                              hyp1 := true                                    (1.6.14)
```

zeros of P(n,a,b,c,d)

```
> xnAW:= sort([solve(qsimpcomb(AW(n,x,a,b,c,d,q)),x)]);
xnAW := [-0.99010807231259815903313516815177506042 87,                       (1.6.15)
     -0.95547387874410582361569392849721651998 22,
     -0.89004168524847189146062523699105023658 63,
     -0.78864125060164886287391577144313427676 53,
     -0.64724759148100638034396100057692154052 66,
     -0.46373931960934832138583863510527863343 26,
     -0.23859313535147107841862713455819925476 07,
     0.02437031471983945387747690809194930762185,
     0.31723359184508663285685216589157945149 87,
     0.62799901685436743922633981796281920473 95]
```

zeros of P(n-1,a,b,c,d)

```
> xnm1AW:= sort([solve(qsimpcomb(AW(n-1,x,a,b,c,d,q)),x)]);
xnm1AW := [-0.98905384520052150280245867595056623487 15,                      (1.6.16)
     -0.95020210899564874979558142412490890806 55,
     -0.87590103189812520837941259553726526874 54,
     -0.75945038535509068184217220473957936159 75,
     -0.59551085654049382571301436096719953822 24,
     -0.38114114219043490902033764280101660762 46,
```

```
    -0.11679093345553168322425627856152855372 34,
    0.19274862699268765876865631346042609690 67,
    0.53870261650065299954009476988424327309 22]
```

zeros of P(n,a/q,b,c,d)

```
> ynAW:= sort([solve(qsimpcomb(AW(n,x,a/q,b,c,d,q)),x)]);
```

```
ynAW := [-1.0012351518419913668213703304795431507 04,                    (1.6.17)
    -0.97815455849755366891573684348693315529 43,
    -0.92600273670937824069132477322789523140 98,
    -0.83775820861906858326435749793062232439 18,
    -0.70804023304851800219771052871547665575 22,
    -0.53318030695827373464948153330372965838 15,
    -0.31206737595204672751672928053164969100 39,
    -0.04700988105285459452247061772528506507 079,
    0.25541477043818762904424294778423134424 62,
    0.58453637495011987265850857162024694771 06]
```

```
> seq(evalb(ynAW[i]<xnAW[i] and xnAW[i]<xnm1AW[i] and xnm1AW[i]
  <ynAW[i+1] and ynAW[i+1]<xnAW[i+1]),i=1..n-1 )
```

```
              true, true, true, true, true, true, true, true, true          (1.6.18)
```

```
> unassign('a,b,c,d,q,n')
```

the monic q-Racah

```
> QR:=(n,x,alpha,beta,gamma,delta,q)-> qpochhammer(alpha*q, q,
  n)*qpochhammer(beta*delta*q, q, n)*qpochhammer(gamma*q, q, n)
  /qpochhammer(alpha*beta*q^(n+1), q, n)*add(qpochhammer(q^(-
  n),q,m)*qpochhammer(alpha*beta*q^(n+1),q,m)*mul(1-x*q^k+
  gamma*delta*q^(2*k+1),k=0..m-1)   *q^m /(qpochhammer(alpha*q,
  q,m)*qpochhammer(beta*delta*q,q,m)*qpochhammer(gamma*q,q,m)*
  qpochhammer(q,q,m)  )   ,m=0..n);
```

$$QR := (n, x, \alpha, \beta, \gamma, \delta, q) \rightarrow \frac{1}{qpochhammer(\alpha \beta q^{n+1}, q, n)} \Big(qpochhammer(\alpha q, q, n)\ qpochhammer(\beta \delta q, q, n)\ qpochhammer(\gamma q, q, n)\ add\big(\big(qpochhammer(q^{-n}, q, m)\ qpochhammer(\alpha \beta q^{n+1}, q, m)\ mul(1 - x q^{k} + \gamma \delta q^{2k+1}, k = 0 .. m-1)\ q^{m}\big) \big/ \big(qpochhammer(\alpha q, q, m)\ qpochhammer(\beta \delta q, q, m)\ qpochhammer(\gamma q, q, m)\ qpochhammer(q, q, m)\big), m = 0 .. n\big)\Big) \tag{1.7.1}$$

```
> FQR:=qpochhammer(alpha*q, q, n)*qpochhammer(beta*delta*q, q,
  n)*qpochhammer(gamma*q, q, n)/qpochhammer(alpha*beta*q^(n+1),
  q, n)*qhyperterm([q^(-n),alpha*beta*q^(n+1),q^(-x),gamma*
  delta*q^(x+1)],[alpha*q,beta*delta*q,gamma*q],q,q,k)
```

$$FQR := \big(qpochhammer(\alpha q, q, n)\ qpochhammer(\beta \delta q, q, n)\ qpochhammer(\gamma q, q, n)\ qpochhammer(q^{-n}, q, k)\ qpochhammer(\alpha \beta q^{n+1}, q, k)\ qpochhammer(q^{-x}, q, \tag{1.7.2}$$

$$k)\ \mathit{qpochhammer}\left(\gamma\,\delta\,q^{x+1}, q, k\right)\ q^{k}\Big) \Big/ \Big(\mathit{qpochhammer}\left(\alpha\,\beta\,q^{n+1}, q, n\right)\ \mathit{qpochhammer}(\alpha\,q, q, k)\ \mathit{qpochhammer}(\beta\,\delta\,q, q, k)\ \mathit{qpochhammer}(\gamma\,q, q, k)\ \mathit{qpochhammer}(q, q, k)\Big)$$

orthogonal on (1+gamma*delta*q, q^(-N)+gamma*delta*q^(N+1)) or for x in (0,N)

Equation (24)

```
> QRreca1:=subs(_C1=1,qMixRec(FQR,q,k,P(n),alpha,0,-1)):
> QRreca:=lhs(QRreca1)=combine(map(qsimpcomb,rhs(QRreca1)),
  power)
```

$$\mathit{QRreca} := P\left(n, \frac{\alpha}{q}\right) = P(n, \alpha) - \frac{\left(q^{n}-1\right)\left(\beta\,q^{n}-1\right)\left(\beta\,\delta\,q^{n}-1\right)\left(\gamma\,q^{n}-1\right)\,q\,\alpha\,P(n-1, \alpha)}{\left(\alpha\,\beta\,q^{2\,n}-1\right)\left(\alpha\,\beta\,q^{2\,n}-q\right)} \tag{1.7.3}$$

```
> QRrecb1:=subs(_C1=1,qMixRec(FQR,q,k,P(n),beta,0,-1)):
> QRrecb:=lhs(QRrecb1)=combine(map(qsimpcomb,rhs(QRrecb1)),
  power)
```

$$\mathit{QRrecb} := P\left(n, \frac{\beta}{q}\right) = P(n, \beta) - \frac{\left(q^{n}-1\right)\left(\alpha\,q^{n}-1\right)\left(\alpha\,q^{n}-\delta\right)\left(\gamma\,q^{n}-1\right)\,q\,\beta\,P(n-1, \beta)}{\left(\alpha\,\beta\,q^{2\,n}-1\right)\left(\alpha\,\beta\,q^{2\,n}-q\right)} \tag{1.7.4}$$

the monic al-Salam-Carlitz II

```
> ASCII:=(n,x,alpha,q)->(-alpha)^n*q^(-binomial(n,2))*add
  (qphihyperterm([q^(-n),x],[],q,q^(n)/alpha,j),j=0..n);
```

$$\mathit{ASCII} := (n, x, \alpha, q) \rightarrow (-\alpha)^{n}\, q^{-\mathrm{binomial}(n,\,2)}\ \mathit{add}\left(\mathit{qphihyperterm}\left(\left[q^{-n}, x\right], [\ ], q, \frac{q^{n}}{\alpha}, j\right), j = 0..n\right) \tag{1.8.1}$$

```
> Fasc2:=(-alpha)^n*q^(-binomial(n,2))*(qphihyperterm([q^(-n),
  x],[],q,q^(n)/alpha,k)):
```

Orthogonal for 0<alpha q<1 on (1, infinity)

```
> eq16a1:=qMixRec(Fasc2,q,k,V(n),alpha,0,-1)
```

$$\mathit{eq16a1} := V\left(n, \frac{\alpha}{q}\right) = -\frac{\left(q^{n}\,x-\alpha\right)\,q^{-n}\,V(n, \alpha)}{\alpha-x} - \frac{\alpha\left(q^{n}-1\right)\,V(n-1, \alpha)\,q^{1-2\,n}}{\alpha-x} \tag{1.8.2}$$

We conclude here that the Al-Salam-Carlitz II polynomial family is not quasi-orthogonal

the monic q-Meixner

```
> QM:=(n,x,beta,gamma,q)->1/((-1)^n*q^(n^2)/gamma^n/qpochhammer
  (beta*q, q, n))*add(qphihyperterm([q^(-n),x],[beta*q],q,-q^
  (n+1)/gamma,j),j=0..n);
```

$$QM := (n, x, \beta, \gamma, q) \rightarrow \frac{1}{(-1)^n q^{n^2}} \left(\gamma^n \mathit{qpochhammer}(\beta q, q, n)\ \mathit{add}\left(\mathit{qphihyperterm}\left([q^{-n}, x], [\beta q], q, -\frac{q^{n+1}}{\gamma}, j\right), j=0..n\right)\right) \tag{1.9.1}$$

```
> Fqm:=1/((-1)^n*q^(n^2)/gamma^n/qpochhammer(beta*q, q, n))*
  (qphihyperterm([q^(-n),x],[beta*q],q,-q^(n+1)/gamma,k)):
```

orthogonal for 0 \leq beta*q<1 and gamma>0 on (1,infinity).

```
> qMixRec(Fqm,q,k,M(n),beta,0,-1)
```

$$M\left(n, \frac{\beta}{q}\right) = \frac{\left(q^n x + \beta\gamma\right) q^{-n} M(n, \beta)}{\beta\gamma + x} - \frac{\left(q^n - 1\right) \gamma M(n-1, \beta) \left(q^n + \gamma\right) q^{-3n+1} \beta}{\beta\gamma + x} \tag{1.9.2}$$

We conclude that the q-Meixner polynomials are not quasi-orthogonal

the monic little q-Laguerre / Wall (special case from little q-Jacobi)

```
> LQLW:=(n,x,alpha,q)->1/((-1)^n*q^(-binomial(n, 2))
  /qpochhammer(alpha*q, q, n))*add(qphihyperterm([q^(-n),0],
  [alpha*q],q,q*x,j),j=0..n);
```

$$LQLW := (n, x, \alpha, q) \rightarrow \frac{1}{(-1)^n q^{-\mathrm{binomial}(n,\, 2)}} \left(\mathit{qpochhammer}(\alpha q, q, n)\ \mathit{add}\left(\mathit{qphihyperterm}\left([q^{-n}, 0], [\alpha q], q, q x, j\right), j=0..n\right)\right) \tag{1.10.1}$$

```
> Flqlw:=1/((-1)^n*q^(-binomial(n, 2))/qpochhammer(alpha*q, q,
  n))*(qphihyperterm([q^(-n),0],[alpha*q],q,q*x,k)):
```

Orthogonal for 0<alpha q<1

```
> eq11a1:=qMixRec(Flqlw,q,k,p(n),alpha,0,-1):
> eq11a2:=subs(_C1=1,eq11a1):
> eq11a:=lhs(eq11a2)=combine(map(qsimpcomb,rhs(eq11a2)),power)
```

$$eq11a := p\left(n, \frac{\alpha}{q}\right) = p(n, \alpha) - \alpha \left(q^n - 1\right) p(n-1, \alpha)\, q^{n-1} \tag{1.10.2}$$

the monic Affine q-Krawtchouk (special case from q-Hahn)

```
> AQK:=(n,x,p,N,q)->1/(1/(qpochhammer(p*q, q, n)*qpochhammer(q^
  (-N), q, n)))*add(qphihyperterm([q^(-n),x,0],[p*q,q^(-N)],q,
```

```
q,j),j=0..n);
```

$$AQK := (n, x, p, N, q) \rightarrow \mathit{qpochhammer}(p\,q, q, n)\ \mathit{qpochhammer}(q^{-N}, q, n)\ \mathit{add}(\mathit{qphihyperterm}([q^{-n}, x, 0], [p\,q, q^{-N}], q, q, j), j = 0\,..n) \tag{1.11.1}$$

```
> Faqk:=1/(1/(qpochhammer(p*q, q, n)*qpochhammer(q^(-NN), q, n)
))*(qphihyperterm([q^(-n),x,0],[p*q,q^(-NN)],q,q,k)):
```

Orthogonal for 0<pq<1

```
> eq10a1:=qMixRec(Faqk,q,k,K(n),p,0,-1):
> eq10a2:=subs({_C1=1,NN=N},eq10a1):
> eq10a:=lhs(eq10a2)=combine(map(qsimpcomb,rhs(eq10a2)),power)
```

$$eq10a := K\left(n, \frac{p}{q}\right) = K(n, p) - p\,(q^{n} - 1)\,(-q^{N+1} + q^{n})\,K(n-1, p)\,q^{-N-1} \tag{1.11.2}$$

the monic quantum q-Krawtchouk (special case from q-Hahn)

```
> QQK:=(n,x,p,N,q)->1/(p^n*q^(n^2)/qpochhammer(q^(-N), q, n))*
  add(qphihyperterm([q^(-n),x],[q^(-N)],q,p*q^(n+1),j),j=0..n);
```

$$QQK := (n, x, p, N, q) \rightarrow \frac{1}{p^{n}\,q^{n^2}}\left(\mathit{qpochhammer}(q^{-N}, q, n)\ \mathit{add}(\mathit{qphihyperterm}([q^{-n}, x], [q^{-N}], q, p\,q^{n+1}, j), j = 0\,..n)\right) \tag{1.12.1}$$

```
> Fqqk:=1/(p^n*q^(n^2)/qpochhammer(q^(-NN), q, n))*
  (qphihyperterm([q^(-n),x],[q^(-NN)],q,p*q^(n+1),k)):
```

orthogonal for p>q^(-N) on (1, q^(-N))

```
> eq81:=qMixRec(Fqqk,q,k,K(n),p,0,-1):
> eq82:=subs({_C1=1,NN=N},eq81):
> eq8:=lhs(eq82)=combine(map(qsimpcomb,rhs(eq82)),power)
```

$$eq8 := K\left(n, \frac{p}{q}\right) = K(n, p) - \frac{(q^{n} - 1)\,(-q^{N+1} + q^{n})\,K(n-1, p)\,q^{-2n-N}}{p} \tag{1.12.2}$$

the monic q-Krawtchouk

```
> QK:=(n,x,p,N,q)->1/(qpochhammer(-p*q^n, q, n)/qpochhammer(q^
  (-N), q, n))*add(qphihyperterm([q^(-n),x,-p*q^n],[q^(-N),0],
  q,q,j),j=0..n);
```

$$QK := (n, x, p, N, q) \rightarrow \frac{1}{\mathit{qpochhammer}(-p\,q^{n}, q, n)}\left(\mathit{qpochhammer}(q^{-N}, q, n)\ \mathit{add}(\mathit{qphihyperterm}([q^{-n}, x, -p\,q^{n}], [q^{-N}, 0], q, q, j), j = 0\,..n)\right) \tag{1.13.1}$$

```
> Fqk:=1/(qpochhammer(-p*q^n, q, n)/qpochhammer(q^(-NN), q, n))
  *(qphihyperterm([q^(-n),x,-p*q^n],[q^(-NN),0],q,q,k)):
```

Orthogonal for p>0

```
> eq9a1:=qMixRec(Fqk,q,k,K(n),p,0,-1):
> eq9a2:=subs({_C1=1,NN=N},eq9a1):
> eq9a:=lhs(eq9a2)=combine(map(qsimpcomb,rhs(eq9a2)),power)
```

$$eq9a := K\left(n, \frac{p}{q}\right) = K(n,p) - \frac{p\,\left(q^{n}-1\right)\,\left(-q^{N+1}+q^{n}\right)\,K(n-1,p)\,q^{n+1-N}}{\left(q^{2n}p+q^{2}\right)\,\left(q^{2n}p+q\right)} \tag{1.13.2}$$

the monic alternative q-Charlier

```
> AQC:=(n,x,alpha,q)->1/((-1)^n*q^(-binomial(n,2))*qpochhammer
  (-alpha*q^n, q, n))*add(qphihyperterm([q^(-n),-alpha*q^(n)],
  [0],q,q*x,j),j=0..n);
```

$$AQC := (n, x, \alpha, q) \rightarrow \frac{add\left(qphihyperterm\left(\left[q^{-n}, -\alpha\,q^{n}\right], [0], q, q\,x, j\right), j=0..n\right)}{(-1)^{n}\,q^{-\mathrm{binomial}(n,\,2)}\,qpochhammer\left(-\alpha\,q^{n}, q, n\right)} \tag{1.14.1}$$

```
> Faqc:=1/((-1)^n*q^(-binomial(n,2))*qpochhammer(-alpha*q^n, q,
  n))*(qphihyperterm([q^(-n),-alpha*q^(n)],[0],q,q*x,k)):
```

Orthogonal for alpha>0

```
> eq13a1:=qMixRec(Faqc,q,k,y(n),alpha,0,-1):
> eq13a2:=subs(_C1=0,eq13a1):
> eq13a:=lhs(eq13a2)=combine(map(qsimpcomb,rhs(eq13a2)),power)
```

$$eq13a := y\left(n, \frac{\alpha}{q}\right) = y(n,\alpha) - \frac{\alpha\,\left(q^{n}-1\right)\,y(n-1,\alpha)\,q^{2n+1}}{\left(\alpha\,q^{2n}+q\right)\,\left(\alpha\,q^{2n}+q^{2}\right)} \tag{1.14.2}$$

the monic q-Charlier

```
> QC:=(n,x,alpha,q)->1/((-1)^n*q^(n^2)/alpha^n)*add
  (qphihyperterm([q^(-n),x],[0],q,-q^(n+1)/alpha,j),j=0..n);
```

$$QC := (n, x, \alpha, q) \rightarrow \frac{\alpha^{n}\,add\left(qphihyperterm\left(\left[q^{-n}, x\right], [0], q, -\frac{q^{n+1}}{\alpha}, j\right), j=0..n\right)}{(-1)^{n}\,q^{n^{2}}} \tag{1.15.1}$$

```
> Fqc:=1/((-1)^n*q^(n^2)/alpha^n)*(qphihyperterm([q^(-n),x],
  [0],q,-q^(n+1)/alpha,k)):
```

Orthogonal for alpha>0

```
> eq14a1:=qMixRec(Fqc,q,k,C(n),alpha,0,1):
> eq14a2:=subs({_C1=1},eq14a1):
> eq14a:=lhs(eq14a2)=combine(map(qsimpcomb,rhs(eq14a2)),power)
```

$$eq14a := C(n, \alpha\,q) = C(n,\alpha) - \alpha\,\left(q^{n}-1\right)\,C(n-1,\alpha)\,q^{1-2n} \tag{1.15.2}$$

```
> read "hsum17.mpl";
            Package "Hypergeometric Summation", Maple V - Maple 17
            Copyright 1998-2013, Wolfram Koepf, University of Kassel      (2)
```

Quasi-orthogonality of the Bessel and the classical O.P. of the quadratic lattice

```
> Mixedrec:=proc(F,k,Sn,alpha,beta,shift)
  local n,S,a,b,sigma,rat,p,q,r,upd,deg,f,j,jj,l,var,req,sol,
  sol2,num,den,J;
  if type(Sn,function) then S:=op(0,Sn); n:=op(1,Sn) else n:=Sn
  end if;
    for J from 1 to MAXORDER do
   a:=subs({alpha=alpha+shift,beta=beta+shift},F)-add(sigma[j]*
  subs(n=n-j,F),j=0..J);
   rat:=ratio(a,k);
   if not type(rat,ratpoly(anything,k)) then
     ERROR(`Algorithm not applicable`)
   fi;
   # p:=1: q:=subs(k=k-1,numer(rat)): r:=subs(k=k-1,denom(rat)):
   p:=1: q:=numer(rat): r:=denom(rat):
   upd:=update(p,q,r,k);
   p:=op(1,upd): q:=op(2,upd): r:=op(3,upd):
   deg:=degreebound(p,q,r,k);
  # Maple 13: if deg>=0 then
   if deg>=-1 then
     f:=add(b[j]*k^j,j=0..deg);
     var:={seq(sigma[jj],jj=0..J),seq(b[jj],jj=0..deg)};
     req:=collect(subs(k=k+1,q)*f-r*subs(k=k-1,f)-p,k);
     sol:={solve({coeffs(req,k)},var)};
     if not(sol={} or {seq(op(2,op(l,op(1,sol))),l=1..nops(op(1,
  sol)))}={0}) then
       if beta=0 then
        sol2:=add(sigma[j]*S(n-j,alpha),j=0..J);
        sol2:=subs(op(1,sol),sol2);
        RETURN( S(n,alpha+shift)=map(factor,sol2));
       else
        sol2:=add(sigma[j]*S(n-j,alpha,beta),j=0..J);
        sol2:=subs(op(1,sol),sol2);
        RETURN( S(n,alpha+shift,beta+shift)=map(factor,sol2));
       fi;

     fi;
   fi;
  od;
  ERROR(cat(`Algorithm finds no derivative rule of order <= `,
  MAXORDER))
```

```
end:
```

the monic Wilson polynomials

```
> Wilson:=(n,x,a,b,c,d)->(-1)^n*pochhammer(a+b,n)*pochhammer(a+
  c,n)*pochhammer(a+d,n)/pochhammer(n+a+b+c+d-1,n)*add
  (pochhammer(-n, k)*pochhammer(n+a+b+c+d-1, k)*mul(a^2+2*a*j+
  j^2+x,j=0..k-1) /(pochhammer(a+b, k)*pochhammer(a+c, k)*
  pochhammer(a+d, k)*factorial(k)), k=0..n)
```

$$Wilson := (n, x, a, b, c, d) \rightarrow \frac{1}{\mathrm{pochhammer}(n+a+b+c+d-1, n)} \Big((-1)^n \, \mathrm{pochhammer}(a+b, n) \, \mathrm{pochhammer}(a+c, n) \, \mathrm{pochhammer}(a+d, n) \, add\Big(\big(\mathrm{pochhammer}(-n, k) \, \mathrm{pochhammer}(n+a+b+c+d-1, k) \, mul(a^2 + 2\,a\,j + j^2 + x, j = 0..k-1)\big) \Big/ \big(\mathrm{pochhammer}(a+b, k) \, \mathrm{pochhammer}(a+c, k) \, \mathrm{pochhammer}(a+d, k) \, k!\big), k = 0..n \Big) \Big) \tag{2.1.1}$$

```
> FWilson:= (-1)^n*pochhammer(a+b,n)*pochhammer(a+c,n)*
  pochhammer(a+d,n)/pochhammer(n+a+b+c+d-1,n)*hyperterm([-n,n+
  a+b+c+d-1, a+I*x, a-I*x],[a+b, a+c, a+d],1,k):
```

Orthogonal on (0,infinity) if Re(a,b,c,d)>0 and non-real parameters occur in conjugate pairs (for example c=bar(a), d=bar(b) if a,b are complex)

Proposition 29

```
> recWil1:=Mixedrec(FWilson,k,W(n),a,0,-1)
```

$$recWil1 := W(n, a-1) = W(n, a) + \frac{n\,(c+d+n-1)\,(b+d+n-1)\,(b+c+n-1)\,W(n-1, a)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.1.2}$$

```
> recWil21:=Mixedrec(FWilson,k,W(n),b,0,-1):
```

```
> recWil22:=subs(coeff(rhs(recWil21),W(n, b))=1,recWil21):
```

```
> recWil2:=lhs(recWil22)=collect(rhs(recWil22),[W(n, b), W(n-1,
  b)], simpcomb)
```

$$recWil2 := W(n, b-1) = W(n, b) + \frac{(c+d+n-1)\,(a-1+d+n)\,(a-1+c+n)\,n\,W(n-1, b)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.1.3}$$

```
> recWil31:=Mixedrec(FWilson,k,W(n),c,0,-1):
```

```
> recWil32:=subs(coeff(rhs(recWil31),W(n, c))=1,recWil31):
```

```
> recWil3:=lhs(recWil32)=collect(rhs(recWil32),[W(n, c), W(n-1,
  c)], simpcomb)
```

$$recWil3 := W(n, c-1) = W(n, c) + \frac{(b+d+n-1)\,(a-1+d+n)\,(a-1+b+n)\,n\,W(n-1, c)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.1.4}$$

```
> recWil41:=Mixedrec(FWilson,k,W(n),d,0,-1):
```

```
> recWil42:=subs(coeff(rhs(recWil41),W(n, d))=1,recWil41):
```

```
> recWil4:=lhs(recWil42)=collect(rhs(recWil42),[W(n, d), W(n-1,
  d)], simpcomb)
```

$$recWil4 := W(n, d-1) = W(n, d) \tag{2.1.5}$$

$$+\frac{(b+c+n-1)\,(a-1+c+n)\,(a-1+b+n)\,n\,W(n-1,d)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)}$$

Theorem 31

```
> a:=0.65;b:=5; c:=2.5; d:=25; n:=10
```

$$a := 0.65$$
$$b := 5$$
$$c := 2.5$$
$$d := 25$$
$$n := 10 \tag{2.1.6}$$

```
> hyp:=b>0 and c>0 and d>0 and 0<a and a<1;
```

$$hyp := true \tag{2.1.7}$$

zeros of P_n(a,b,c,d)

```
> xnWs:= sort([solve(qsimpcomb(Wilson(n,x,a,b,c,d)),x)]);
```

$$\begin{aligned}xnWs := [&2.6374451395897674497410543540923406095\,45,\\ &8.2326130238826928594246924743749160054\,52,\\ &16.857742915360916919147322118122844187\,22,\\ &29.069911216700249531321432769546816062\,40,\\ &45.656269941633199749599834662258888448\,22,\\ &67.708341039897756104664337368583580298\,85,\\ &96.814920527751197568951834667190784306\,43,\\ &135.50344602186595073306352476874047531\,88,\\ &188.43732134219207408278404425712261664\,86,\\ &267.14650496015845306581805159222480263\,06]\end{aligned} \tag{2.1.8}$$

zeros of P_{n-1}(a,b,c,d)

```
> xnm1Ws:= sort([solve(qsimpcomb(Wilson(n-1,x,a,b,c,d)),x)]);
```

$$\begin{aligned}xnm1Ws := [&2.8724356402240774709052625691214552576\,09,\\ &8.9386510399309437073948998400180732174\,16,\\ &18.356088990121504011700435256735848633\,11,\\ &31.829256601576906090707241063037579449\,25,\\ &50.381489150775340549765494169847446929\,83,\\ &75.513189754544690829372626586056330691\,76,\\ &109.60835432187042360589464457338883495\,37,\\ &157.05194682458172196734757812245045879\,92,\\ &228.64899459397357793171344548974071672\,77]\end{aligned} \tag{2.1.9}$$

zeros of P_n(a-1,b,c,d)

```
> ynWs:= sort([solve(qsimpcomb(Wilson(n,x,a-1,b,c,d)),x)]);
```

$$\begin{aligned}ynWs := [&0.75810871089181731965218792373710286871\,52,\\ &5.1795470503325144440320583203305009982\,15,\\ &12.532760284691927971124025347977206751\,55,\\ &23.340294217318394905880123796151048464\,00,\\ &38.359377699881161468628258608432495603\,90,\\ &58.650390481057644753812037155044124945\,38,\\ &85.758823862867437674717530708465443374\,58,\end{aligned} \tag{2.1.10}$$

```
122.1404879604011465169239352576329809261,
172.3192083423090455542996665585546431908,
247.4614998947354497697935860943623890677]
```

```
> seq(evalb(ynWs[i]<xnWs[i] and xnWs[i]<xnm1Ws[i] and xnm1Ws[i]
  <ynWs[i+1] and ynWs[i+1]<xnWs[i+1] ), i=1..n-1)
```

$$true, true, true, true, true, true, true, true, true \tag{2.1.11}$$

```
> unassign('a,b,c,d,n')
```

the monic Racah polynomials

```
> Racah:=(n,X,alpha, beta, gamma, delta)-> pochhammer(alpha+1,
  n)*pochhammer(beta+delta+1,n)*pochhammer(gamma+1,n)
  /pochhammer(n+alpha+beta+1,n)*add(pochhammer(-n, k)*
  pochhammer(n+alpha+beta+1, k)*mul(-X+j*(gamma+delta+j+1)   ,j=
  0..k-1)/(pochhammer(alpha+1, k)*pochhammer(beta+delta+1, k)*
  pochhammer(gamma+1, k)*factorial(k))   ,k=0..n)
```

$$Racah := (n, X, \alpha, \beta, \gamma, \delta) \rightarrow \frac{1}{\mathrm{pochhammer}(n+\alpha+\beta+1, n)} \Bigg(\mathrm{pochhammer}(\alpha+1, n)\ \mathrm{pochhammer}(\beta+\delta+1, n)\ \mathrm{pochhammer}(\gamma+1, n)\ add\Big(\big(\mathrm{pochhammer}(-n, k)\ \mathrm{pochhammer}(n+\alpha+\beta+1, k)\ mul(-X+j\,(\gamma+\delta+j+1), j=0..k-1)\big) \Big/ \big(\mathrm{pochhammer}(\alpha+1, k)\ \mathrm{pochhammer}(\beta+\delta+1, k)\ \mathrm{pochhammer}(\gamma+1, k)\ k!\big), k=0..n\Big)\Bigg) \tag{2.2.1}$$

```
> FRac:=pochhammer(alpha+1,n)*pochhammer(beta+delta+1,n)*
  pochhammer(gamma+1,n)/pochhammer(n+alpha+beta+1,n)*hyperterm(
  [-n,n+alpha+beta+1,-x,x+gamma+delta+1],[alpha+1, beta+
  delta+1, gamma+1],1  ,k)
```

$$FRac := \big(\mathrm{pochhammer}(\alpha+1, n)\ \mathrm{pochhammer}(\beta+\delta+1, n)\ \mathrm{pochhammer}(\gamma+1, n)\ \mathrm{pochhammer}(-n, k)\ \mathrm{pochhammer}(n+\alpha+\beta+1, k)\ \mathrm{pochhammer}(-x, k)\ \mathrm{pochhammer}(x+\gamma+\delta+1, k)\big) \Big/ \big(\mathrm{pochhammer}(n+\alpha+\beta+1, n)\ \mathrm{pochhammer}(\alpha+1, k)\ \mathrm{pochhammer}(\beta+\delta+1, k)\ \mathrm{pochhammer}(\gamma+1, k)\ k!\big) \tag{2.2.2}$$

Orthogonal on (0, N*(N+gamma+delta+1)) for alpha+1=-N or beta+delta+1=-N or gamma+1=-N. Shift only alpha and beta not to change x

```
> recRac11:=Mixedrec(FRac,k,R(n),alpha,0,-1):
> recRac12:=subs(coeff(rhs(recRac11),R(n, alpha))=1,recRac11):
> recRac1:=lhs(recRac12)=collect(rhs(recRac12),[R(n, alpha), R
  (n-1,alpha)], simpcomb)
```

$$recRac1 := R(n, \alpha-1) = R(n, \alpha) - \frac{(\beta+n)\ (\beta+\delta+n)\ (\gamma+n)\ n\ R(n-1, \alpha)}{(2\,n+\alpha+\beta)\ (2\,n+\alpha+\beta-1)} \tag{2.2.3}$$

```
> recRac21:=Mixedrec(FRac,k,R(n),beta,0,-1):
> recRac22:=subs(coeff(rhs(recRac21),R(n, beta))=1,recRac21):
> recRac2:=lhs(recRac22)=collect(rhs(recRac22),[R(n, beta), R
  (n-1,beta)], simpcomb)
```

$$recRac2 := R(n, \beta-1) = R(n, \beta) - \frac{(\alpha+n)\ (\alpha-\delta+n)\ (\gamma+n)\ n\ R(n-1, \beta)}{(2\,n+\alpha+\beta)\ (2\,n+\alpha+\beta-1)} \tag{2.2.4}$$

the monic continuous Hahn

```
> CHahn:=(n,x,a,b,c,d)->I^n*pochhammer(a+c,n)*pochhammer(a+d,n)
  /pochhammer(n+a+b+c+d-1,n)*add(hyperterm([-n,n+a+b+c+d-1,a+I*
  x],[a+c,a+d],1 , k),k=0..n)
```

$$CHahn := (n, x, a, b, c, d) \to \frac{1}{\mathrm{pochhammer}(n+a+b+c+d-1, n)} \left(\mathrm{I}^n\, \mathrm{pochhammer}(a+c, n)\, \mathrm{pochhammer}(a+d, n)\, add(hyperterm([-n, n+a+b+c+d-1, a+\mathrm{I}\,x], [a+c, a+d], 1, k), k=0..n)\right) \tag{2.3.1}$$

```
> Fch:=I^n*pochhammer(a+c,n)*pochhammer(a+d,n)/pochhammer(n+a+
  b+c+d-1,n)*(hyperterm([-n,n+a+b+c+d-1,a+I*x],[a+c,a+d],1 , k)
  ):
```

Orthogonal on (-infinity, infinity) for Re(a, b, c, d)>0 and c=bar(a), d=bar(b)

Proposition 34

```
> eqch1:=Mixedrec(Fch,k,P(n),a,0,-1)
```

$$eqch1 := P(n, a-1) = P(n, a) + \frac{\mathrm{I}\,P(n-1, a)\,(b+c+n-1)\,(b+d+n-1)\,n}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.2}$$

```
> Eqch1 := P(n, a-1,b,c,d) = P(n, a,b,c,d)+I*P(n-1, a,b,c,d)*
  (b+c+n-1)*(b+d+n-1)*n/((2*n+a-2+b+c+d)*(2*n+a-3+b+c+d))
```

$$Eqch1 := P(n, a-1, b, c, d) = P(n, a, b, c, d) + \frac{\mathrm{I}\,P(n-1, a, b, c, d)\,(b+c+n-1)\,(b+d+n-1)\,n}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.3}$$

```
> eqch2:=Mixedrec(Fch,k,P(n),b,0,-1)
```

$$eqch2 := P(n, b-1) = P(n, b) + \frac{\mathrm{I}\,n\,(a-1+d+n)\,(a-1+c+n)\,P(n-1, b)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.4}$$

```
> Eqch2 := P(n, a,b-1,c,d) = P(n, a,b,c,d)+I*n*(a-1+d+n)*(a-1+
  c+n)*P(n-1, a,b,c,d)/((2*n+a-3+b+c+d)*(2*n+a-2+b+c+d))
```

$$Eqch2 := P(n, a, b-1, c, d) = P(n, a, b, c, d) + \frac{\mathrm{I}\,n\,(a-1+d+n)\,(a-1+c+n)\,P(n-1, a, b, c, d)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.5}$$

```
> eqch3:=Mixedrec(Fch,k,P(n),c,0,-1)
```

$$eqch3 := P(n, c-1) = P(n, c) - \frac{\mathrm{I}\,(b+d+n-1)\,(a-1+d+n)\,n\,P(n-1, c)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.6}$$

```
> Eqch3 := P(n, a,b,c-1,d) = P(n, a,b,c,d)-I*(b+d+n-1)*(a-1+d+
  n)*n*P(n-1,a,b,c,d)/((2*n+a-3+b+c+d)*(2*n+a-2+b+c+d))
```

$$Eqch3 := P(n, a, b, c-1, d) = P(n, a, b, c, d) - \frac{\mathrm{I}\,(b+d+n-1)\,(a-1+d+n)\,n\,P(n-1, a, b, c, d)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.7}$$

```
> eqch4:=Mixedrec(Fch,k,P(n),d,0,-1)
```

$$eqch4 := P(n, d-1) = P(n, d) \tag{2.3.8}$$

$$-\frac{\mathrm{I}\,(b+c+n-1)\,(a-1+c+n)\,n\,P(n-1,d)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)}$$

```
> Eqch4  := P(n, a,b,c,d-1) = P(n, a,b,c,d)-I*(b+c+n-1)*(a-1+c+
  n)*n*P(n-1, a,b,c,d)/((2*n+a-3+b+c+d)*(2*n+a-2+b+c+d))
```

$$Eqch4 := P(n,a,b,c,d-1) = P(n,a,b,c,d) - \frac{\mathrm{I}\,(b+c+n-1)\,(a-1+c+n)\,n\,P(n-1,a,b,c,d)}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)} \tag{2.3.9}$$

Corollary 35

```
> Eqch1A1:=subs(c=c-1,Eqch1):
```

```
> Eqch1A2:=subs({Eqch3, subs(n=n-1,Eqch3)}, Eqch1A1):
```

```
> Eqch1A:=lhs(Eqch1A2)=collect(rhs(Eqch1A2), [P(n, a, b, c, d),
  P(n-1, a, b, c, d),P(n-2, a, b, c, d)], simpcomb)
```

$$Eqch1A := P(n,a-1,b,c-1,d) = P(n,a,b,c,d) - \frac{\mathrm{I}\,(a+d-b-c)\,(b+d+n-1)\,n\,P(n-1,a,b,c,d)}{(2\,n+a-2+b+c+d)\,(a+b+c+d+2\,n-4)} + \Big((b+d+n-2)\,(a+d+n-2)\,(n-1)\,P(n-2,a,b,c,d)\,(b+c+n-2)\,(b+d+n-1)\,n\Big)\Big/\Big((2\,n+a-5+b+c+d)\,(a+b+c+d+2\,n-4)^2\,(2\,n+a-3+b+c+d)\Big) \tag{2.3.10}$$

```
> Eqch2A1:=subs(d=d-1,Eqch2):
```

```
> Eqch2A2:=subs({Eqch4, subs(n=n-1,Eqch4)}, Eqch2A1):
```

```
> Eqch2A:=lhs(Eqch2A2)=collect(rhs(Eqch2A2), [P(n, a, b, c, d),
  P(n-1, a, b, c, d),P(n-2, a, b, c, d)], simpcomb)
```

$$Eqch2A := P(n,a,b-1,c,d-1) = P(n,a,b,c,d) + \frac{\mathrm{I}\,(a+d-b-c)\,(a-1+c+n)\,n\,P(n-1,a,b,c,d)}{(2\,n+a-2+b+c+d)\,(a+b+c+d+2\,n-4)} + \Big(n\,(a+d+n-2)\,(a-1+c+n)\,(b+c+n-2)\,(a+c+n-2)\,(n-1)\,P(n-2,a,b,c,d)\Big)\Big/\Big((2\,n+a-5+b+c+d)\,(a+b+c+d+2\,n-4)^2\,(2\,n+a-3+b+c+d)\Big) \tag{2.3.11}$$

the Jacobi polynomials

```
> Jacobi := (n, alpha, beta, x) -> pochhammer(alpha+1, n)/n! *
  add(hyperterm([-n,n+alpha+beta+1],[alpha+1],(1-x)/2,k), k = 0
  .. n);
```

$$Jacobi := (n,\alpha,\beta,x) \rightarrow \frac{1}{n!}\left(\mathrm{pochhammer}(\alpha+1,n)\ add\left(hyperterm\left(\left[-n,n+\alpha+\beta+1\right],[\alpha+1],\frac{1}{2}-\frac{1}{2}\,x,k\right),k=0..n\right)\right) \tag{2.4.1}$$

```
> FJac:=pochhammer(alpha+1, n)/n! *(hyperterm([-n,n+alpha+
  beta+1],[alpha+1],(1-x)/2,k));
```

$$FJac := \frac{1}{n!\ \mathrm{pochhammer}(\alpha+1,k)\ k!}\Big(\mathrm{pochhammer}(\alpha+1,n)\ \mathrm{pochhammer}(-n, \tag{2.4.2}$$

$$k)\ \mathrm{pochhammer}(n+\alpha+\beta+1, k)\ \left(-\frac{1}{2}\,x+\frac{1}{2}\right)^{k}\Bigg)$$

Orthogonal on (-1, 1) for alpha>-1 and beta>-1

```
> rec11Jac:=Mixedrec(FJac,k,L(n),alpha,0,-1)
```

$$rec11Jac := L(n, \alpha-1) = \frac{(n+\alpha+\beta)\,L(n,\alpha)}{2\,n+\alpha+\beta} - \frac{(\beta+n)\,L(n-1,\alpha)}{2\,n+\alpha+\beta} \tag{2.4.3}$$

```
> rec12Jac:=Mixedrec(FJac,k,L(n),beta,0,-1)
```

$$rec12Jac := L(n, \beta-1) = \frac{(n+\alpha+\beta)\,L(n,\beta)}{2\,n+\alpha+\beta} + \frac{(\alpha+n)\,L(n-1,\beta)}{2\,n+\alpha+\beta} \tag{2.4.4}$$

the Gegenbauer polynomials

```
> Gegenbauer :=(n, lambda, x) -> pochhammer(2*lambda, n)/n!*
  add(hyperterm([-n, n+2*lambda],[lambda+1/2],(1-x)/2,k), k = 0
  .. n);
```

$$Gegenbauer := (n,\lambda,x) \rightarrow \frac{1}{n!}\Bigg(\mathrm{pochhammer}(2\,\lambda, n)\ add\Bigg(hyperterm\Bigg([-n, n+2\,\lambda], \left[\lambda+\frac{1}{2}\right], \frac{1}{2}-\frac{1}{2}\,x, k\Bigg), k=0\,..n\Bigg)\Bigg) \tag{2.5.1}$$

```
> FGe:=pochhammer(2*lambda, n)/n!*(hyperterm([-n, n+2*lambda],
  [lambda+1/2],(1-x)/2,k));
```

$$FGe := \frac{1}{n!\ \mathrm{pochhammer}\left(\lambda+\frac{1}{2}, k\right)\ k!}\Bigg(\mathrm{pochhammer}(2\,\lambda, n)\ \mathrm{pochhammer}(-n, k)\ \mathrm{pochhammer}(n+2\,\lambda, k)\ \left(-\frac{1}{2}\,x+\frac{1}{2}\right)^{k}\Bigg) \tag{2.5.2}$$

Orthogonal on (-1,1) for lambda>-1/2

```
> rec11Gegen:=Mixedrec(FGe,k,L(n),lambda,0,-1)
```

$$rec11Gegen := L(n, \lambda-1) = \frac{2\,(\lambda-1)\,L(n,\lambda)}{n+2\,\lambda-2} - \frac{2\,(\lambda-1)\,x\,L(n-1,\lambda)}{n+2\,\lambda-2} \tag{2.5.3}$$

the Laguerre polynomials

```
> Laguerre :=  (n, alpha, x) -> pochhammer(alpha+1, n)/n!*add
  (hyperterm([-n],[alpha+1],x,k), k = 0 .. n);
```

$$Laguerre := (n,\alpha,x) \rightarrow \frac{\mathrm{pochhammer}(\alpha+1, n)\ add(hyperterm([-n], [\alpha+1], x, k), k=0\,..n)}{n!} \tag{2.6.1}$$

```
> FLag:=pochhammer(alpha+1, n)/n!*(hyperterm([-n],[alpha+1],x,
  k));
```

$$FLag := \frac{\mathrm{pochhammer}(\alpha+1, n)\ \mathrm{pochhammer}(-n, k)\ x^{k}}{n!\ \mathrm{pochhammer}(\alpha+1, k)\ k!} \tag{2.6.2}$$

Orthogonal on (0, infinity) for alpha>-1

```
> rec11Jac:=Mixedrec(FLag,k,L(n),alpha,0,-1)
```

$$rec11Jac := L(n, \alpha - 1) = L(n, \alpha) - L(n - 1, \alpha) \tag{2.6.3}$$

the monic Bessel polynomials

```
> Bessel :=  (n, alpha, x) -> 2^n/pochhammer(n+alpha+1,n)*add
  (hyperterm([-n,n+alpha+1],[],-x/2,k), k = 0 .. n) ;
```

$$Bessel := (n, \alpha, x) \rightarrow \frac{2^n\, add\left(hyperterm\left([-n, n+\alpha+1], [\,], -\frac{1}{2}\, x, k\right), k=0\,..n\right)}{\mathrm{pochhammer}(n+\alpha+1, n)} \tag{2.7.1}$$

```
> FBess:=2^n/pochhammer(n+alpha+1,n)*hyperterm([-n,n+alpha+1],
  [],-x/2,k)
```

$$FBess := \frac{2^n\, \mathrm{pochhammer}(-n, k)\, \mathrm{pochhammer}(\alpha+1+n, k)\, \left(-\frac{1}{2}\, x\right)^k}{\mathrm{pochhammer}(\alpha+1+n, n)\, k!} \tag{2.7.2}$$

Orthogonal on (0, infinity) for n=0,1,...,N, alpha<-2N-1

```
> recBess1:=Mixedrec(FBess,k,L(n),alpha,0,1)
```

$$recBess1 := L(n, \alpha+1) = \frac{\left(\alpha^2\, x + 4\,\alpha\, n\, x + 4\, n^2\, x + \alpha\, x + 2\, n\, x - 2\, n\right) L(n, \alpha)}{x\, (\alpha+1+2\, n)\, (\alpha+2\, n)} + \frac{4\, n\, (\alpha+n)\, L(n-1, \alpha)}{x\, (\alpha+1+2\, n)\, (\alpha+2\, n-1)\, (\alpha+2\, n)^2} \tag{2.7.3}$$

We conclude that the Bessel polynomials are not quasi-orthogonal

the monic Hahn polynomials

```
> Hahn:=(n,x,alpha,beta,N)->pochhammer(alpha+1,n)*pochhammer(-
  N,n)/pochhammer(n+alpha+beta+1,n)*add(hyperterm([-n,n+1+
  alpha+beta,-x],[alpha+1,-N],1,k),k=0..n);
```

$$Hahn := (n, x, \alpha, \beta, N) \rightarrow \frac{1}{\mathrm{pochhammer}(n+\alpha+\beta+1, n)} \left(\mathrm{pochhammer}(\alpha+1, n)\, \mathrm{pochhammer}(-N, n)\, add\left(hyperterm\left([-n, n+\alpha+\beta+1, -x], [\alpha+1, -N], 1, k\right), k=0\,..n\right)\right) \tag{2.8.1}$$

```
> FHahn:=pochhammer(alpha+1,n)*pochhammer(-N,n)/pochhammer(n+
  alpha+beta+1,n)*(hyperterm([-n,n+1+alpha+beta,-x],[alpha+1,-
  N],1,k));
```

$$FHahn := \left(\mathrm{pochhammer}(\alpha+1, n)\, \mathrm{pochhammer}(-N, n)\, \mathrm{pochhammer}(-n, k)\, \mathrm{pochhammer}(n+\alpha+\beta+1, k)\, \mathrm{pochhammer}(-x, k)\right) \Big/ \left(\mathrm{pochhammer}(n+\alpha+\beta+1, n)\, \mathrm{pochhammer}(\alpha+1, k)\, \mathrm{pochhammer}(-N, k)\, k!\right) \tag{2.8.2}$$

Orthogonal on (0,N) for alpha>-1 and beta>-1

```
> recHahn1:=Mixedrec(FHahn,k,L(n),alpha,0,-1)
```

$$recHahn1 := L(n, \alpha - 1) = L(n, \alpha) + \frac{n\,(\beta + n)\,(-n + 1 + N)\,L(n - 1, \alpha)}{(2\,n + \alpha + \beta)\,(2\,n + \alpha + \beta - 1)} \qquad \textbf{(2.8.3)}$$

```
> recHahn2:=Mixedrec(FHahn,k,L(n),beta,0,-1)
```

$$recHahn2 := L(n, \beta - 1) = L(n, \beta) - \frac{n\,(\alpha + n)\,(-n + 1 + N)\,L(n - 1, \beta)}{(2\,n + \alpha + \beta)\,(2\,n + \alpha + \beta - 1)} \qquad \textbf{(2.8.4)}$$

the monic Meixner polynomials

```
> Meixner:=(n,x,gamma,mu)->pochhammer(gamma,n)*(mu/(mu-1))^n*
  add(hyperterm([-n,-x],[gamma],1-1/mu,m),m=0..n );
```

$$Meixner := (n, x, \gamma, \mu) \rightarrow \mathrm{pochhammer}(\gamma, n) \left(\frac{\mu}{\mu - 1}\right)^{n} add\left(hyperterm\left([-n, -x], [\gamma], 1 - \frac{1}{\mu}, m\right), m = 0\,..n\right) \qquad \textbf{(2.9.1)}$$

```
> FMeix:=pochhammer(gamma,n)*(mu/(mu-1))^n*(hyperterm([-n,-x],
  [gamma],1-1/mu,k)):
```

Orthogonal on (0, infinity) for gamma>0 and 0<mu<1

```
> recMeix1:=Mixedrec(FMeix,k,M(n),gamma,0,-1)
```

$$recMeix1 := M(n, \gamma - 1) = M(n, \gamma) - \frac{\mu\, n\, M(n - 1, \gamma)}{\mu - 1} \qquad \textbf{(2.9.2)}$$

the monic Charlier polynomials

```
> Charlier:=(n,x,alpha)->(-alpha)^n*add(hyperterm([-n,-x],[],
  -1/alpha,k) ,k=0..n );
```

$$Charlier := (n, x, \alpha) \rightarrow (-\alpha)^{n}\, add\left(hyperterm\left([-n, -x], [\,], -\frac{1}{\alpha}, k\right), k = 0\,..n\right) \qquad \textbf{(2.10.1)}$$

```
> FChar:=(-alpha)^n*(hyperterm([-n,-x],[],-1/alpha,k)):
```

Orthogonal on (0, infinity) for alpha>0

```
> recChar:=Mixedrec(FChar,k,C(n),alpha,0,-1)
Error, (in Mixedrec) Algorithm not applicable
```

the monic Krawtchouk polynomials

```
> Krawtchouk:=(n,x,p,N)->pochhammer(-N,n)*p^n*add(hyperterm([-
  n,-x],[-N],1/p,k)  ,k=0..n );
```

$$Krawtchouk := (n, x, p, N) \rightarrow \mathrm{pochhammer}(-N, n)\, p^{n}\, add(hyperterm([-n, -x], [\qquad \textbf{(2.11.1)}$$

$$-N],\ \frac{1}{p},\ k\Big),\ k=0\,..n\Big)$$

```
> FKrawt:=pochhammer(-N,n)*p^n*(hyperterm([-n,-x],[-N],1/p,k));
```

$$FKrawt := \frac{\mathrm{pochhammer}(-N,\, n)\; p^{n}\; \mathrm{pochhammer}(-n,\, k)\; \mathrm{pochhammer}(-x,\, k)\; \left(\frac{1}{p}\right)^{k}}{\mathrm{pochhammer}(-N,\, k)\; k!} \qquad \textbf{(2.11.2)}$$

Orthogonal on (0, N) for 0<p<1

```
> recKraw:=Mixedrec(FKrawt,k,M(n),p,0,+1)
Error, (in Mixedrec) Algorithm not applicable
```